SOLUTION OF DIFFUSIVE TRANSIENT PROBLEMS IN ANNULAR SECTOR GEOMETRIES BY THE INTEGRAL TRANSFORM

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***Abstract.** In this work the solution of diffusive transient problems in domains represented by annular sector cross section cylinders submitted the boundary conditions of first kind is carried out. In order to facilitate the analytical treatment and the application of the boundary conditions, a Conformal Transform is used to change the domain into a more suitable coordinate system. To solve the energy equation resulting after this transformation was applied the Generalized Integral Transformed Technique (GITT). As a result of this new transformation it was obtained a coupled differential equation system that can be solved through classic numerical methods. Thus, for the determination of the evolution of the temperature field it was made use of the inversion formulas of all the transformations realized. Physical parameters of interest were, then, calculated and compared for several annular sector cross section formats.*

***Keywords**: Conformal Transform, Integral Transform, transient heat diffusion.*

1. INTRODUCTION

The differential equations that govern the principles of conservation of mass, energy and momentum are, in rule, of complex nature. Therefore, the application of classic analytical techniques for the attainment of solutions always was limited at problems which present simpler mathematics structures.

With the evolution of equipments and computational tools, the scientific community is developing larges efforts for the consolidation of several numerical techniques that allow the attainment of solutions to problems which present more complex structure in the engineering sciences. More recently, hybrid analytical-numerical approach come being implemented in many areas of interest to permit the attainment of numerical results more accurate and in minor time computational.

In this sense, the Generalized Integral Transform Technique - GITT, (Cotta, 1998), is a method with this characteristics and has been used successfully to solve several problems of the heat and mass transfer. In respect to process purely diffusives, the GITT come being used with success to solve several problems such as those that involving irregular domains (Aparecido et al, 1989, Cotta & Ramos, 1998, Maia et. al., 2004, Pelegrini et. al., 2004 and Alves et. al., 2004); three-dimensional and non linear diffusive problems (Cotta & Özisik, 1987, Mikhailov & Cotta, 1996, Serfaty, 1997); diffusive problems with moving boundaries (Aparecido et. al., 1989, Mikhailov & Cotta, 1996 and Diniz et. al., 1999) and others.

Thus, in continuity to this line of investigation, the GITT will be utilized in the present work to obtain the solution of transients diffusive problems in a two-dimensional domain non conventional. More specifically, will be analyzed cylinders or slugs of cross section with annular sector shape. To the formulation of the proposal problem will be considered diffusives means with thermo-physics properties constants, uniform initial temperature profile and conditions of temperature prescribed in the contour. However, to facility the application of the GITT, will be a variables change through a orthogonal coordinates system obtained of a conformal transformation adequate. The temperature distribution and physical parameters of interest, such as average temperature and time constant will be, so, determined for slugs with annular sector of several shapes.

2. OBJECTIVES

To make use of the conformal transformation and integral transformed to obtain solutions of two-dimensional transient diffusive problems represented by cylinders of cross section with annular sector shape.

3. ANALYSIS

Para o problema proposto, será estabelecido um meio difusivo isotrópico, sem fontes e com perfil de temperatura inicial uniforme. Neste modelo, a equação da difusão em meios cilíndricos com seção de domínio *Ω* e contorno, *Γ*, é dada por:

$$\nabla \bullet \nabla T(x,y,t) = \frac{1}{\alpha}\frac{\partial T(x,y,t)}{\partial t}, \ \{(x,y) \in \Gamma, \ t > 0\} \tag{1a}$$

$$T(x,y,t) = T_p, \quad \{(x,y) \in \Gamma, \ t > 0\}, \tag{1b}$$

$$T(x,y,0) = T_i, \quad \{(x,y) \in \Omega\}, \tag{1c}$$

onde, T_p representa a condição de temperatura prescrita no contorno e T_i representa a condição de temperatura inicial uniforme no domínio. A Eq. (1) pode ser reescrita na forma como segue:

