A PROCEDURE FOR EXACT SOLUTION OF NONLINEAR CONDUCTION HEAT TRANSFER PROBLEMS IN A BODY WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

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Abstract. *In this work a systematic procedure is proposed for simulating the conduction heat transfer process in a solid with a strong dependence of the thermal conductivity on the temperature. Most of this kind of problem is traduced in nonlinear partial differential equations, subjected to classical linear boundary conditions, such as Newton's law of cooling. These problems will be solved with the aid of a Kirchoff Transform by means of a sequence of very simple linear problems. The proposed procedure provides the exact solution of the problem and induces finite dimensional approximations that can be used for computational simulations. Some typical cases will be simulated by employing the Finite Element Method.*

***Keywords**: Temperature Dependent Thermal Conductivity, Nonlinear Equations System, Heat Transfer, Finite Element Method, Kirchoff Transform.*

1. INTRODUCTION

Most of the conduction heat transfer phenomena are described under the assumption of temperature independent thermal conductivity. Such hypothesis is mathematically convenient because, in general, gives rise to linear partial differential equations. Nevertheless, the thermal conductivity is always a temperature dependent function. Many times, neglecting such dependence (assuming constant thermal conductivity), we have an inadequate mathematical description of the conduction heat transfer process.

For example, composite materials thermal conductivities have a strong dependence on the temperature, which is well shown in Mantovani and Franceschi (2003). In this work the one dimensional problem allowed the analytical solution, but depending on the boundary conditions it could not be possible. Many authors – most of them, actually – solve conduction heat problems with temperature dependent thermal conductivity by means of numerical methods, obtaining approximated solutions (carrying out an inherent error).

The main objective of this work is to provide a reliable and systematic procedure for describing the conduction heat transfer in a rigid solid, with temperature dependent thermal conductivity, subjected to linear boundary conditions (Newton's law of cooling). This procedure is exact and employs only the tools utilized in problems in which the thermal conductivity is assumed to be a constant.

Considering a conduction heat transfer problem subjected to Newton's law of cooling, the first step consists of employing the Kirchoff transform in order to change the original problem into another one consisting of a linear partial differential equation subjected to a nonlinear boundary condition. The second step consists of regarding the new problem as the limit of a sequence of linear problems.

So, the problems with temperature dependent thermal conductivity will be regarded as the limit of a sequence whose elements are solution of linear PDE's systems.

2. GOVERNING EQUATIONS

Let us consider a rigid, opaque and isotropic body at rest with domain represented by bounded open set Ω with boundary $\partial\Omega$. The steady-state heat transfer process inside this body is mathematically described by the following elliptic partial differential equation (Slattery, 1972)

$$div\left(k\ grad\,T\right)+\dot{q}=0 \quad in \quad \Omega \tag{1}$$

where T, $\dot{q}$ and k denote, respectively, the temperature field, the internal heat generation rate (per unit volume) and the thermal conductivity. In this work the thermal conductivity is assumed to be a function of the local temperature. In other words,

$$k = \hat{k}(T) \tag{2}$$

Assuming that the body boundary and the environment exchanges energy according to Newton's law of cooling, the boundary conditions associated with equation (1) are given by

$$-(k\ grad\,T) \cdot \mathbf{n} = h(T - T_\infty) \quad on \quad \partial\Omega \tag{3}$$

in which $\mathbf{n}$ is the unit outward normal (defined on $\partial\Omega$), h is the convection heat transfer coefficient and T_∞ is a temperature of reference

The resulting problem (nonlinear) is given by

$$\begin{aligned} div(k\ grad\,T) + \dot{q} = 0 \quad in \quad \Omega \\ -(k\ grad\,T) \cdot \mathbf{n} = h(T - T_\infty) \quad on \quad \partial\Omega \end{aligned} \tag{4}$$

