ENRICHED METHODS FOR VIBRATION ANALYSIS OF FRAMED STRUCTURES

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Abstract. *This work deals with the enriched methods applied to vibration analysis of framed structures. These methods have as main feature the enrichment of the shape functions space of the classical FEM by adding other non polynomial functions. The variational problem of free vibration of bars and beams is formulated and the main aspects of some enriched methods are discussed. The Assumed Mode Method (AMM), the Composite Element Method (CEM), the p Fourier element and the Generalized Finite Element Method (GFEM) are presented. These approaches result in hierarchical refinements. The application of these enriched methods in vibration analysis of framed structures is investigated. The eigenvalues obtained by enriched methods are compared with those obtained by analytical solution and by h and p-versions of FEM. The numerical results show that GFEM is the enriched method that has the highest rates of convergence and the better results.*

Keywords*: enriched methods, vibration analysis, free vibration, generalized finite element method.*

1. INTRODUCTION

The Finite Element Method (FEM) is the numerical method commonly used in vibration analysis of structures. Its approximated solution can be improved using two refinement techniques: *h* and *p*-versions. The *h*-version consists of the refinement of element mesh; the *p*-version may be understood as the increase in the number of shape functions in the element domain without any change in the mesh. The conventional *p*-version of FEM consists of increasing the polynomial degree in the solution. The *h*-version of FEM gives good results for the lowest frequencies but demands great computational cost to work up the accuracy for the higher frequencies. The accuracy of the FEM can be improved applying the polynomial *p* refinement.

Some enriched methods based on the FEM have been developed in last 20 years seeking to increase the accuracy of the solutions for the higher frequencies with lower computational cost. Engels (1992), Ganesan and Engels (1992) present the Assumed Mode Method (AMM) which is obtained adding to the FEM shape functions set, some interface restrained assumed modes. The Composite Element Method (CEM) (Zeng, 1998a, b and c) is obtained by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from analytical solutions of simple vibration problems. A modified CEM applied to analysis of beams is proposed by Lu and Law (2007). The use of products between polynomials and Fourier series instead of polynomials alone in the element shape functions is recommended by Leung and Chan (1998). They develop the Fourier *p*-element applied to the vibration analysis of bars, beams and plates. A formulation of the Generalized Finite Element Method (GFEM), which was conceived on the basis of the Partition of Unity Method (Melenk and Babuska, 1996) allowing the inclusion of a priori knowledge about the fundamental solution of the governing differential equation, was developed by Arndt, Machado and Scremin (2007 and 2009) for vibration analysis of bars and beams. These methods have the same characteristics and they will be called enriched methods in this work.

This work discusses and compares the enriched methods in free vibration analysis of framed structures.

2. STRUCTURAL FREE VIBRATION PROBLEM

The structural free vibration problem is an eigenvalue problem with variational statement: find a pair (λ, u), with $u \in H(\Omega)$ and $\lambda \in \mathbf{R}$, so that

$$B(u,w) = \lambda F(u,w),\ \forall w \in H \tag{1}$$

where $B: H \times H \mapsto \mathbf{R}$ and $F: H \times H \mapsto \mathbf{R}$ are bilinear forms.

In numerical methods, finite dimensional subspaces of approximation $H^h \subset H(\Omega)$ are chosen and the variational statement becomes: find $\lambda_h \in \mathbf{R}$ and $u_h \in H^h(\Omega)$ so that

$$B(u_h, w) = \lambda_h F(u_h, w), \ \forall w \in H^h \quad (2)$$

So the free vibration of a bar becomes an eigenvalue problem with variational statement: find a pair (λ, u), with $u \in H^1(0,L)$ and $\lambda \in \mathbf{R}$, which satisfies Eq. (1) when H space is $H^1(0,L)$, $\lambda = \omega^2$ and L is the bar length. The bilinear forms B and F in Eq. (1) for Dirichlet and Neumann boundary conditions are:

$$B(u, w) = \int_0^L EA \frac{du}{dx}\frac{dw}{dx} dx \quad (3)$$

$$F(u, w) = \int_0^L \rho A u w dx \quad (4)$$

where A is the cross section area, E is the Young modulus, ρ is the specific mass and u is the axial displacement.