$$\nabla^2 \theta(X,Y,\tau) = \frac{\partial \theta(X,Y,\tau)}{\partial \tau}, \quad \{(X,Y) \in \Omega, \ \tau > 0\}, \tag{2a}$$

$$\theta(X,Y,\tau) = \theta_p = 0, \quad \{(X,Y) \in \Gamma, \ \tau > 0\}, \tag{2b}$$

$$\theta(X,Y,0) = \theta_i = 1, \quad \{(X,Y) \in \Omega\}, \tag{2c}$$

Com

$$X = x/L_{ref}, \quad Y = y/L_{ref}, \quad \tau = t\alpha/L_{ref}^2, \tag{2d,e,f}$$

$$\theta(X,Y,\tau) = \frac{T(X,Y,\tau) - T_p}{T_i - T_p}, \tag{2g}$$

O parâmetro L_{ref} representa um comprimento de referência que é aqui definido como:

$$L_{ref} = 2A_s/Per, \tag{3}$$

onde A_s representa a área da seção transversal do cilindro e Per o perímetro.

3.1. Transformação de Coordenadas

Para facilitar a aplicação das condições de contorno e o tratamento analítico será feita uma mudança de variáveis a partir da seguinte Transformação Conforme:

$$Z = r_e e^{i\omega}, \tag{4a}$$

$$Z = x + iy, \qquad w = u + iv \tag{4b,c}$$

Esta relação permite transformar o domínio do setor anular no plano (x, y) em um domínio retangular no plano (u,v) conforme ilustrado na Fig. 1.

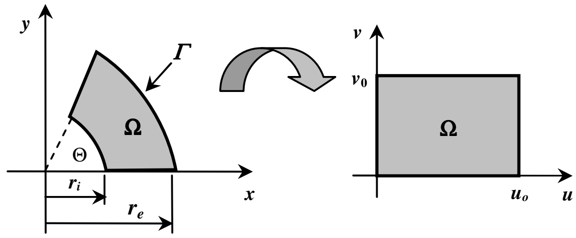

Figura 1. Transformação do domínio de setor de anel no plano (x,y) para o plano (u,v).

As relações de transformação de coordenadas, os coeficientes métricos h_u e h_v e o Jacobiano $J(u,v)$ são dadas por:

$$x = r_e\, e^{-v} \cos(u), \qquad y = r_e\, e^{-v} sen(u). \tag{5a,b}$$

$$h_u(u,v) = h_v(u,v) = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2} = r_e\, e^{-v}, \tag{6a,b}$$

Este novo sistema de coordenadas é ortogonal. Como pode ser observado, a abertura angular do setor de anel é definida pelas retas $u = 0$ e $u = u_0$ (u_0 é o ângulo Θ), o arco externo pela reta $v = 0$ e o arco interno pela reta $v = v_0$, com $v_0 = ln(r_e/r_i)$. Assim, para caracterizar a geometria do problema proposto serão considerados os parâmetros geométricos abertura angular $\Theta = u_0$ e espessura do anel definida pela razão de aspecto ρ_{asp}:

$$\rho_{asp} = r_i / r_e \tag{8}$$

A equação da difusão e as condições iniciais e de contorno no novo sistema de coordenadas são dadas por:

$$\frac{\partial^2\theta(u,v,\tau)}{\partial u^2} + \frac{\partial^2\theta(u,v,\tau)}{\partial v^2} = J(u,v)\frac{\partial\theta(u,v,\tau)}{\partial\tau}, \tag{9a}$$

$$\{0 \le u \le u_o, \quad 0 \le v \le v_0, \quad \tau > 0\}, \tag{9b}$$

$$\theta(u,v,0) = 1, \quad \{0 \le u \le u_o, \quad 0 \le v \le v_0\}, \tag{10a}$$

$$\theta(u,v,\tau) = 0, \quad \{0 \le v \le v_o, \quad u = 0, \quad \tau > 0\}, \tag{10b}$$

$$\theta(u,v,\tau) = 0, \quad \{0 \le v \le v_o, \quad u = u_o, \quad \tau > 0\}, \tag{10c}$$

$$\theta(u,v,\tau) = 0, \quad \{0 \le u \le u_o, \quad v = 0, \quad \tau > 0\}, \tag{10d}$$

$$\theta(u,v,\tau) = 0, \quad \{0 \le u \le u_o, \quad v = v_o, \quad \tau > 0\}. \tag{10e}$$