3. THE KIRCHOFF TRANSFORM

Since the thermal conductivity is always positive-valued, we define the new variable ω as the Kirchoff transform (Wylie, 1960):

$$\omega = \int_{T_0}^{T} \hat{k}(\xi)\, d\xi = \hat{f}(T) \tag{5}$$

being an invertible function of T. This definition allows us to write

$$grad\,\omega = k\ grad\,T \tag{6}$$

So, the original problem can be rewritten as follows

$$\begin{aligned} div(grad\,\omega) + \dot{q} = 0 \quad in \quad \Omega \\ -(grad\,\omega) \cdot \mathbf{n} = h\left(\hat{f}^{-1}(\omega) - T_\infty\right) \quad on \quad \partial\Omega \end{aligned} \tag{7}$$

where $T \equiv \hat{f}^{-1}(\omega)$. Although above problem remains nonlinear, this nonlinearity takes place only on the boundary, not in the PDE. It is to be noticed that, since the thermal conductivity is everywhere positive,

$$k > 0 \quad \Rightarrow \quad \frac{d\omega}{dT} > 0 \quad \text{and} \quad \frac{dT}{d\omega} > 0 \quad \text{everywhere} \tag{8}$$

so, the temperature is a strictly increasing function of ω.

For instance, if we have

$$k = \begin{cases} k_1 = \text{constant} & \text{if} \quad T < T_0 \\ k_2 = \text{constant} & \text{if} \quad T \ge T_0 \end{cases} \tag{9}$$

it is possible to show that

$$T \equiv \hat{f}^{-1}(\omega) = |\omega| \left[\frac{1}{2k_2} - \frac{1}{2k_1} \right] + \omega \left[\frac{1}{2k_2} + \frac{1}{2k_1} \right] + T_0 \tag{10}$$

This function will be employed in a further example.

4. CONSTRUCTING THE SOLUTION FROM A SEQUENCE OF LINEAR PROBLEMS

The solution of

$$\begin{aligned} & div\left(grad\,\omega\right)+\vec{q}=0 \quad in \quad \Omega \\ & -\left(grad\,\omega\right)\cdot\mathbf{n}=h\left(\hat{f}^{-1}\left(\omega\right)-T_\infty\right) \quad on \quad \partial\Omega \end{aligned} \tag{11}$$

can be represented by the limit of the nondecreasing sequence $\left[\Phi_0,\Phi_1,\Phi_2,...\right]$, whose elements are obtained from the solution of the linear problems below

$$\begin{aligned} & div\left(grad\,\Phi_{i+1}\right)+\vec{q}=0 \quad in \quad \Omega \\ & -\left(grad\,\Phi_{i+1}\right)\cdot\mathbf{n}=\alpha\Phi_{i+1}+\beta_i \quad on \quad \partial\Omega \\ & \beta_i=h\left(\hat{f}^{-1}\left(\Phi_i\right)-T_\infty\right)-\alpha\Phi_i \end{aligned} \tag{12}$$

in which α is a sufficiently large constant, and $\Phi_0 \equiv 0$.

It is remarkable that, for each i, the function Φ_{i+1} is the unknown and the function Φ_i is known. So, $\hat{f}^{-1}\left(\Phi_i\right)$ is always known in (12), being evaluated from the following equation

$$\Phi_i=\int_{T_0}^{\hat{f}^{-1}(\Phi_i)}\hat{k}(\zeta)d\zeta \tag{13}$$

For each spatial position, the root of the above equation is unique. This uniqueness is supported by equation (8).

The constant α must be large enough for ensuring that, at any point of Ω, $\Phi_{i+1}\geq\Phi_i$. In Gama (2000) it is provided an upper bound for the constant α. For the problem considered in this work it is sufficient to choose α such that

$$\alpha\geq\frac{h}{k_{MIN}} \tag{14}$$

where k_{MIN} is the minimum value of the thermal conductivity.