Otherwise the free vibration of a beam is an eigenvalue problem with variational statement: find a pair (λ, v), with $v \in H^2(0,L)$ and $\lambda \in \mathbf{R}$, which satisfies Eq. (1) when H space is $H^2(0,L)$, $u = v$, $\lambda = \omega^2$ and L is the beam length. In this case the bilinear forms B and F in Eq. (1) for Dirichlet and Neumann boundary conditions are:

$$B(v, w) = \int_0^L EI \frac{d^2 v}{dx^2}\frac{d^2 w}{dx^2} dx \quad (5)$$

$$F(v, w) = \int_0^L \rho A v w dx \quad (6)$$

where I is the second moment of area, A is the cross section area, E is the Young modulus, ρ is the specific mass and v is the beam lateral displacement.

3. ENRICHED METHODS

Several methods found in the literature have as main feature the enrichment of the shape functions space of the classical FEM by adding other non polynomial functions. In this work such methods will be called enriched methods. Actually the Assumed Mode Method (AMM) of Ganesan and Engels (1992), the Composite Element Method (CEM) of Zeng (1998a, b and c) and the Fourier *p*-element of Leung and Chan (1998) are enriched methods. The Generalized Finite Element Method (GFEM) of Arndt, Machado and Scremin (2007 and 2009), although based on the Partition of Unity Method, can be considered an enriched method. Their main characteristics are: (a) they use non polynomial shape functions; (b) the introduction of boundary conditions follows the standard finite element procedure; (c) hierarchical *p* refinements are easily implemented and (d) they present more accurate results than conventional *h*-version of FEM.

The approximated solution of the enriched methods, in the element domain, is obtained by:

$$u_h^e = u_{FEM}^e + u_{ENRICHED}^e \quad (7)$$

or in matrix form

$$u_h^e = \mathbf{N}^T \mathbf{q} + \mathbf{Ø}^T \overline{\mathbf{q}} \quad (8)$$

where u_{FEM}^e is the FEM displacement field based on nodal degrees of freedom, $u_{ENRICHED}^e$ is the enriched displacement field based on field degrees of freedom, $\mathbf{q}$ is the conventional FEM degrees of freedom vector, the vector $\mathbf{N}$ contains the classical FEM shape functions and the vectors $\mathbf{Ø}$ and $\overline{\mathbf{q}}$ contain the enrichment functions and the field degrees of freedom, respectively. The vectors $\mathbf{Ø}$ and $\overline{\mathbf{q}}$ can be defined by:

$$\overline{\mathbf{Ø}}^{\mathrm{T}}(\xi) = [F_1 \quad F_2 \quad \dots \quad F_r \quad \dots \quad F_n] \quad (9)$$

$$\overline{\mathbf{q}}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \tag{10}$$

$$\xi = \frac{x}{L_e} \tag{11}$$

where F_r are the enrichment functions, c_r are the field degrees of freedom and L_e is the element length. Different sets of enrichment functions produce different enriched methods. The enrichment functions spaces of the main enriched methods are described as follows.

3.1. Enriched C^0 elements

C^0 elements are used in free vibration analysis of bars and shafts. In this section the enriched C^0 elements are described. In all these enriched methods the FEM displacement field corresponds to the classical FEM with two node elements and linear Lagrangian shape functions. Only the enrichment functions are different.

In the AMM proposed by Engels (1992) the enrichment functions are the normalized analytical solutions of the free vibration problem of a fixed-fixed bar in the form

$$F_r = C \sin(r\pi\xi), \quad r = 1,2,\ldots \tag{12}$$

where C is the mass normalization constant given by

$$C = \sqrt{\frac{2}{\rho A L_e}} \,. \tag{13}$$

The CEM enrichment functions proposed by Zeng (1998b) are trigonometric functions in the form

$$F_r = \sin(r\pi\xi), \quad r = 1,2,\ldots \tag{14}$$

They differ to those of AMM just by the normalization.

The enrichment functions used by Leung and Chan (1998) in the bar Fourier *p*-element and by Zeng (1998b) in the CEM are the same.