3.2. Aplicação da TTIG

Para a obtenção dos perfis de temperatura a transformada integral será aplicada sobre a equação da difusão. Devido a sua característica bidimensional, o potencial $\theta(u,v)$ será escrito em termos de uma expansão em autofunções normalizadas obtidas de problemas auxiliares de autovalor para cada coordenada espacial (Aparecido, 1997). Neste sentido, a aplicação da transformada integral será feita por partes, para cada um dos problemas propostos.

Considere o seguinte problema auxiliar de autovalor:

$$\frac{d^2\psi(u)}{du^2}+\mu^2\psi(u)=0\ ,\quad \{0<u<u_o\}, \tag{11}$$

$$\psi(0)=0\ ,\quad \psi(u_o)=0\ . \tag{12a,b}$$

Os autovalores e as autofunções associados a este problema são:

$$\mu_i = i\pi/u_o \qquad i = 1,2,3... \tag{13a}$$

$$\psi_i(u)=sen(\mu_i u). \tag{13b}$$

As autofunções acima são ortogonais e permitem o desenvolvimento do seguinte par transformada-inversa:

$$\bar{\theta}_i(v,\tau)=\int_0^{u_o} K_i(u)\,\theta(u,v,\tau)\,du\ ,\quad \text{transformada,} \tag{14a}$$

$$\theta(u,v,\tau)=\sum_{i=1}^{\infty}K_i(u)\bar{\theta}_i(v,\tau),\quad \text{inversa.} \tag{14b}$$

onde $\bar{\theta}_i(v,\tau)$ é o potencial transformado em u e $K_i(u)$ são as autofunções normalizadas, dadas por:

$$K_i(u)=\frac{\psi_i(u)}{N_i^{1/2}}, \tag{15a}$$

$$N_i=\int_0^{u_o}\psi_i^2(u)\,du\ =\ u_o/2 \tag{15b}$$

Efetuando o produto interno das autofunções normalizadas $K_i(u)$ com a equação da difusão dada pela Eq. (9) e fazendo uso das condições de contorno dadas pelas Eqs. (10b) a (10e) e da equação que define o problema auxiliar de autovalor, Eq. (11), obtém-se:

$$\sum_{j=1}^{\infty}A_{ij}(v)\frac{\partial\bar{\theta}_j(v,\tau)}{\partial\tau}+\mu_i^2\bar{\theta}_i(v,\tau)=\frac{\partial^2\bar{\theta}_i(v,\tau)}{\partial v^2}\ ,\ i=1,2,3... \tag{16a}$$

$$A_{ij}(v)=\int_0^{u_o}K_i(u)K_j(u)\,J(u,v)\,du\ . \tag{16b}$$

Para proceder a transformação integral relativo a coordenada *v*, considere o seguinte problema de autovalor:

$$\frac{d^2\phi(v)}{dv^2}+\lambda^2\phi(v)=\ 0\ ,\quad \{0<\phi<v_0\}, \tag{17}$$

$$\phi(0)=0\ ,\quad \phi(v_0)=0\ . \tag{18a,b}$$

Os autovalores e as autofunções para este novo problema auxiliar de autovalor são similares ao anterior:

$$\lambda_m=m\pi/2\ ,\qquad m=1,2,3... \tag{19a}$$

$$\phi_m(v)=cos(\lambda_m v). \tag{19b}$$

As autofunções $\phi_m(v)$ são ortogonais e permitem o desenvolvimento do seguinte par transformada-inversa:

$$\tilde{\tilde{\theta}}_{im}(\tau)=\int_0^{v_o}\int_0^{u_o} K_i(u)\,Z_m(v)\,\theta(u,v,\tau)\,du\,dv \quad \text{,transformada} \tag{20a}$$

$$\theta(u,v,\tau)=\sum_{i=1}^{\infty}\sum_{m=1}^{\infty} K_i(u)\,Z_m(v)\,\tilde{\tilde{\theta}}_{im}(\tau), \qquad \text{inversa .} \tag{20b}$$

