

## A THREE-DIMENSIONAL DESCRIPTION OF SHAPE MEMORY ALLOY THERMOMECHANICAL BEHAVIOR

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**Abstract.** Literature presents numerous constitutive models that describe the phenomenological features of the thermomechanical behavior of shape memory alloys (SMAs). The present paper introduces a novel three-dimensional constitutive model that describes the martensitic phase transformations within the scope of standard generalized materials. The model considers four macroscopic phases associated with austenitic phase and three variants of martensite. The use of these phases are motivated by one-dimensional models and each one of them can be induced either by volumetric or by deviatoric strains. Besides, plasticity and transformation induced plasticity (TRIP) are also of concern. Numerical simulations show that the proposed model is able to capture the general thermomechanical behavior of SMAs.

**Keywords:** Shape Memory Alloy, constitutive model, transformation induced plasticity, plasticity.

### 1. INTRODUCTION

Shape memory alloys (SMAs) have unique characteristics that make possible their application in different areas of human knowledge. The thermomechanical behavior of these materials presents complex responses related to distinct phenomena. The constitutive modeling of SMAs is related to the macroscopic phenomenological features being based on mechanics of continua and thermodynamics (Popov & Lagoudas, 2007; Paiva & Savi, 2006). Savi & Paiva (2005) and Lagoudas (2008) presented an overview of the SMA modeling with emphasis on phenomenological constitutive models. The intent of this article is to present a three-dimensional constitutive model to describe thermomechanical behavior of SMAs including plasticity and transformation induced plasticity (TRIP).

Regarding the plastic behavior of SMAs Baêta-Neves et al. (2004) and Paiva et al. (2005) presented one-dimensional models that couple phase transformation and plasticity. Recently, Hartl et al. (2010) exploited the idea of plastic behavior of SMAs investigating the interaction between phase transformations and yield surface. The classical plasticity is a different phenomenon from the so-called transformation induced plasticity (TRIP). While the classical plasticity results from an applied stress or a temperature change, TRIP is caused by the variation of phase proportions, even for low-stress levels. Therefore, it is possible to observe plastic strains due to TRIP even inside the yield surface (Leblond et al., 1989, Gautier et al. 1989; Gautier, 1998, Fischer et al. 2000, 1996, Sato & Tanaka, 1988). Concerning the TRIP effect, this is characterized by the growth of a nonlinear irreversible deformation, while the phase transformations is occurring in the solid state and results of internal tensions arising either by volume change (Greenwood & Johnson, 1965) or change in shape (Magee, 1966, Marketz & Fischer, 1994).

This paper proposes a three-dimensional constitutive model developed in the framework of continuum mechanics and generalized standard materials that is inspired on the one-dimensional model that is able to describe TRIP and plasticity behavior of SMAs in a flexible way (Paiva et al., 2005). This one-dimensional model is built upon the Fremnod's model (Fremnod, 1996). The numerical simulations are performed for uniaxial and multiaxial tests considering a single point response that shows the capability of the proposed model to capture the general thermomechanical behavior of SMAs.

### 2. CONSTITUTIVE EQUATIONS

The constitutive modeling of SMAs can be done within the framework of generalized standard materials under the assumption that the thermodynamic state of the material can be completely defined by a finite number of state variables, see Lemaitre & Chaboche (1990). Under this assumption, the thermomechanical behavior can be described by the Helmholtz free energy density,  $\psi$ , and the pseudo-potential of dissipation,  $\Phi$ .

Experimental studies have revealed key aspects of the thermomechanical behavior of SMAs. Basically there are two possible phases: austenite and martensite. In the martensitic phase, different orientations of crystallographic deformation of the plates constitute what is known as martensitic variants. In the case of three-dimensional environment, there are 24 possible martensitic variants that are organized into six groups of four plate variants by group card. As the crystal structure of martensite is less symmetrical than the austenite, only one variant is created on the reverse transformation (Zhang et al., 1991; Schroeder & Wayman, 1977).

The three-dimensional description of the thermomechanical behavior of SMAs is usually inspired on one-dimensional models using a limited number of martensitic variants. Motivated by the one-dimensional models, the proposed model considers four macroscopic phases: austenite ( $A$ ), the twinned martensite ( $M$ ), which is stable in the absence of a stress field, and two other martensitic phases ( $M^+$  and  $M^-$ ). The definition of the Helmholtz free energy density proposes different expressions for each of the macroscopic phases, assuming that they are functions of strain, and temperature  $T$ .

$$\begin{aligned} M^+ : \rho\psi_1(\varepsilon_{ij}^e, T, \vartheta, \zeta_{ij}) &= \frac{1}{2}(\lambda^M(\varepsilon_{kk}^e)^2 + 2\mu^M\varepsilon_{ij}^e\varepsilon_{ij}^e) - \alpha\Gamma - \Lambda_M - \Omega_{ij}^M(T - T_0)\varepsilon_{ij}^e + \frac{1}{2}K^M\vartheta^2 + \frac{1}{2HM}\zeta_{ij}\zeta_{ij} \\ M^- : \rho\psi_2(\varepsilon_{ij}^e, T, \vartheta, \zeta_{ij}) &= \frac{1}{2}(\lambda^M(\varepsilon_{kk}^e)^2 + 2\mu^M\varepsilon_{ij}^e\varepsilon_{ij}^e) + \alpha\Gamma - \Lambda_M - \Omega_{ij}^M(T - T_0)\varepsilon_{ij}^e + \frac{1}{2}K^M\vartheta^2 + \frac{1}{2HM}\zeta_{ij}\zeta_{ij} \\ A : \rho\psi_3(\varepsilon_{ij}^e, T, \vartheta, \zeta_{ij}) &= \frac{1}{2}(\lambda^A(\varepsilon_{kk}^e)^2 + 2\mu^A\varepsilon_{ij}^e\varepsilon_{ij}^e) - \Lambda_A - \Omega_{ij}^A(T - T_0)\varepsilon_{ij}^e + \frac{1}{2}K^A\vartheta^2 + \frac{1}{2HA}\zeta_{ij}\zeta_{ij} \\ M : \rho\psi_4(\varepsilon_{ij}^e, T, \vartheta, \zeta_{ij}) &= \frac{1}{2}(\lambda^M(\varepsilon_{kk}^e)^2 + 2\mu^M\varepsilon_{ij}^e\varepsilon_{ij}^e) + \Lambda_M - \Omega_{ij}^M(T - T_0)\varepsilon_{ij}^e + \frac{1}{2}K^M\vartheta^2 + \frac{1}{2HM}\zeta_{ij}\zeta_{ij} \end{aligned} \quad (1)$$