The GFEM enriched displacement field proposed by Arndt, Machado and Scremin (2007) is

$$u^e_{ENRICHED}(\xi) = \sum_{i=1}^{2} \eta_i(\xi) \left[\sum_{j=1}^{n_l} \gamma_j(\xi)\; a_{ij} \right] \tag{15}$$

$$\eta_1 = 1 - \xi \tag{16}$$

$$\eta_2 = \xi \tag{17}$$

$$\gamma_j = \sin(j\pi\xi) \;, \quad j = 1,2,\ldots,n_l \tag{18}$$

where η_i are partition of unity functions, γ_j are trigonometric functions, n_l is the number of enrichment levels and a_{ij} are the field degrees of freedom related to the functions γ_j .

It is noteworthy that all the shape functions vanish at element nodes. This feature allows the introduction of boundary conditions following the standard finite element procedure.

3.2. Enriched C^1 elements

C^1 elements are used in free vibration analysis of Euler-Bernoulli beams. In this section the enriched C^1 elements are described. The FEM displacement field in the enriched methods corresponds to the classical FEM with two node elements and cubic Hermitian shape functions. The enrichment functions are described below.

In the AMM four different enrichment functions are proposed. Engels (1992) used analytical free vibration normal modes of a clamped-clamped beam in the classical form

$$F_r = C_r \left\{ \sinh(\lambda_r \xi) - \sin(\lambda_r \xi) - \alpha_r \left[\cosh(\lambda_r \xi) - \cos(\lambda_r \xi) \right] \right\}, \quad r = 1,2,\ldots \tag{19}$$

$$C_r = \frac{1}{\sqrt{\rho A L_e \alpha_r^2}} \tag{20}$$

$$\alpha_r = \frac{\sinh(\lambda_r) - \sin(\lambda_r)}{\cosh(\lambda_r) - \cos(\lambda_r)} \tag{21}$$

where C_r is the mass normalization constant for the rth mode and λ_r are the eigenvalues associated to the analytical solution obtained by the following characteristic equation

$$\cos(\lambda_r)\cosh(\lambda_r) - 1 = 0 \tag{22}$$

Alternatively, Ganesan and Engels (1992) propose enrichment functions based on the same analytical solution but in the form presented by Gartner and Olgac (1982) given by

$$F_r = \frac{1}{\sqrt{\rho A L_e}} \left[\cos(\lambda_r \xi) - \frac{1 + (-1)^r e^{-\lambda_r}}{1 - (-1)^r e^{-\lambda_r}} \sin(\lambda_r \xi) - \frac{e^{-\lambda_r \xi} - (-1)^r e^{-\lambda_r (1-\xi)}}{1 - (-1)^r e^{-\lambda_r}} \right] \tag{23}$$

where λ_r are the eigenvalues obtained by solving the equation

$$\cos(\lambda_r) - \frac{2e^{-\lambda_r}}{1 + e^{-2\lambda_r}} = 0 \tag{24}$$

According to Ganesan and Engels (1992), neglecting $e^{-\lambda_r}$ because $e^{-\lambda_r} << 1$, Eq. (23) can be approximated as

$$F_r = \frac{1}{\sqrt{\rho A L_e}} \left[\sqrt{2} \sin\left(\lambda_r \xi - \frac{\pi}{4} \right) + e^{-\lambda_r \xi} - (-1)^r e^{-\lambda_r (1-\xi)} \right] \tag{25}$$

and λ_r can be approximated by $(r + 1/2)\pi$ to within 0,01% error for $i \geq 2$.

Ganesan and Engels (1992) also propose trigonometric enrichment functions in the following form

$$F_r = \cos[(r-1)\pi\xi] - \cos[(r+1)\pi\xi] \tag{26}$$

The Composite Element Method (CEM), proposed by Zeng (1998c), uses enrichment functions given by:

$$F_r = \sin(\lambda_r \xi) - \sinh(\lambda_r \xi) - \frac{\sin\lambda_r - \sinh\lambda_r}{\cos\lambda_r - \cosh\lambda_r} \left[\cos(\lambda_r \xi) - \cosh(\lambda_r \xi) \right] \tag{27}$$

corresponding to the clamped-clamped beam free vibration solution where λ_r are the eigenvalues obtained by the solution of Eq. (22).