Aqui, $Z_m(v)$ são as autofunções normalizadas e são dadas por:

$$Z_m(v)=\frac{\phi_m(v)}{M_m^{1/2}}, \tag{21a}$$

$$M_m=\int_0^{v_o}\phi_m^2(v)\,dv=\frac{v_o}{2}. \tag{21b}$$

A transformação integral sobre a coordenada *v* é feita efetuando-se o produto interno das autofunções normalizadas $z_m(v)$ com a equação diferencial transformada na coordenada *u*, Eq.(16a). Em seguida, fazendo uso das condições de contorno e das propriedades de ortogonalidade das autofunções correspondentes ao problema auxiliar de autovalor em *v*, obtém-se a seguinte relação para o potencial transformado $\tilde{\tilde{\theta}}_{im}(\tau)$:

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} B_{ijmn}\frac{d\tilde{\tilde{\theta}}_{jn}(\tau)}{d\tau}+\left[\mu_i^2+\lambda_m^2\right]\tilde{\tilde{\theta}}_{im}(\tau)=0, \qquad i,m=1,2,3... \tag{22a}$$

$$B_{ijmn}=\int_0^{v_o}\int_0^{u_o} K_i(u)\,K_j(u)\,Z_m(v)\,Z_n(v)\,J(u,v)\,du\,dv, \tag{22b}$$

que deve satisfazer a condição inicial transformada, que é dada por:

$$\tilde{\tilde{\theta}}_{im}(0)=\int_0^{v_o}\int_0^{u_o} K_i(u)\,Z_m(v)\,\theta_i(u,v)\,dv\,du \quad . \tag{22c}$$

O potencial transformado $\tilde{\tilde{\theta}}_{im}(\tau)$ pode ser obtido numericamente quando se trunca a expansão para uma dada ordem *M* e *N*:

$$\sum_{n=1}^{M}\sum_{j=1}^{N} B_{ijmn}\frac{d\tilde{\tilde{\theta}}_{jn}(\tau)}{d\tau}+\left[\mu_i^2+\lambda_m^2\right]\tilde{\tilde{\theta}}_{im}(\tau)=0\,. \tag{23}$$

O potencial temperatura $\theta\,(u,v,\tau)$ é obtido, então, através da fórmula de inversão dada pela Eq.(20b),

$$\theta_{elip}(u,v,\tau)=\sum_{i=1}^{M}\sum_{m=1}^{N} K_i(u)\,Z_m(v)\,\tilde{\tilde{\theta}}_{im}(\tau). \tag{24}$$

3.3. Temperatura média e energia interna específica

A temperatura média no domínio em um dado instante τ é dada por:

$$\theta_m(\tau)=\frac{\lfloor T_m(\tau)-T_p\rfloor}{T_i-T_p}=\frac{1}{A_s}\int_{A_s}\frac{\lfloor T_m(X,Y,\tau)-T_p\rfloor}{T_i-T_p}dA \ , \tag{25a}$$

$$\theta_m(\tau)=\frac{1}{A_s}\int_0^{v_o}\int_0^{u_o}\theta(u,v,\tau)\,J(u,v)\,du\,dv \ . \tag{25b}$$

Estabelecendo como referência para a energia interna o estado em regime permanente, a energia interna específica em um dado instante é dada por:

$$U(\tau)=\frac{1}{\rho V}\int_V \rho\, c_p\,[T(X,Y,\tau)-T(X,Y,\infty)]dV \ . \tag{26}$$