Here the indices  $A$  and  $M$  are related to the martensitic and austenitic phases, respectively;  $\lambda$  and  $\mu$  are the Lamé coefficients,  $\alpha$  is a parameter related to the hysteresis loop, and  $\Lambda_M$  and  $\Lambda_A$  are temperature functions that define the stress level of phase transformation;  $\Omega_{ij}$  is a tensor related to the thermal expansion coefficients,  $T_0$  is a reference temperature that indicates stress-free state;  $K$  is the plastic modulus,  $H$  is the kinematic hardening coefficient; the internal variable  $\vartheta$  is related to isotropic hardening and  $\zeta_{ij}$  is kinematic hardening tensor; finally,  $\rho$  is the material density. Moreover, it is defined an equivalent strain field as,

$$\Gamma = \frac{1}{3}\varepsilon_{kk}^e + \frac{2}{3}|\sqrt{3J_2^e}|\text{sign}(\varepsilon_{kk}^e) \quad (2)$$

that is dependent of:

$$J_2^e = \frac{1}{6}\{(\varepsilon_{11}^e - \varepsilon_{22}^e)^2 + (\varepsilon_{22}^e - \varepsilon_{33}^e)^2 + (\varepsilon_{33}^e - \varepsilon_{11}^e)^2 + 6((\varepsilon_{12}^e)^2 + (\varepsilon_{23}^e)^2 + (\varepsilon_{13}^e)^2)\} \quad (3)$$

$$\varepsilon_{kk}^e = \varepsilon_{11}^e + \varepsilon_{22}^e + \varepsilon_{33}^e \quad (4)$$

$$\text{where } \text{sign}(\varepsilon_{kk}^e) = \begin{cases} +1, & \text{if } \varepsilon_{kk}^e \geq 0 \\ -1, & \text{if } \varepsilon_{kk}^e < 0 \end{cases}$$

The equivalent strain field,  $\Gamma$ , contributes to the phase transformations taking into account that phase transformations can be induced either by volume expansion (represented by the first term), or by deviatoric effect (represented by the second term). This hypothesis is based on experimental observations showing that both effects induce the phase transformation. Torsion tests indicate that the experimental stress-strain curves are qualitatively similar to those obtained in tensile tests (Jackson et al., 1972; Manach & Favier, 1997; Aguiar et al., 2010). Under this assumption, the equivalent field  $\Gamma$  can be interpreted as an inductor of phase transformations, which defines what kind of martensitic variant is induced. Note that if  $\Gamma \geq 0$  the  $M^+$  variant is induced. On the other hand, the  $M^-$  variant is induced when  $\Gamma < 0$ . Furthermore, it should be noted that, since the sign of shear stress does not appear in this inductor, it has a neutral influence, tending to follow the influence of volume expansion.

At this point, it is necessary to define the free energy density of the mixture, setting the volume fraction of martensite variants  $\beta_1$  and  $\beta_2$ , associated with de twinned martensite ( $M^+$  and  $M^-$ , respectively) and  $\beta_3$ , for the austenite ( $A$ ). The fourth phase is associated with twinned martensite ( $M$ ) and its volume fraction.

$$\rho\psi(\varepsilon_{ij}^e, T, \vartheta, \zeta_{ij}, \beta_1, \beta_2, \beta_3, \beta_4) = \rho \sum_{n=1}^4 \beta_n \psi_n(\varepsilon_{ij}^e, T, \vartheta, \zeta_{ij}) + I_\Theta(\beta_1, \beta_2, \beta_3, \beta_4) \quad (5)$$

where  $I_\Theta(\beta_1, \beta_2, \beta_3, \beta_4)$  is the indicator function associated with the convex set  $\Theta$  (Rockafellar, 1970) establishing the conditions for coexistence of phases:

$$\Theta = \left\{ \beta_n \in \Re \left| \begin{array}{l} 0 \leq \beta_n \leq 1 (n = 1, 2, 3, 4); \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1 \\ \beta_1 = \beta_2 = 0 \text{ if } \sigma_{ij} = 0 \text{ and } \beta_1^s = \beta_2^s = 0 \end{array} \right. \right\} \quad (6)$$

The restrictions  $0 \leq \beta_n \leq 1 (n = 1, 2, 3, 4)$  and  $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$  are related to the coexistence of phases. Moreover, the phases  $M^+$  and  $M^-$  should not appear for a stress-free state. In order to consider this aspect of the physical behavior, it is adopted the restriction  $\beta_1 = \beta_2 = 0$  if  $\sigma_{ij} = 0$  and  $\beta_1^s = \beta_2^s = 0$  where  $\beta_1^s$  and  $\beta_2^s$  provide information about the loading history and corresponds respectively to the values of  $\beta_1$  and  $\beta_2$  when each phase

transformation begins. Now, it is possible to use the condition  $\beta_4 = 1 - \beta_1 - \beta_2 - \beta_3$  to define a free energy density in terms of only three macroscopic phases:

$$\rho\psi(\varepsilon_{ij}^e, T, \vartheta, \varsigma_{ij}, \beta_1, \beta_2, \beta_3, \beta_4) = \rho\{\beta_1(\psi_1 - \psi_4) + \beta_2(\psi_2 - \psi_4) + \beta_3(\psi_3 - \psi_4) + \psi_4\} + I_\pi(\beta_1, \beta_2, \beta_3) \quad (7)$$

The indicator function  $I_\pi(\beta_1, \beta_2, \beta_3)$  is related to the convex set defined as follows, which can be geometrically interpreted by a tetrahedron in the  $\beta_1, \beta_2, \beta_3$ -space, shown in Fig.1.

$$\pi = \left\{ \beta_n \in \Re \left| \begin{array}{l} 0 \leq \beta_m \leq 1 (m = 1, 2, 3); \beta_1 + \beta_2 + \beta_3 \leq 1 \\ \beta_1 = \beta_2 = 0 \text{ if } \sigma_{ij} = 0 \text{ and } \beta_1^s = \beta_2^s = 0 \end{array} \right. \right\} \quad (8)$$

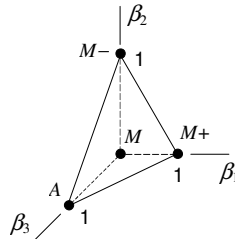


Figure 1. Tetrahedron of the constraints  $\pi$ .