Leung and Chan (1998) propose two types of enrichment functions based on the Fourier series: the cosine version

$$F_r = 1 - \cos(r\pi\xi) \tag{28}$$

and the sine version

$$F_r = \xi(1-\xi)\sin(r\pi\xi) \tag{29}$$

The cosine version is the simplest but is not recommended when modeling a free of shear forces structure with only one element. Leung and Chan (1998) also noted that the cosine version fails to predict the clamped-hinged and clamped-clamped modes of beams.

The GFEM enriched displacement field proposed by Arndt, Machado and Scremin (2009) is

$$u^{e}_{ENRICHED}(\xi)=\sum_{i=1}^{2}\eta_i(\xi)\left[\sum_{j=1}^{n_l}\gamma_j(\xi)\ a_{ij}\right] \tag{30}$$

$$\gamma_j=\cos(\lambda_j\xi)-\frac{1+(-1)^j e^{-\lambda_j}}{1-(-1)^j e^{-\lambda_j}}\sin(\lambda_j\xi)-\frac{e^{-\lambda_j\xi}-(-1)^j e^{-\lambda_j(1-\xi)}}{1-(-1)^j e^{-\lambda_j}} \tag{31}$$

where η_i are the linear partition of unity functions defined in Eq. (16) and Eq. (17), n_l is the number of enrichment levels, a_{ij} are the field degrees of freedom and λ_j are the eigenvalues obtained by the solution of Eq. (24).

It is noteworthy that all these shape functions and their first derivatives vanish at element nodes. Again this feature allows the introduction of boundary conditions following the standard finite element procedure.

4. APPLICATION

Numerical solutions for a bar and a beam are given below to compare the different enriched methods. To check the efficiency of these methods the results were compared to those obtained by *h* and *p*-versions of FEM.

The number of degrees of freedom (ndof) considered in each analysis is the total number of effective degrees of freedom after introduction of boundary conditions.

4.1. Uniform fixed-free bar

The free axial vibration of a fixed-free bar (Fig. 1) with length *L*, elasticity modulus *E*, mass density ρ and uniform cross section area *A*, has exact natural frequencies (ω_r) given by:

$$\omega_r=\frac{(2r-1)\pi}{2L}\sqrt{\frac{E}{\rho}}\quad ,\ r=1,2,\ldots \tag{32}$$

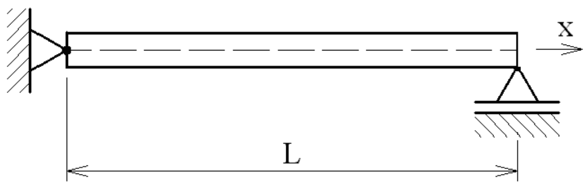

Figure 1. Uniform fixed-free bar

In order to compare the exact solution with the approximated ones, in this example a non-dimensional eigenvalue χ_r given by:

$$\chi_r=\frac{\rho L^2\omega_r^{\ 2}}{E} \tag{33}$$

will be used.

To check the efficiency of the enriched methods the results were compared to those obtained by *h*-version of FEM and by *p*-version of FEM. In the analyses by *p*-version of FEM and by all the enriched methods, the bar was described geometrically by one element and the successive refinements were obtained increasing the number of shape functions. Figures 2 to 4 present the behavior of relative error for the six earliest eigenvalues in logarithmic scale.