Desta forma, a energia interna relativa, definida como sendo a relação entre a energia interna no instante τ e a energia interna no instante inicial, pode ser determinada pela temperatura média adimensional:

$$U^*(\tau)=\frac{U(\tau)}{U(0)}=\frac{U}{U_i}=\theta_{med}(\tau)\,, \tag{27}$$

3.4. Constante de Tempo

Para a análise dos problemas abordados é conveniente que se estabeleça um parâmetro apropriado capaz de verificar o comportamento transiente da difusão de calor em função da razão de aspecto do cilindro de seção elíptica e retangular. Para tal fim, defini-se aqui a constante de tempo τ_{med} como sendo o parâmetro que determina o tempo necessário para que a energia interna relativa $U^*(\tau)$ esteja a 1/e do seu valor em regime permanente:

$$U^*(\tau_{med})=U(\tau_{med})/U_i=1/e=0{,}36788\,. \tag{28}$$

4. RESULTS AND DISCUSSIONS

In order, to determine the coefficients $\tilde{\tilde{\theta}}_{jn}$, the series expansion has been truncated to several choices of values for M and N. Parameters B_{ijmn} have been numerically calculated by a Gauss quadrature method (36 quadrature points) and the equation system resultant has been solved by using the routine DIVPAG of the IMSL Library (1979). It was observed that the series convergence to compute temperature distribution becomes slower in the transient beginning, ($\tau < 0.01$), and when the aspect ratio and angular overture are small ($\rho_{asp} \to 0$, $\Theta \to 0$). For this cases is necessary a high number of terms to get stable results over 3 or 4 decimal numeric places ($N = M > 30$). When τ growing is verified that the series converge to 4 or 5 decimal places when its is truncated to an order of approximately $M = N = 15$. Anyway, even considering a high number of terms in the series the computer processing time is small (nearly 30 sec in a PC with clock of 2.0 MHz).

The results obtained for the average temperature are presented in Tables 1 and 2.

Table 1. Evolution of average temperature for slugs with aspect ratio $\rho_{asp} = 0.5$.

τ	Θ = 15°	Θ = 45°	Θ = 90°	Θ = 180°	Θ = 270°
0.0001	0.968	0.969	0.969	0.967	0.967
0.0002	0.962	0.963	0.962	0.961	0.960
0.0005	0.946	0.948	0.948	0.945	0.944
0.0010	0.928	0.929	0.929	0.927	0.925
0.0020	0.9003	0.9015	0.9010	0.8991	0.8974
0.0050	0.8454	0.8467	0.8457	0.8438	0.8422
0.0100	0.7846	0.7869	0.7849	0.7816	0.7796
0.0200	0.7014	0.7060	0.7021	0.6956	0.6919
0.0500	0.5467	0.5581	0.5484	0.5323	0.5234

0.1000	0.3903	0.4119	0.3930	0.3632	0.3475
0.2000	0.2129	0.2430	0.2143	0.1754	0.1575
0.5000	0.0385	0.0539	0.0376	0.0209	0.0152
1.0000	0.0024	0.0044	0.0021	0.0006	0.0003

Table 2. Evolution of average temperature for slugs with angular overture $\Theta = 45º$.

τ	$\rho_{asp} = 0.1$	$\rho_{asp} = 0.3$	$\rho_{asp} = 0.5$	$\rho_{asp} = 0.7$	$\rho_{asp} = 0.9$
0.0001	0.956	0.966	0.969	0.968	0.967
0.0002	0.951	0.961	0.962	0.961	0.959
0.0005	0.939	0.947	0.948	0.946	0.942
0.0010	0.923	0.929	0.929	0.927	0.922
0.0020	0.8984	0.9012	0.9010	0.8994	0.8942
0.0050	0.8457	0.8465	0.8457	0.8441	0.8386
0.0100	0.7861	0.7864	0.7849	0.7819	0.7755
0.0200	0.7047	0.7050	0.7021	0.6961	0.6862
0.0500	0.5547	0.5556	0.5484	0.5336	0.5105
0.1000	0.4058	0.4069	0.3930	0.3652	0.3255
0.2000	0.2354	0.2354	0.2143	0.1774	0.1341
0.5000	0.0500	0.0495	0.0376	0.0215	0.0096
1.0000	0.0038	0.0038	0.0021	0.0007	0.0001