Therefore, the free energy density of the mixture has the following form:

$$\begin{aligned} \rho\psi(\varepsilon_{ij}^e, T, \vartheta, \varsigma_{ij}, \beta_1, \beta_2, \beta_3) = & \alpha\Gamma(\beta_2 - \beta_1) - \Lambda(\beta_1 + \beta_2) + \left\{ \frac{1}{2}(\lambda^A - \lambda^M)(\varepsilon_{kk}^e)^2 + (\mu^A - \mu^M)\varepsilon_{ij}^e\varepsilon_{ij}^e - \right. \\ & (\Omega_{ij}^A - \Omega_{ij}^M)(T - T_0)\varepsilon_{ij}^e - \Lambda_3 + \frac{1}{2}(K^A - K^M)\vartheta^2 + \left( \frac{1}{2H^A} - \frac{1}{2H^M} \right) \varsigma_{ij}\varsigma_{ij} \left. \right\} \beta_3 + \left\{ \frac{1}{2}\lambda^M(\varepsilon_{kk}^e)^2 + \mu^M\varepsilon_{ij}^e\varepsilon_{ij}^e \right\} - \\ & \Omega_{ij}^M(T - T_0)\varepsilon_{ij}^e + \Lambda_3 + \frac{1}{2}K^M\vartheta^2 + \frac{1}{2H^M}\varsigma_{ij}\varsigma_{ij} + I_\pi(\beta_1, \beta_2, \beta_3) \end{aligned} \quad (9)$$

where  $\Lambda = 2\Lambda_M$  and  $\Lambda_3 = \Lambda_M + \Lambda_A$ . The elastic strain is defined by establishing an additive decomposition given by:

$$\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^p - \varepsilon_{ij}^{tp} + \alpha_{ij}^h(\beta_2 - \beta_1)\text{sign}(\varepsilon_{kk}^e) \quad (10)$$

where  $\varepsilon_{ij}^p$  is the plastic deformation,  $\varepsilon_{ij}^{tp}$  is the TRIP deformation and  $\alpha_{ij}^h$  is responsible for the horizontal size of the hysteresis loop.

From the generalized standard material approach, the thermodynamical forces associated with each internal variable are defined as follows (Lemaitre & Chaboche, 1990):

$$\sigma_{ij}(\varepsilon_{ij}^e, T, \beta_1, \beta_2, \beta_3) = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}^e} = \lambda \varepsilon_{kk}^e \delta_{ij} + 2\mu \varepsilon_{ij}^e + \alpha \omega_{ij}(\beta_2 - \beta_1) - \Omega_{ij}(T - T_0) \quad (11)$$

$$B_1 \in -\rho \partial_{\beta_1} \psi = \Gamma \alpha + \Lambda + P_1 - \alpha_{ij}^h \Omega_{ij}(T - T_0) - \partial_{\beta_1} I_\pi \quad (12)$$

$$B_2 \in -\rho \partial_{\beta_2} \psi = -\Gamma \alpha + \Lambda - P_2 + \alpha_{ij}^h \Omega_{ij}(T - T_0) - \partial_{\beta_2} I_\pi \quad (13)$$

$$B_3 \in -\rho \partial_{\beta_3} \psi = \Lambda_3 + P_1 + \varepsilon_{ij}^e (\Omega_{ij}^A - \Omega_{ij}^M)(T - T_0) - \frac{1}{2}(K^A - K^M)\vartheta^2 - \left( \frac{1}{2H^A} - \frac{1}{2H^M} \right) \varsigma_{ij}\varsigma_{ij} - \partial_{\beta_3} I_\pi \quad (14)$$

$$R_{ij} = -\rho \frac{\partial \psi}{\partial \varepsilon_{ij}^{tp}} = \lambda \varepsilon_{kk}^e \delta_{ij} + 2\mu \varepsilon_{ij}^e + \alpha \omega_{ij}(\beta_2 - \beta_1) - \Omega_{ij}^M(T - T_0) = \sigma_{ij} \quad (15)$$

$$X_{ij} = -\rho \frac{\partial \psi}{\partial \varepsilon_{ij}^p} = \lambda \varepsilon_{kk}^e \delta_{ij} + 2\mu \varepsilon_{ij}^e + \alpha \omega_{ij}(\beta_2 - \beta_1) - \Omega_{ij}^M(T - T_0) = \sigma_{ij} \quad (16)$$

$$Y \in -\rho \frac{\partial \psi}{\partial \vartheta} = K \vartheta \quad (17)$$

$$Z_{ij} \in -\rho \frac{\partial \psi}{\partial \varsigma_{ij}} = -\frac{1}{H} \varsigma_{ij} \quad (18)$$

$$S_i \in -\rho \frac{\partial \psi}{\partial \xi_i} = -\partial \xi_i J_\tau \quad (19)$$

where  $B_i, R, X_{ij}, Y, Z_{ij}$  and  $S_i$  are thermodynamics forces while  $\sigma_{ij}$  represents the stress tensor. The preceding equations have used the following auxiliary parameters:

$$P_1 = (\lambda \varepsilon_{kk}^e \delta_{ij} + 2\mu \varepsilon_{ij}^e) \alpha_{ij}^h + \alpha(\beta_2 - \beta_1) \left\{ \frac{1}{3} \alpha_{kk}^h + \frac{2}{\sqrt{3}} \frac{J_\alpha}{J_2^e} \text{sign}(\varepsilon_{kk}^e) \right\} \quad (20)$$

$$P_2 = -(\lambda \varepsilon_{kk}^e \delta_{ij} + 2\mu \varepsilon_{ij}^e) \alpha_{ij}^h - \alpha(\beta_2 - \beta_1) \left\{ \frac{1}{3} \alpha_{kk}^h + \frac{2}{\sqrt{3J_2^e}} \text{sign}(\varepsilon_{kk}^e) \right\} \quad (21)$$

$$P_3 = \left\{ \left[ -\frac{1}{2} (\lambda^A (\varepsilon_{kk}^e)^2 + 2\mu^A \varepsilon_{ij}^e \varepsilon_{ij}^e) + \frac{1}{2} (\lambda^M (\varepsilon_{kk}^e)^2 + 2\mu^M \varepsilon_{ij}^e \varepsilon_{ij}^e) \right] \right\} \quad (22)$$