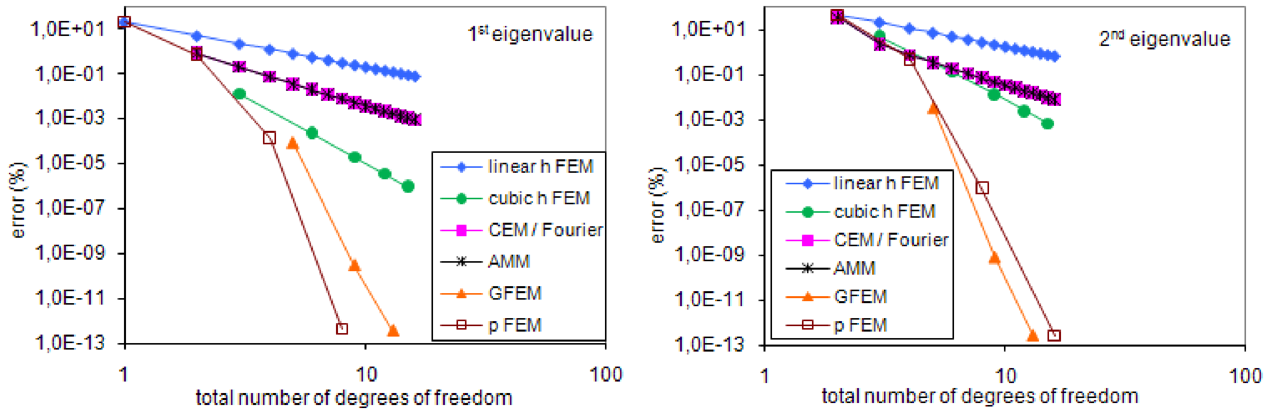

Figure 2. Relative error (%) for the 1st and 2nd bar eigenvalues

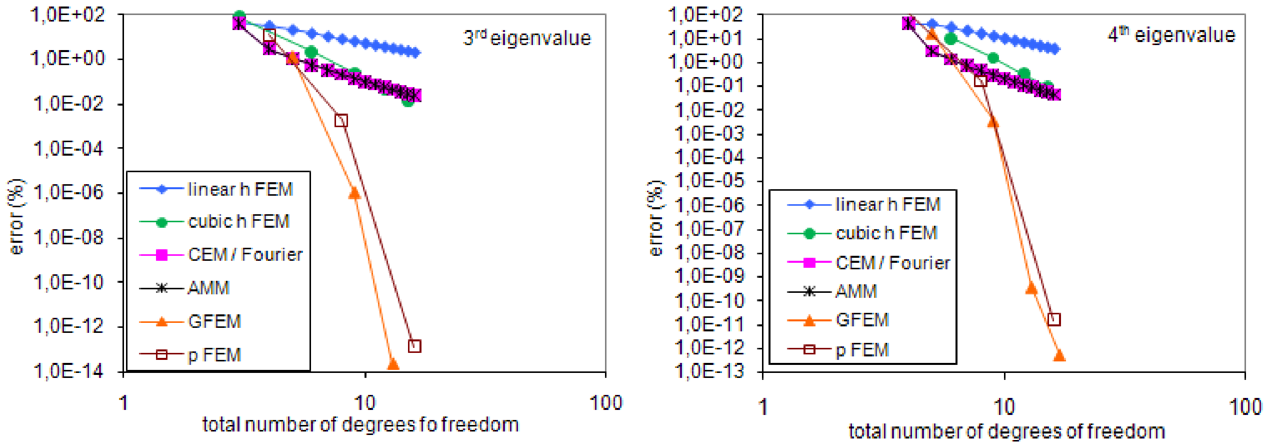

Figure 3. Relative error (%) for the 3rd and 4th bar eigenvalues

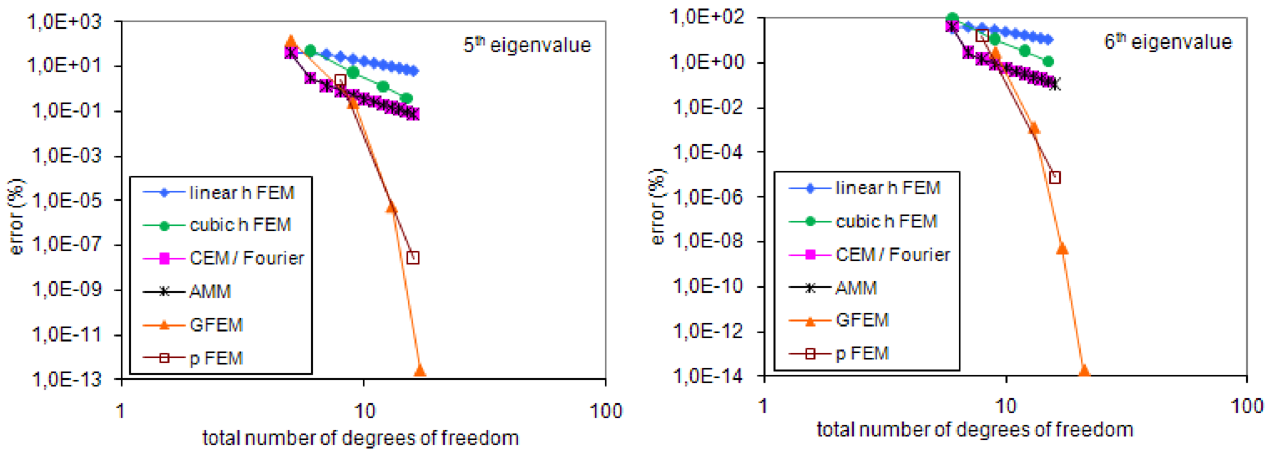

Figure 4. Relative error (%) for the 5th and 6th bar eigenvalues

Analyzing the results obtained for the fixed-free bar, one observes that the results obtained by CEM / Fourier and by AMM are identical and all enriched methods show convergence rates greater than the linear *h* refinement of the FEM. The cubic *h*-version of FEM shows better results than CEM / Fourier and AMM just for three earliest eigenvalues and it shows worst results than GFEM for all eigenvalues. The hierarchical *p* refinement of the FEM has greater accuracy than CEM / Fourier and AMM. Otherwise, the GFEM showed worst precision than the *p* version of the FEM just for the first eigenvalue.

4.2. Uniform clamped-free beam