From the results obtained it may be observed that the thermal development occur almost entirety in the interval $0 < \tau < 1$. The results obtained for the time constant τ_{aver} are presented in Table 3. In the Figures 2 and 3 are presented the comportment of the time constant in respect to the geometric parameters ρ_{asp} and Θ.

Table 3. Time constant in function of the aspect ratio ρ_{asp} and angular overture Θ.

ρ_{asp}	$\Theta = 15°$	$\Theta = 45°$	$\Theta = 90°$	$\Theta = 180°$	$\Theta = 270°$
0.10	0.1120	0.1160	0.1171	0.1124	0.1068
0.20	0.1051	0.1160	0.1184	0.1101	0.1034
0.30	0.1036	0.1183	0.1177	0.1065	0.0998
0.40	0.1054	0.1204	0.1149	0.1024	0.0962
0.50	0.1093	0.1205	0.1105	0.0983	0.0929
0.60	0.1146	0.1177	0.1051	0.0942	0.0898
0.70	0.1198	0.1117	0.0990	0.0903	0.0870
0.80	0.1210	0.1029	0.0926	0.0865	0.0842
0.90	0.1096	0.0920	0.0861	0.0826	0.0812

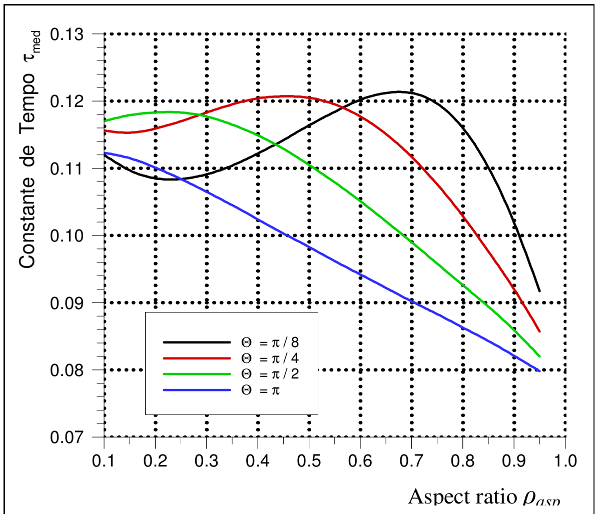

Figure. 2. Time constant in function of the aspect ratio ρ_{asp} for the slugs with several angular overtures Θ.

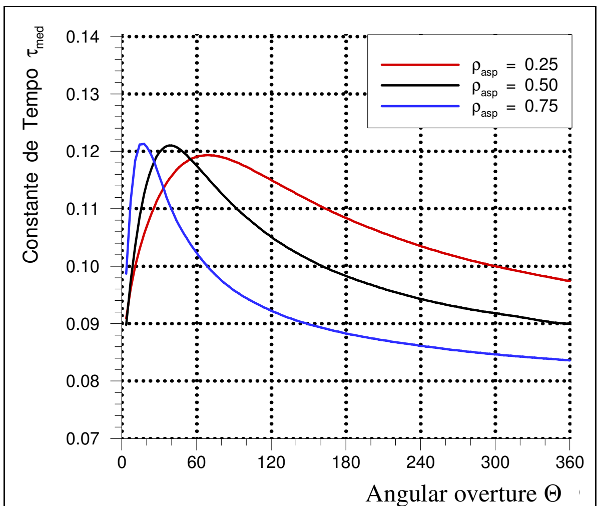

Fig. 3. Time constant in function of the angular overtures Θ for the slugs with several aspect ratio ρ_{asp} .