5. CONVERGENCE

The limit of the sequence $\left[\Phi_0,\Phi_1,\Phi_2,...\right]$, denoted here by Φ_∞ exists and is, in fact, a solution of the problem. To prove this assertion, let us begin showing that Φ_∞ is a solution of (7). In other words

$$\begin{aligned} & div\left(grad\,\Phi_\infty\right)+\vec{q}=0 \quad in \quad \Omega \\ & -\left(grad\,\Phi_\infty\right)\cdot\mathbf{n}=h\left(\hat{f}^{-1}\left(\Phi_\infty\right)-T_\infty\right) \quad on \quad \partial\Omega \end{aligned} \tag{15}$$

Since β_∞ is given by

$$\beta_\infty = h\left(\hat{f}^{-1}(\Phi_\infty) - T_\infty \right) - \alpha\Phi_\infty \tag{16}$$

we have that (12) and (15) coincide. So, Φ_∞ is a solution. Now, taking into account that the sequence is nondecreasing and has an upper bound, we ensure the convergence, once that the solution of (15) belongs to the same space of the solutions of (12) for each i (Helmberg, 1974 and Wylie, 1960).

6. AN EXAMPLE

Since the proposed method provides the exact solution, an analytical solution will be shown for a simple problem with thermal conductivity depending on the temperature: spherical body with uniform heat generation, surrounded by the same medium, with $h=1$ and $T_\infty=1$ - in some system of units. The values used don't necessarily have realistic physical coherence, as the focus is to demonstrate the procedure's applicability. The parameters are conveniently chosen for the analytical development.

$$\begin{aligned} &\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 k\frac{dT}{dr}\right)\right]+1=0 \quad for \quad 0\le r<1 \\ &-k\frac{dT}{dr}=T \qquad at \quad r=1 \end{aligned} \tag{17}$$

in which it is assumed that T represents an absolute temperature and $k=3T+2$. The solution is easily reached and given by

$$T=-\frac{2}{3}+\left[\frac{\left(1-r^2\right)}{9}+1\right]^{1/2} \tag{18}$$

Now, let us employ the proposed procedure. With the Kirchoff transform, the problem yields

$$\begin{aligned} &\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\omega}{dr}\right)\right]+1=0 \quad for \quad 0\le r<1 \\ &-\frac{d\omega}{dr}=\hat{f}^{-1}(\omega)=-\frac{2}{3}+\left[\frac{4}{9}+\frac{2\omega}{3}\right]^{1/2} \qquad at \quad r=1 \end{aligned} \tag{19}$$

The linear procedure for reaching the elements of the sequence is represented as follows

$$\begin{aligned} &\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{i+1}}{dr}\right)\right]+1=0 \quad for \quad 0\le r<1 \\ &-\frac{d\Phi_{i+1}}{dr}=\alpha\Phi_{i+1}+\beta_i \qquad at \quad r=1 \\ &\beta_i=-\frac{2}{3}+\left[\frac{4}{9}+\frac{2\Phi_i}{3}\right]^{1/2}-\alpha\Phi_i \end{aligned} \tag{20}$$

or, simply as

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{i+1}}{dr}\right)\right]+1=0 \quad for \quad 0\le r<1$$
$$-\frac{d\Phi_{i+1}}{dr}=\alpha\Phi_{i+1}-\alpha\Phi_i-\frac{2}{3}+\left[\frac{4}{9}+\frac{2\Phi_i}{3}\right]^{1/2} \qquad at \quad r=1 \tag{21}$$

Since the problem makes sense only for $T>0$, we have that $k_{MIN}>2$. So, we can work with any $\alpha\ge 1/2$. We shall use $\alpha=3$!

The general solution of equation

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_{i+1}}{dr}\right)\right]+1=0 \quad for \quad 0\le r<1 \tag{22}$$

is

$$\Phi_{i+1}=-\frac{r^2}{6}+C_{i+1} \quad for \quad 0\le r<1 \tag{23}$$

where the constant C_{i+1}, for $i>0$, is obtained from the boundary condition. In other words,

$$\frac{1}{3}=3\left[-\frac{1}{6}+C_{i+1}\right]-3\left[-\frac{1}{6}+C_i\right]-\frac{2}{3}+\left[\frac{4}{9}+\frac{2}{3}\left(-\frac{1}{6}+C_i\right)\right]^{1/2} \tag{24}$$