The free vibration of an uniform clamped-free beam in lateral motion (Fig. 5), with length L, second moment of area I, elasticity modulus E, mass density ρ and cross section area A, is analyzed in order to compare the enriched methods.

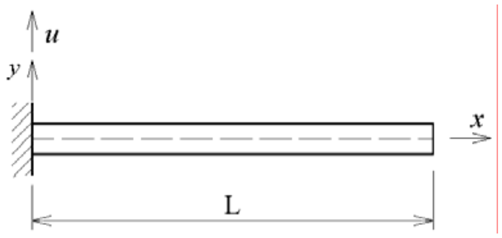

Figure 5. Uniform clamped-free beam

The analytical natural frequencies (ω_r) of this beam are obtained by the solution of the equation:

$$\cos(\kappa_r L)\cosh(\kappa_r L)+1=0 \quad , r=1,2,\ldots \tag{34}$$

$$\kappa_r = \sqrt[4]{\frac{\omega_r^{\ 2}\rho A}{EI}} \tag{35}$$

To check the efficiency of the enriched methods the results were compared to those obtained by h-version of FEM and by p-version of FEM. The eigenvalue $\chi_r = \kappa_r.L$ is used to compare the analytical solution with the approximated ones. In the analyses by p-version of FEM and by all the enriched methods, the beam was described geometrically by one element and the successive refinements were obtained increasing the number of shape functions. Figures 6 to 9 present the evolution of relative error for the eight earliest eigenvalues in logarithmic scale.