It may be observed that the problem geometry rule more influence on the parameter time constant in slim slugs.

Finally, to visualize the effect of the temperature field evolution in the course of transient are presented in the Figures 4 to 6 the distribution of the isotherms in an slug with $\rho_{asp} = 0.5$ and $\Theta = 45°$ in the instants $\tau = \tau_{aver}/2$, $\tau = \tau_{aver}$ and $\tau = 2\tau_{aver}$.

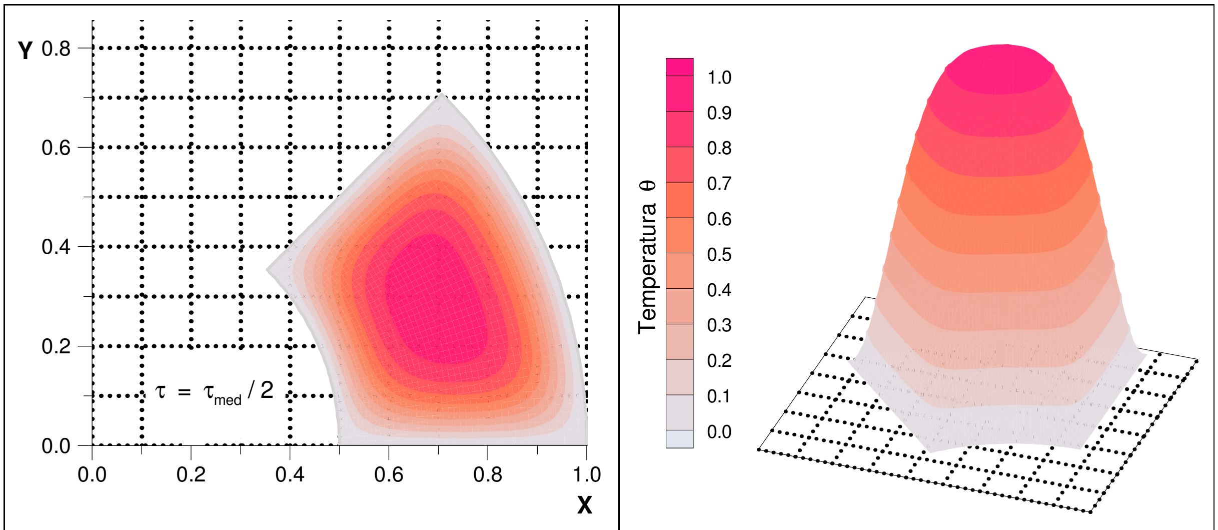

Figure 4. Temperature distribution for $\tau = \tau_{aver} / 2$

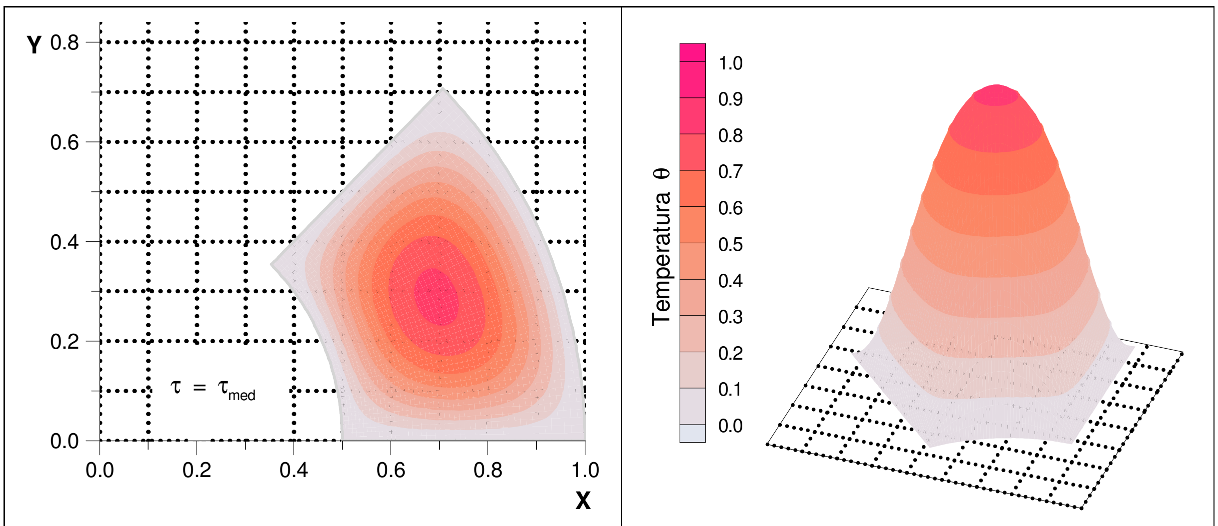

Figure 5. Temperature distribution for $\tau = \tau_{aver}$

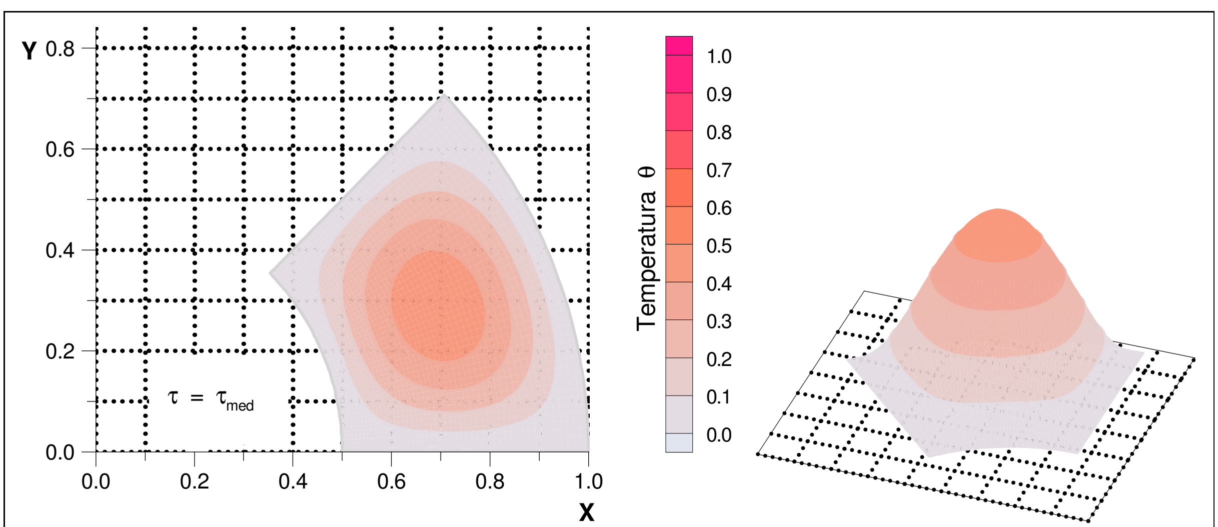

Figure 6. Temperature distribution for $\tau = 2\tau_{aver}$

5. CONCLUSIONS

In this work, transient heat diffusion problems it was analyzed in slugs of the cross section with annular sector shape. To easy the analytical handling of the diffusion equation and the application of the boundary conditions was make use of an orthogonal system coordinates obtained through an adequate conformal transformation. The field temperature was obtained through the GITT applied on the diffusion equation in the new coordinates system. It was observed that the temperature potential series presented slower convergence in the initial of the transient, mainly to the slim slugs. Thermal parameters of the interest, as the average temperature evolution and time constant were calculated for slugs with cross section of several geometries. To finish, was verified that the combination of the techniques proposals is efficient to obtain formals and accuracy solutions of complex diffusive problems, with small computer processing time.

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