Where

$$J^\alpha = \frac{1}{6} \{ (\alpha_{11}^h - \alpha_{22}^h) (\varepsilon_{11}^e - \varepsilon_{22}^e) + (\alpha_{22}^h - \alpha_{33}^h) (\varepsilon_{22}^e - \varepsilon_{33}^e) + (\alpha_{33}^h - \alpha_{11}^h) (\varepsilon_{33}^e - \varepsilon_{11}^e) + 6(\alpha_{12}^h \varepsilon_{12}^e + \alpha_{13}^h \varepsilon_{13}^e + \alpha_{23}^h \varepsilon_{23}^e) \} \quad (23)$$

Here the  $\partial_{\beta_m}(\ )$  represents the subdifferential with respect to  $\beta_m$ . Note that the material parameter is given by a kind of rule of mixtures, being defined as follows:

$$\begin{aligned} \lambda &= \lambda^M + \beta_3 (\lambda^A - \lambda^M) \\ \mu &= \mu^M + \beta_3 (\mu^A - \mu^M) \\ \Omega_{ij} &= \Omega_{ij}^M + \beta_3 (\Omega_{ij}^A - \Omega_{ij}^M) \\ K &= K^M + \beta_3 (K^A - K^M) \\ \frac{1}{H} &= \frac{1}{H^M} + \beta_3 \left( \frac{1}{H^A} - \frac{1}{H^M} \right) \end{aligned} \quad (24)$$

It is also important to observe that,

$$\omega_{ij} = \frac{1}{3} \delta_{ij} + \left[ \frac{3\varepsilon_{ij}^e - \varepsilon_{kk}^e \delta_{ij}}{3\sqrt{3J_2^e}} \right] \text{sign}(\varepsilon_{kk}^e) \quad (25)$$

Moreover, the functions  $\Lambda$  and  $\Lambda_3$  are temperature dependent as follows:

$$\Lambda = 2\Lambda_M = \begin{cases} -L_0 + \frac{L}{T_M} (T - T_M) & \text{if } T > T_M \\ -L_0 & \text{if } T \leq T_M \end{cases} \quad (26)$$

$$\Lambda_3 = \Lambda_M + \Lambda_A = \begin{cases} -L_0^A + \frac{L^A}{T_M} (T - T_M) & \text{if } T > T_M \\ -L_0^A & \text{if } T \leq T_M \end{cases} \quad (27)$$

where  $T_M$  is the temperature below which the martensitic phase becomes stable. Besides,  $L_0$ ,  $L$ ,  $L_0^A$  and  $L^A$  are parameters related to phase transformation critical stresses. Note that, based on the previous definition, the phase transformation stress level is constant for  $T < T_M$

Since  $\lambda \varepsilon_{kk}^e \delta_{ij} + 2\mu \varepsilon_{ij}^e = E_{ijkl} \varepsilon_{kl}$ , it is possible to rewrite the stress-strain relation as follows:

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl} + \alpha \omega_{ij} (\beta_2 - \beta_1) - \Omega_{ij} (T - T_0) \quad (28)$$

where  $E_{ijkl} = E_{ijkl}^M + \beta_3 (E_{ijkl}^A - E_{ijkl}^M)$ . In case of isotropic materials, Lamé coefficients can be expressed in terms of engineering constants as follows:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad G = \frac{E}{2(1+\nu)} \quad (29)$$

where  $E$  is the elastic modulus,  $G$  is the shear modulus and  $\nu$  is the Poisson ratio.

The thermomechanical behavior of SMAs is intrinsically dissipative and therefore, it is important to establish the pseudo-potential of dissipation that allows the description of dissipative materials. By assuming that this potential may be split into mechanical and thermal parts, its mechanical part may be considered as follows:

$$\begin{aligned} \Phi_m^* (B_i, X_{ij}, Y, Z_{ij}, R, S_i) &= \frac{1}{2\eta_1} (B_1 + \eta^{ci} Y + \eta_{ij}^{ck} Z_{ij})^2 + \frac{1}{2\eta_2} (B_2 + \eta^{ci} Y + \eta_{ij}^{ck} Z_{ij})^2 + \frac{1}{2\eta_3} (B_3 - \eta^{ci} Y - \eta_{ij}^{ck} Z_{ij})^2 + \\ &R^2 \{ (M_{31}\beta_1 + M_{13}\beta_3) \dot{\beta}_1 + (M_{32}\beta_1 + M_{23}\beta_3) \dot{\beta}_2 + [M_{43}\beta_3 + M_{34}(1 - \beta_1 - \beta_2 - \beta_3)] \dot{\beta}_3 \} + |\dot{\beta}_1| S_1 + |\dot{\beta}_2| S_2 + \\ &|\dot{\beta}_3| S_3 + I_\chi (B_n) + I_f (X_{ij}, Y, Z_{ij}, R, S_i) \end{aligned} \quad (30)$$

where  $I_\chi = I_\chi(\dot{\beta}_1, \dot{\beta}_2, \dot{\beta}_3)$  is the indicator function related to the convex set  $\chi$ , that provides constraints associated with phase transformation evolution. This set establishes conditions for the correct description of internal sub-loops due to

incomplete phase transformations and also to eliminate the phase transformations  $M+ \rightarrow M$  or  $M- \rightarrow M$  (Savi & Paiva, 2005). Basically, for  $\dot{\sigma}_{ij} \neq 0$ :

$$\chi = \left\{ \dot{\beta}_n \in \Re \left| \begin{array}{l} \dot{\Gamma} \dot{\beta}_1 \geq 0; \dot{\Gamma} \dot{\beta}_3 \leq 0 \text{ if } \Gamma > 0 \\ \dot{\Gamma} \dot{\beta}_2 \leq 0; \dot{\Gamma} \dot{\beta}_3 \geq 0 \text{ if } \Gamma < 0 \end{array} \right. \right\} \quad (31)$$