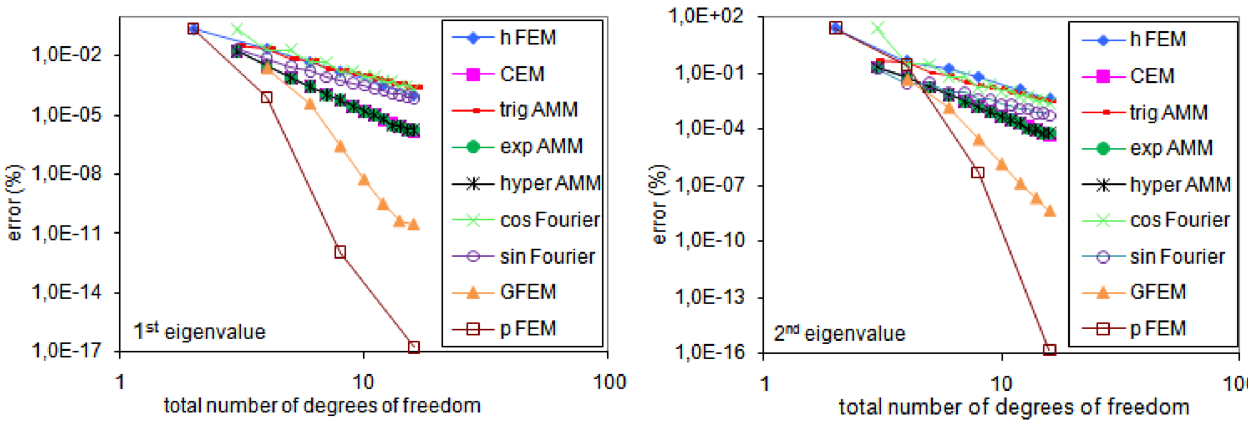

Figure 6. Relative error (%) for the 1st and 2nd beam eigenvalues

Analyzing the results obtained for the clamped-free beam, one observes that all enriched methods show more precise results than the h refinement of the FEM for all eigenvalues except the first. The hyperbolic AMM, the exponential AMM and the CEM present similar results. Their results are more precise than those obtained by sine and cosine versions of p Fourier element and by trigonometric AMM. The GFEM is the enriched method that presents the best results and the highest convergence rates. The GFEM presents more accurate results than the p version of FEM for eigenvalues of order higher than six.