Then,

$$C_{i+1}=C_i+\frac{1}{3}\left[1-\sqrt{\frac{1}{3}+\frac{2}{3}C_i}\right] \tag{25}$$

The constant C_1 is obtained from

$$\frac{1}{3}=3\left[-\frac{1}{6}+C_1\right] \quad\Rightarrow\quad C_1=\frac{5}{18} \tag{26}$$

If we employ $\alpha=20$, then

$$C_{i+1}=C_i+\frac{1}{20}\left[1-\sqrt{\frac{1}{3}+\frac{2}{3}C_i}\right] \tag{27}$$

and the constant C_1 is obtained from

$$\frac{1}{3}=20\left[-\frac{1}{6}+C_1\right] \quad\Rightarrow\quad C_1=\frac{11}{60} \tag{28}$$

Table 1 presents a comparison between the values obtained for C_i with two distinct values of α:

Table 1 – A comparison between results obtained with $\alpha=3$ and $\alpha=20$.

	$\alpha = 3$	$\alpha = 20$
$i =$	$C_i =$	$C_i =$
1	0.371083678318	0.199585905322
2	0.450399853146	0.215439507012
3	0.518403223696	0.230908373418
4	0.577078085519	0.246005928926
5	0.627951577215	0.260744856563
10	0.800503112959	0.329459272507
20	0.940194171974	0.445715284244
50	0.998272514569	0.679342555325
100	0.999995202718	0.866108252119
500	0.999999985099	0.999842551009
1000	0.999999985099	0.999999949824

It can be proven that $C_\infty = 1$. Hence, the limit of the sequence is given by

$$\Phi_\infty = -\frac{r^2}{6} + 1 \quad for \quad 0 \le r < 1 \tag{29}$$

The solution ω is exactly the limit of the sequence. Taking into account that

$$\hat{f}^{-1}(\omega) = T = -\frac{2}{3} + \left[\frac{4}{9} + \frac{2\omega}{3}\right]^{1/2} \tag{30}$$

we have the following result (coincident with the exact solution, previously obtained)

$$T = -\frac{2}{3} + \left[\frac{4}{9} + \frac{2}{3}\left(-\frac{r^2}{6} + 1\right)\right]^{1/2} \tag{31}$$

7. A NUMERICAL EXAMPLE

As the solution is obtained from the limit of a sequence, through an iteration process, most of the problems will be solved by computational methods, making use of classical numerical procedures (finite differences, finite elements etc). Thus, a computational support has been developed parallel to the mathematical modeling in order to provide numerical results of the proposed methodology's convergence, and also graphical results to illustrate the temperature field. The programs were developed in Matlab. The objective is not the comparison between analytical and numerical solutions, but the illustration of the method's performance, in a less simplified problem.

Let us show now some numerical results. The Finite Element Method was employed to solve a two-dimensional problem, in an irregular domain, of conduction heat transfer with the following thermal conductivity:

$$k = \begin{cases} k_1 = 40 \rightarrow T < 20.35 \\ k_2 = 10 \rightarrow T \ge 20.35 \end{cases} \tag{32}$$

The other parameter are: convection heat transfer coefficient $h = 5$, heat source $\dot{q} = 10$, reference temperature $T_\infty = 20$, in some system of units. The boundary is subjected to Newton's law of cooling.

We will assume $\alpha = 6$. The following figures show the iterative evolution of the simulation.