And for  $\dot{\sigma}_{ij} = 0$ :

$$\chi = \left\{ \dot{\beta}_n \in \Re \left| \begin{array}{l} \dot{T} \dot{\beta}_1 \left\{ \begin{array}{l} < 0 \text{ if } \dot{T} > 0, \sigma < \sigma_M^{crit} \text{ and } \beta_1^s \neq 0 \\ = 0, \text{ otherwise} \end{array} \right. \\ \dot{T} \dot{\beta}_2 \left\{ \begin{array}{l} < 0 \text{ if } \dot{T} > 0, \sigma < \sigma_M^{crit} \text{ and } \beta_2^s \neq 0 \\ = 0, \text{ otherwise} \end{array} \right. \\ \dot{T} \dot{\beta}_3 \geq 0 \\ -\dot{\beta}_1^2 - \dot{\beta}_1 \dot{\beta}_3 = 0 \quad \text{or} \quad -\dot{\beta}_2^2 - \dot{\beta}_2 \dot{\beta}_3 = 0 \end{array} \right. \right\} \quad (32)$$

where  $\beta_1^s$  and  $\beta_2^s$  are the values of  $\beta_1$  and  $\beta_2$ , respectively, when the phase transformation begins to take place. Moreover,  $\sigma_M^{crit}$ , which actually has different values for tensile or compressive behaviors, is the critical stress value for  $M \rightarrow M+$  and  $M \rightarrow M-$  phase transformations.

The parameter  $\eta_m$  ( $m = 1, 2, 3$ ) is associated with the internal dissipation of each phase of the material, while  $\eta^{ci}$  and  $\eta_{ij}^{ck}$  are respectively, isotropic and kinematic parameters related to plastic-phase transformation coupling:  $M_{13} = M_{31}, M_{23} = M_{32}$  and  $M_{34} = M_{43}$  are parameters associated with the TRIP effect.

In order to contemplate different aspects of kinetics of phase transformation, parameter  $\eta_i$  may assume different values for cases of loading or unloading behaviors:

$$\left\{ \begin{array}{l} \eta_i = \eta_i^L \text{ if } \dot{\Gamma} > 0 \\ \eta_i = \eta_i^U \text{ if } \dot{\Gamma} < 0 \end{array} \right. \quad (33)$$

where  $\eta_i^L$  and  $\eta_i^U$  are internal dissipation parameters related to variable  $\beta_i$  during loading or unloading process, respectively.

At this point, it is important to define aspects related to classical plasticity. The indicator function  $I_f$  is associated with a function  $f$ , which is related to the yield surface:

$$f = |d_{ij} - \varsigma_{ij}| - \sqrt{\frac{2}{3}(\sigma^Y - K\vartheta)} \quad (34)$$

where  $f$  obeys the conditions of Kuhn-Tucker condition defined by:

$$\gamma \geq 0, \quad f(\sigma_{ij}, \varsigma_{ij}, \vartheta) \leq 0 \quad \text{and} \quad \gamma f(\sigma_{ij}, \varsigma_{ij}, \vartheta) = 0 \quad (35)$$

and the consistency conditions, defined by:

$$\gamma \dot{f}(\sigma_{ij}, \varsigma_{ij}, \vartheta) = 0 \quad (36)$$

Here,  $d_{ij}$  is the deviatoric tensor from  $\sigma_{ij}$ ,  $\sigma^Y$  is the yield stress of Von Mises and  $\gamma$  is the plastic multiplier. The yield stress  $\sigma^Y$  has different values for austenitic and martensitic phases. For high temperatures the value of  $\sigma^Y$  tends to decrease. Therefore, the yield stress is temperature dependent being defined as follows:

$$\left\{ \begin{array}{l} \sigma^Y = \sigma_M^Y \text{ if } T \leq T_M \\ \sigma^Y = \frac{\sigma_M^Y(T_A - T) + \sigma_{A,i}^Y(T - T_M)}{T_A - T_M} \text{ if } T_M < T \leq T_A \\ \sigma^Y = \frac{\sigma_{A,i}^Y(T_F - T) + \sigma_{A,f}^Y(T - T_A)}{T_F - T_A} \text{ if } T_A < T \leq T_F \end{array} \right. \quad (37)$$

where  $T_F$  is a high value reference temperature and  $T_A$  is the temperature above austenite is the only stable phase.

The description of the TRIP is made by considering the parameter variation with respect to a variable that represents the accumulation of phase transformation,  $\xi_i$ . Under this assumption, it is possible to control the stress-strain hysteresis by considering the following definitions:

$$\begin{cases} \alpha = \hat{\alpha} \left[ \frac{N + \exp(-m\alpha\xi_1)}{N+1} \right] & \text{if } \Gamma \geq 0 \\ \alpha = \hat{\alpha} \left[ \frac{N + \exp(-m\alpha\xi_2)}{N+1} \right] & \text{if } \Gamma < 0 \end{cases} \quad (38)$$

$$\begin{cases} \alpha_{ij}^h = \hat{\alpha}_{ij}^h \left[ \frac{N + \exp(-m\alpha_h\xi_1)}{N+1} \right] \frac{\sigma_{ij}}{|\sigma_{ij}|} & \text{if } \Gamma \geq 0 \\ \alpha_{ij}^h = \hat{\alpha}_{ij}^h \left[ \frac{N + \exp(-m\alpha_h\xi_2)}{N+1} \right] \frac{\sigma_{ij}}{|\sigma_{ij}|} & \text{if } \Gamma < 0 \end{cases} \quad (39)$$

where  $\frac{\sigma_{ij}}{|\sigma_{ij}|} = 0$  if  $|\sigma_{ij}| = 0$ .