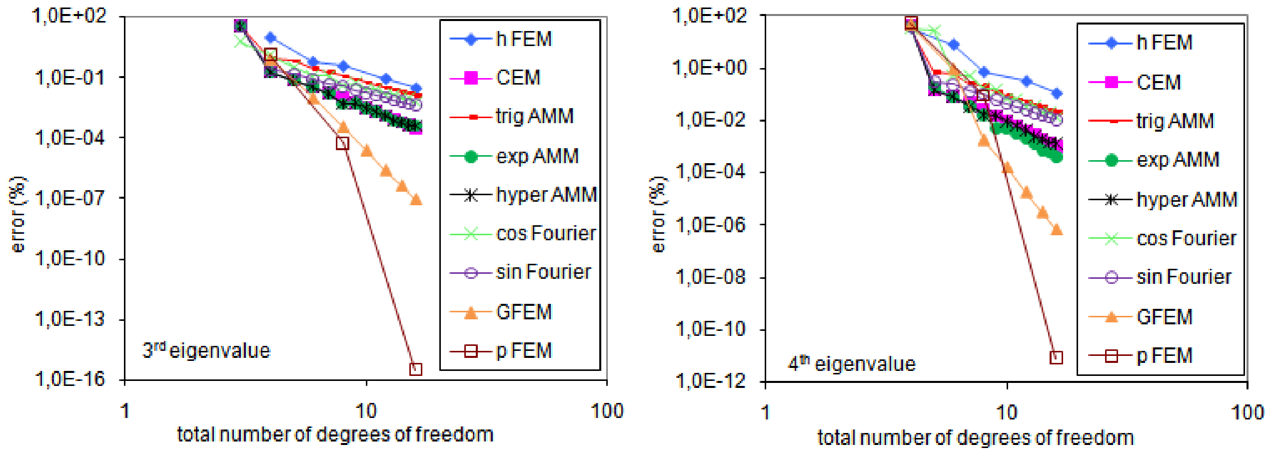

Figure 7. Relative error (%) for the 3rd and 4th beam eigenvalues

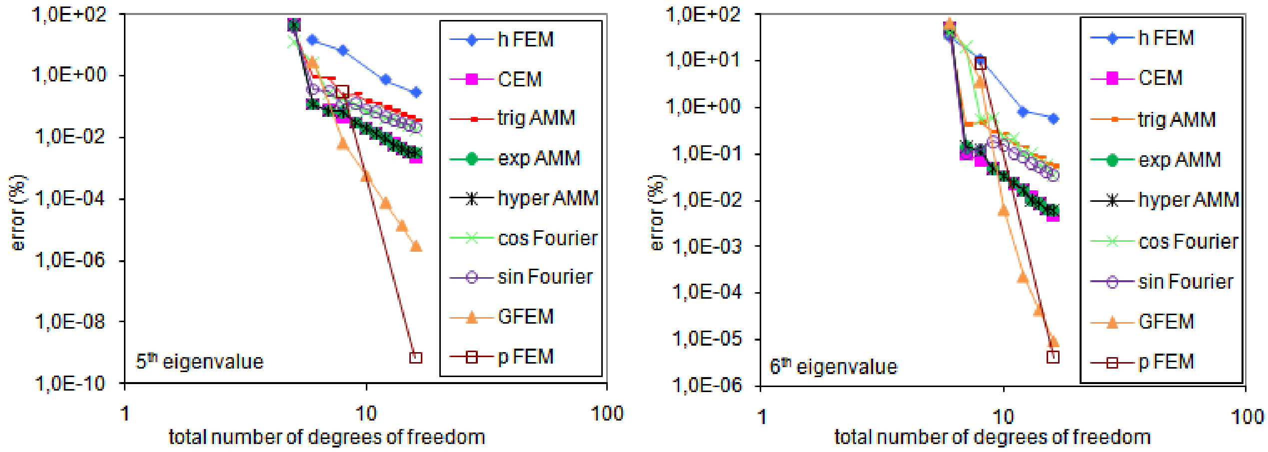

Figure 8. Relative error (%) for the 5th and 6th beam eigenvalues

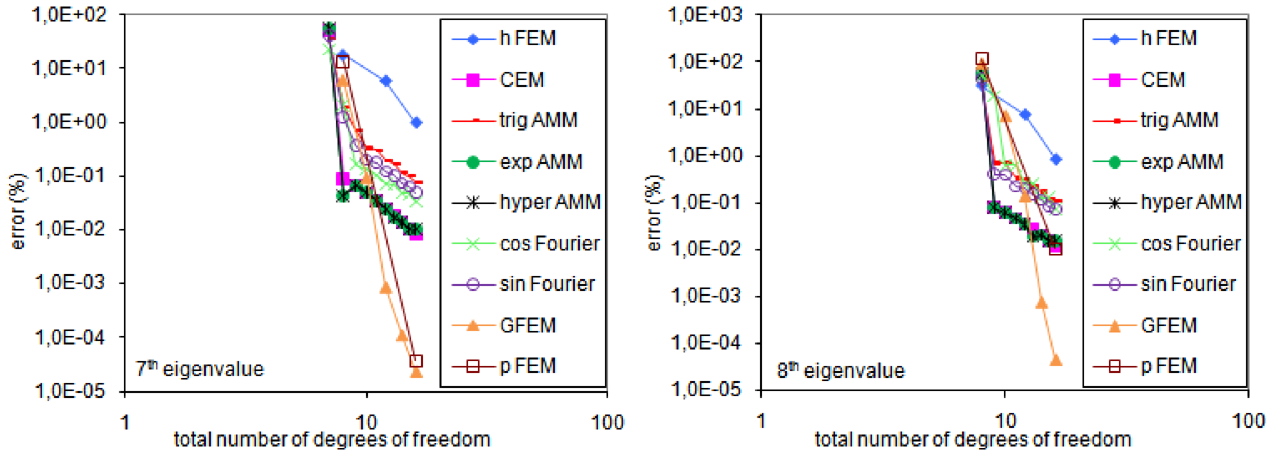

Figure 9. Relative error (%) for the 7th and 8th beam eigenvalues

5. CONCLUSION

This work presents the variational form to the free vibration problem of straight bars and Euler-Bernoulli beams with classical boundary conditions.

The numerical methods that have as main feature the enrichment of the shape functions space of the classical FEM by adding other non polynomial functions were here called enriched methods. Among these methods one can cite:

AMM, CEM, *p* Fourier element and GFEM. The C^0 and C^1 elements of these methods and their properties were discussed in this work.

To compare these methods, some eigenvalues of free vibration of a fixed-free bar and a clamped-free beam were calculated. The enriched methods were compared to *h* and *p* refinements of FEM. The results have shown that the GFEM presents convergence rates greater than those obtained from *h* refinement of FEM and the other enriched methods for all eigenvalues obtained. Moreover the GFEM results for eigenvalues with order higher than one for the bar problem and higher than six for the beam problem were better than those obtained by the *p* refinement of FEM.

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