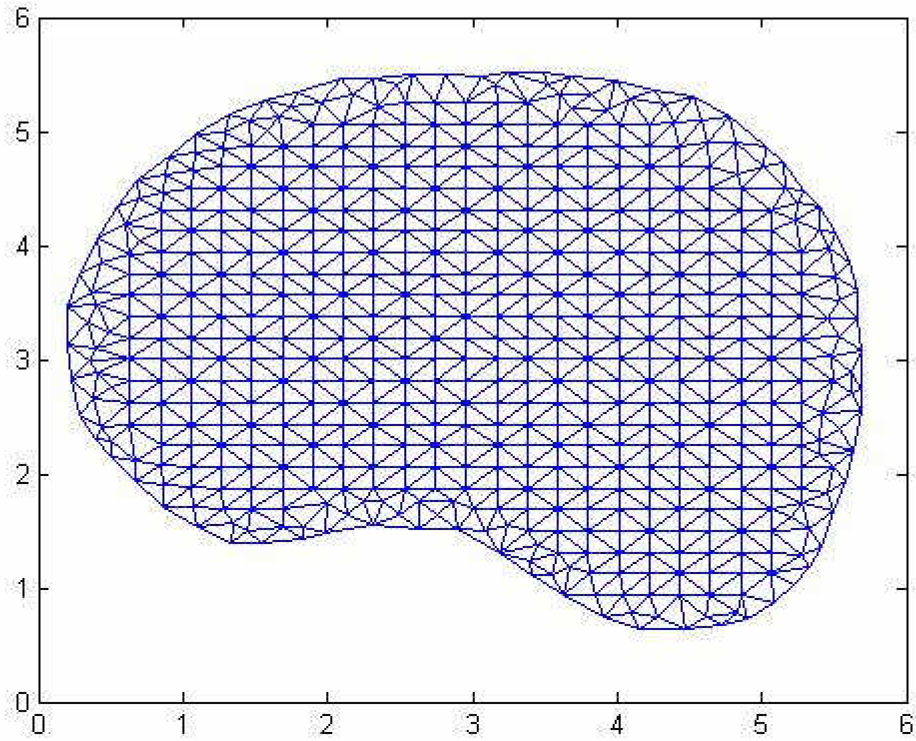

Figure 1. 2-d domain discretized in a triangular mesh

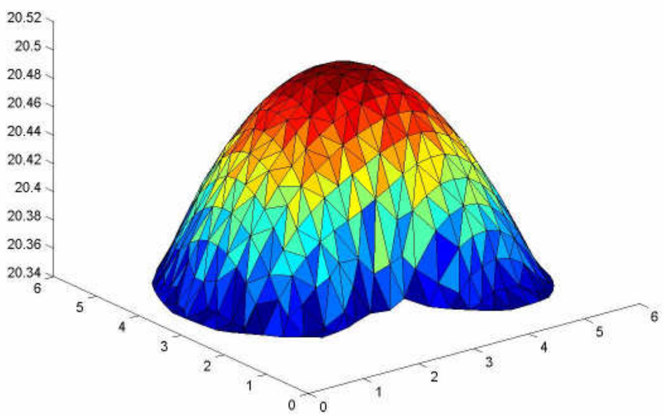

Figure 2. first iteration result

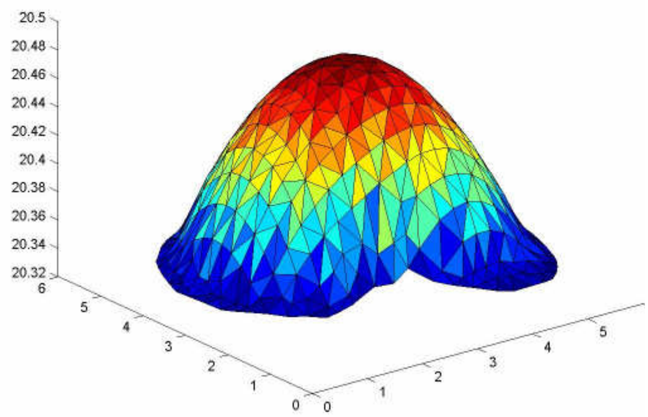

Figure 3. fourth iteration result

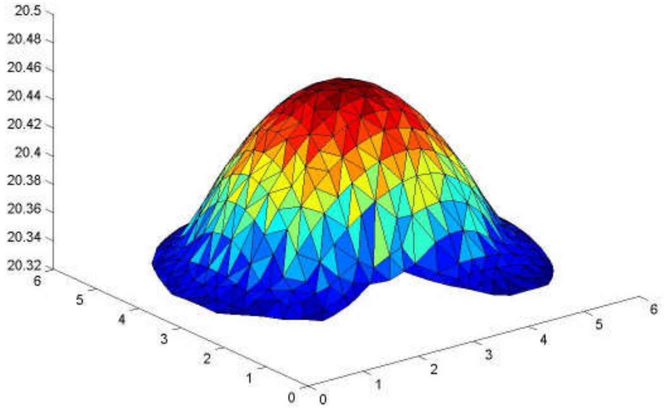

Figure 4. seventh iteration result

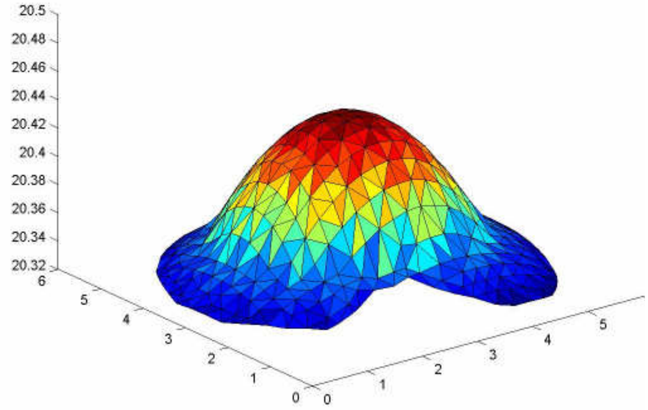

Figure 5. tenth iteration result

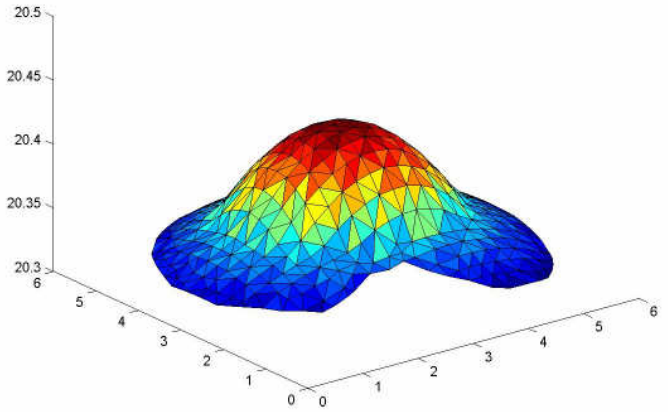

Figure 6. twelveth iteration result

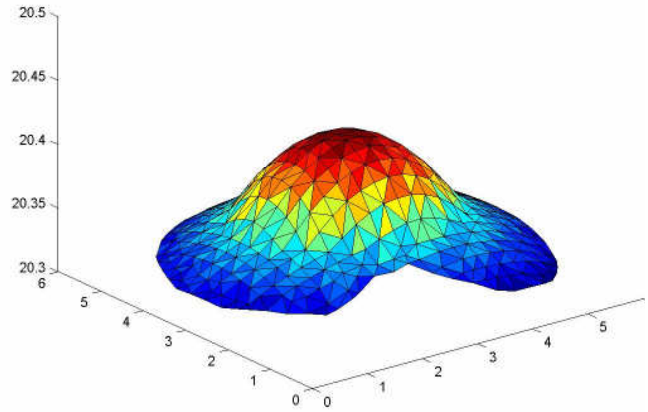

Figure 7. fifteenth result

It is notable that there are two distinguished regions of constant thermal conductivity in the temperature field, which was an expected result due to the temperature dependence of the thermal conductivity employed in this example.

8. Final remarks

The presented procedure proved to be a simple, but efficient subsidy for solving problems of conduction heat transfer with temperature dependent thermal conductivity, by means of classic tools employed on heat transfer linear problems. Despite most of the solutions will be obtained utilizing numerical methods, with an inherent error, the mathematical model provides the exact solution, as long as we reach the exact limit of the sequence. Moreover, the convergence is ensured if we choose α larger or equal to h / k_{MIN} .

9. References

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