Moreover, it is important to define the parameters of functions  $\Lambda_1, \Lambda_2$  e  $\Lambda_3$ , by the same way,

$$L_0^T = \hat{L}_0^T \left[ \frac{N + \exp(-m_L\xi_1)}{N+1} \right] \quad L^T = \hat{L}^T \left[ \frac{N + \exp(-m_L\xi_1)}{N+1} \right] \quad (40)$$

$$L_0^C = \hat{L}_0^C \left[ \frac{N + \exp(-m_L\xi_2)}{N+1} \right] \quad L^C = \hat{L}^C \left[ \frac{N + \exp(-m_L\xi_2)}{N+1} \right] \quad (41)$$

$$L_0^A = \hat{L}_0^A \left[ \frac{N + \exp(-m_L\xi_3)}{N+1} \right] \quad L^A = \hat{L}^A \left[ \frac{N + \exp(-m_L\xi_3)}{N+1} \right] \quad (42)$$

And the TRIP strain is characterized by:

$$M_{13} = \hat{M}_{13} \exp(-m_M\xi_1) \quad M_{31} = \hat{M}_{31} \exp(-m_M\xi_1) \quad (43)$$

$$M_{23} = \hat{M}_{23} \exp(-m_M\xi_2) \quad M_{32} = \hat{M}_{32} \exp(-m_M\xi_2) \quad (44)$$

$$M_{34} = \hat{M}_{34} \exp(-m_M\xi_3) \quad M_{43} = \hat{M}_{43} \exp(-m_M\xi_3) \quad (45)$$

In order to control the amount of TRIP strain at different temperatures, the parameters associated with the TRIP effect should also be temperature dependent. Thus, the parameters assumed to be linearly dependent on temperature. For example, to the following expression is adopted:

$$\hat{M}_{13} = \begin{cases} 0 & \text{if } T < T_{TRIP} \\ \hat{M}_{13}^{Ref} \frac{(T - T_{TRIP})}{(T_F - T_{TRIP})} & \text{if } T \geq T_{TRIP} \end{cases} \quad (46)$$

where  $\hat{M}_{13}^{Ref}$  is a reference value of  $M_{13}$  at  $T = T_F$  and  $T_{TRIP}$  is a temperature below which transformation plasticity (TRIP) should not exist. Similar expressions are used to  $\hat{M}_{31}$ ,  $\hat{M}_{23}$  and  $\hat{M}_{32}$ . For the sake of simplicity, this article considers  $\hat{M}_{34} = \hat{M}_{43} = 0$

These equations form a complete set of constitutive equations, summarized the set equations that follows.

$$\sigma_{ij}(\varepsilon_{ij}^e, T, \beta_1, \beta_2, \beta_3) = E_{ijkl} \varepsilon_{kl} + \alpha \omega_{ij} (\beta_2 - \beta_1) - \Omega_{ij} (T - T_0) \quad (47)$$

$$\dot{\beta}_1 = \frac{1}{\eta_1} \left\{ \Gamma \alpha + \Lambda + P_1 - \alpha_{ij}^h \Omega_{ij} (T - T_0) + \eta^{ci} K \vartheta - \eta_{ij}^{ck} \frac{\zeta_{ij}}{H} - \partial_{\beta_1} I_\pi \right\} + \partial_{\beta_1} I_\chi \quad (48)$$

$$\dot{\beta}_2 = \frac{1}{\eta_2} \left\{ -\Gamma \alpha + \Lambda - P_2 + \alpha_{ij}^h \Omega_{ij} (T - T_0) - \eta^{ci} K \vartheta - \eta_{ij}^{ck} \frac{\zeta_{ij}}{H} - \partial_{\beta_2} I_\pi \right\} + \partial_{\beta_2} I_\chi \quad (49)$$

$$\dot{\beta}_3 = \frac{1}{\eta_3} \left\{ P_3 + \Lambda_3 + \varepsilon_{ij}^e (\Omega_{ij}^A - \Omega_{ij}^M) (T - T_0) - \frac{1}{2} (K^A - K^M) \vartheta^2 - \left( \frac{1}{2H^A} - \frac{1}{2H^M} \right) \zeta_{ij} \zeta_{ij} + \eta^{ci} K \vartheta + \eta_{ij}^{ck} \frac{\zeta_{ij}}{H} - \partial_{\beta_3} I_\pi \right\} + \partial_{\beta_3} I_\chi \quad (50)$$

$$\dot{\varepsilon}_{ij}^{tp} = 2\sigma_{ij} [(M_{31}\beta_1 + M_{13}\beta_3)\dot{\beta}_1 + (M_{32}\beta_1 + M_{23}\beta_3)\dot{\beta}_2 + [M_{43}\beta_3 + M_{34}(1 - \beta_1 - \beta_2 - \beta_3)]\dot{\beta}_3] \quad (51)$$

$$\dot{\xi}_1 = |\dot{\beta}_1| \quad \dot{\xi}_2 = |\dot{\beta}_2| \quad \dot{\xi}_3 = |\dot{\beta}_3| \quad (52)$$

$$\dot{\varepsilon}_{ij}^p = \gamma \text{sign}(d_{ij} - \zeta_{ij}) \quad (53)$$

$$\dot{\vartheta} = \sqrt{\frac{2}{3}} \gamma + \eta^{ci} (\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3) \quad (54)$$

$$\dot{\zeta}_{ij} = \frac{2}{3} H \dot{\varepsilon}_{ij}^p + \eta_{ij}^{ck} (\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3) \quad (55)$$

### 3. NUMERICAL SIMULATIONS

This section presents numerical simulations that are carried out in order to evaluate the model capability to capture the general thermomechanical behavior of SMAs. Initially, TRIP effect is of concern by considering experimental tests by Lagoudas et al. (2003). Basically, the SMA is subjected to a cyclic stress loading. Model parameters are presented in

Table 1 (where plastic parameters are omitted) and Figure 2 shows the comparison between experimental and numerical stress-strain curve. Note a good agreement between results showing the ability of the model to represent the phenomenon of saturation. After a few cycles, there is a stabilization of the SMA behavior and the TRIP effect is no longer observed. This result is particularly important because it enables the understanding of saturation in SMAs, allowing the description of the SMA training necessary to its use in various applications. Figure 3 shows the volume fraction time evolution during the test.

Table 1. Model parameters for TRIP analysis based on experimental results by Lagoudas et al. (2003).

$E_A$ (GPa)	$E_M$ (GPa)	$\Omega_A$ (MPa/K)	$\Omega_M$ (MPa/K)	$\hat{\alpha}_{ij}^h$	$\hat{\alpha}$ (MPa)
72	28.2	0.74	0.17	0.038	200
$L_0^T$ (MPa)	$L_0^T$ (MPa)	$L_0^A$ (MPa)	$L^A$ (MPa)	$T_M$ (K)	$T_0$ (K)
0.1	41.5	0.63	152	291.4	323
$T_A$ (K)	$T_F$ (K)	$T_{TRIP}$ (K)	$n_1^L$ (MPa.tu)	$\eta_1^U$ (MPa.tu)	$n_2^L$ (MPa.tu)
307.5	423	330	0.1	0.04	0.1
$n_2^U$ (MPa.tu)	$n_3^L$ (MPa.tu)	$n_3^U$ (MPa.tu)	$\hat{M}_{13}$ (GPa <sup>-1</sup> )	$\hat{M}_{31}$ (GPa <sup>-1</sup> )	$m_M$
0.04	0.1	0.04	0.063	0.063	0.12
$m_L$	$m_\alpha$	$m_{\alpha_h}$	$m_p$	N	
0.2	0.45	0.007	0.12	2	

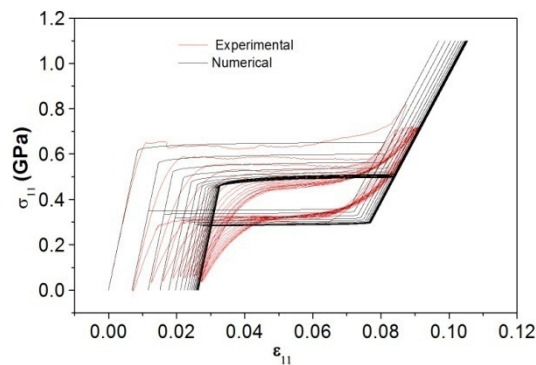


Figure 2. Numerical-experimental comparison to cyclic loading (Lagoudas et al., 2003).

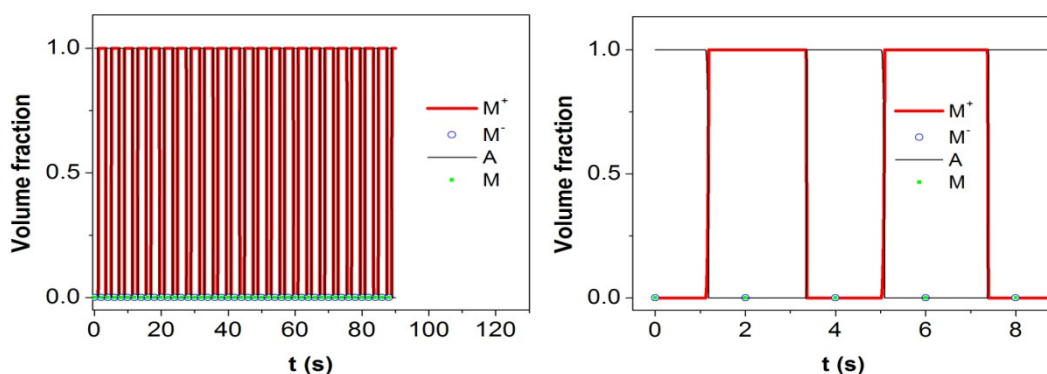


Figure 3. Temporal evolution of volume fractions to cyclic loading.

Phase transformation, plasticity and TRIP are now concerned by considering model parameters presented in Table 2. Initially, a cyclic loading with constant maximum stress of 1.3 GPa is imposed (Figure 4). Figure 4a shows the loading history while Figure 4b presents the stress-strain curve. Note the saturation effect due to TRIP occurs before the plastic behavior is reached. Figure 5 presents a different situation where loading history has maximum stress values that vary progressively from 1.0 GPa to 1.3 GPa. Figure 5a shows the loading history while Figure 5b presents stress-strain curve. Under this new condition, a different evolution occurs and the plastic behavior is preponderant during the first cycles and then the TRIP becomes more important.

Table 2. Model parameters.

$E_A$ (GPa)	$E_M$ (GPa)	$\Omega_A$ (MPa/K)	$\Omega_M$ (MPa/K)	$\hat{\alpha}_{ij}^h$	$\hat{\alpha}$ (MPa)
54	42	0.74	0.17	0.0463	330
$L_0^T$ (MPa)	$L_0^T$ (MPa)	$L_0^A$ (MPa)	$L^A$ (MPa)	$T_M$ (K)	$T_0$ (K)
0.15	41.5	0.63	185	291.4	307
$T_A$ (K)	$T_F$ (K)	$T_{TRIP}$ (K)	$\sigma_M^Y$ (GPa)	$\sigma_{A,i}^Y$ (GPa)	$\sigma_{A,f}^Y$ (GPa)
307.5	423	330	0.5	1.5	1.0
$K_A$ (GPa)	$K_M$ (GPa)	$H_A$ (GPa)	$H_M$ (GPa)	$\eta^{ci}$	$\eta^{ck}$
1.4	0.4	4	1.1	-0.01	-0.01
$n_1^L$ (MPa.tu)	$n_1^U$ (MPa.tu)	$n_2^L$ (MPa.tu)	$n_2^U$ (MPa.tu)	$n_3^L$ (MPa.tu)	$n_3^U$ (MPa.tu)
1	2.7	1	2.7	1	2.7
$\hat{M}_{13}$ (GPa <sup>-1</sup> )	$\hat{M}_{31}$ (GPa <sup>-1</sup> )	$\hat{M}_{23}$ (GPa <sup>-1</sup> )	$\hat{M}_{32}$ (GPa <sup>-1</sup> )	$\hat{M}_{34}$ (GPa <sup>-1</sup> )	$\hat{M}_{43}$ (GPa <sup>-1</sup> )
0.009	0.003	0.009	0.003	0	0
$m_\alpha$	$m_{\alpha_h}$	$m_L$	$m_p$	$m_M$	$v_M$
0.2	0.03	0.1	0.1	0.1	0.3
$v_A$					
0.44					

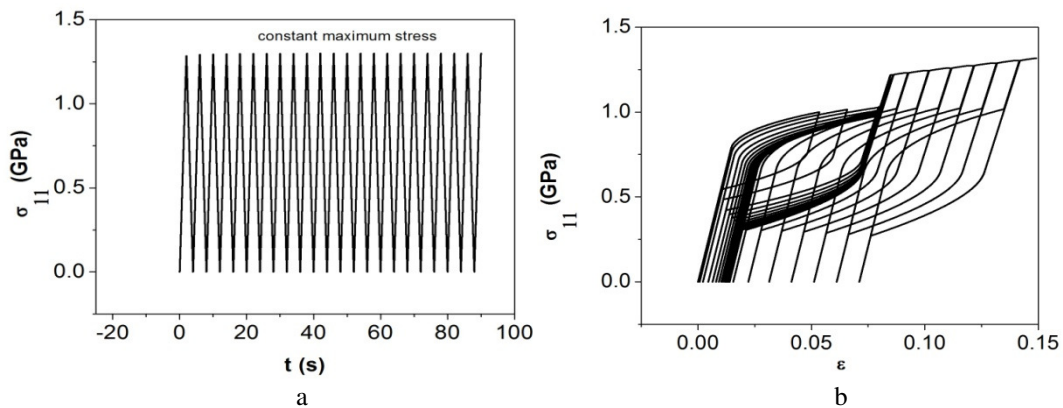


Figure 4. Pseudoelastic and plastic behaviors - TRIP and plastic evolution.  
 (a) Loading history; (b) Stress-strain curve.

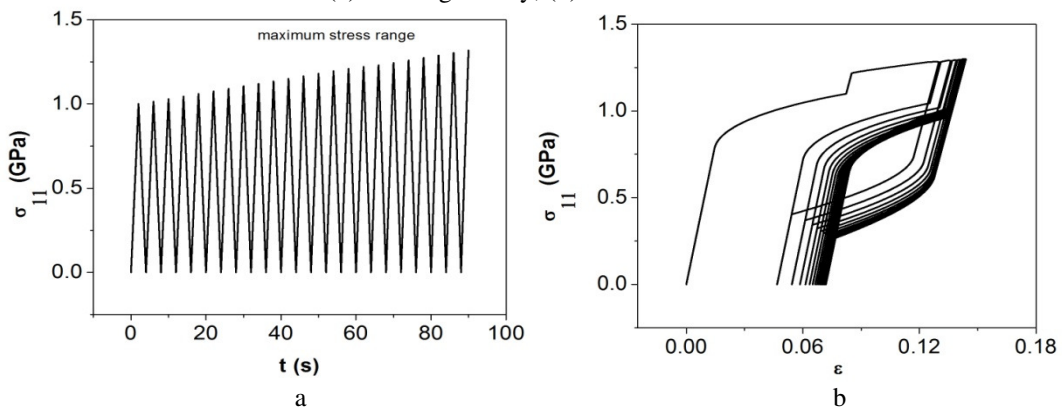


Figure 5. Pseudoelastic and plastic behaviors - Plastic and TRIP evolution.  
 (a) Loading history; (b) Stress-strain curve.

In order to consider a multiaxial test, we now analyze the pure shear test that is used to compare pure shear loading with biaxial loading. Therefore, two different loading are of concern defined by the following stress tensors:



$$\sigma^C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ GPa} \quad \sigma^D = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ GPa} \quad (56)$$

It is assumed a pseudoelastic behavior and therefore, the yield surface is not reached being performed with  $T = T_0$ . This test is interesting to verify the coordinate invariance. Figure 6 shows a comparison between the stress-strain curve for the two loadings considered ( $\sigma_{11} \times \varepsilon_{11}$  and  $\sigma_{12} \times \varepsilon_{12}$ ). Both curves are identical, representing a typical pseudoelastic behavior with a TRIP strain at the end of the test. Figure 8 illustrates the evolution of volume fractions involved in the process. These phenomena are similar in both cases showing the main characteristics of the model where phase transformations are induced by the inductor that includes either volumetric and deviatoric strains. It should be highlighted that both loading histories induce the same volume fraction and besides, the coordinate system invariance.

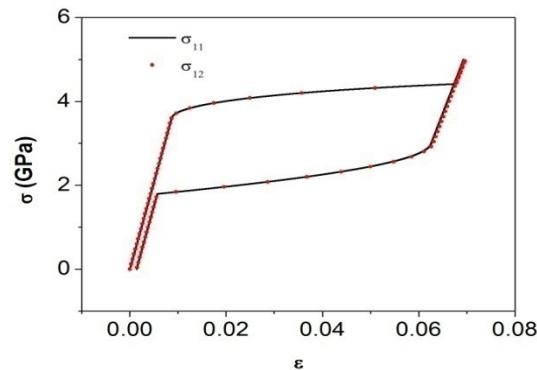


Figure 6. Comparison of loading  $\sigma_{ij}^C$  and  $\sigma_{ij}^D$ .

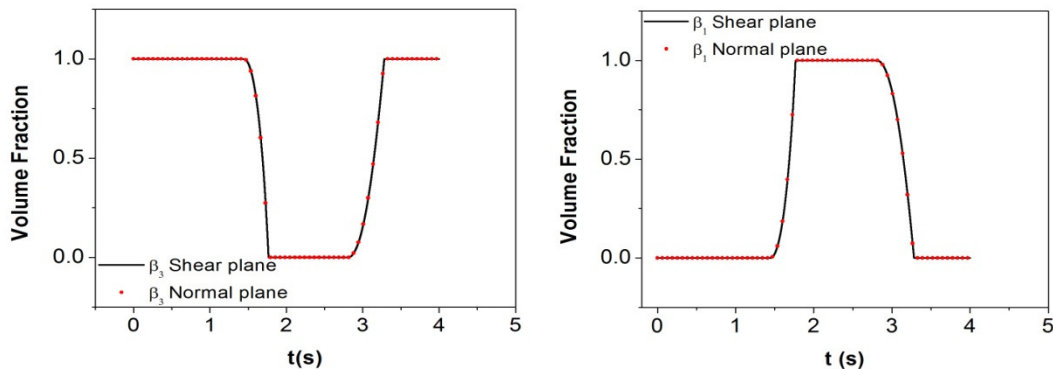


Figure 7. Volume fraction time history for the pseudoelastic effect.

#### 4. CONCLUSIONS

A three dimensional macroscopic constitutive model is proposed to describe SMA thermomechanical behavior including plasticity and TRIP. Numerical simulations are carried out showing different aspects of SMA behavior. TRIP phenomenon is described and compared with experimental data, presenting a good agreement. The occurrence of both TRIP and classical plasticity together can induce interesting behaviors of the SMA. Saturation effect is of special interest due to its importance to training of SMAs. Concerning multiaxial tests, the proposed model is able to capture the general thermomechanical behavior of SMAs. This paper presents a pure shear test showing the coordinate system invariance.

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