REDUCED ORDER MODEL FOR THE NONLINEAR VIBRATION ANALYSIS OF BEAMS AND PORTAL FRAMES

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Abstract. *One of the fundamental results in classical mechanics is that linear systems with n degrees of freedom have n orthogonal vibration modes and n frequencies which are independent of the vibration amplitude, and that any motion of the system can be obtained as a linear combination of these modes. This does not hold for nonlinear systems and amplitude dependent vibrations modes and frequencies must be obtained. One way of obtaining these informations for arbitrary structures is to use a nonlinear finite element software. However this is a cumbersome and time consuming procedure. A better approach is to derive a consistent low dimensional model from which the nonlinear frequencies and mode shapes can be derived. In this paper a procedure for the derivation of low dimensional models for beams and portal frames is proposed. Based on a known linear vibration mode and using the basic ideas of perturbation theory, the nonlinear vibration modes and the respective frequency-amplitude relation are obtained. This allows the analysis of the forced (damped or undamped) vibrations of the structure in non-linear regime. However nonlinear resonance curves usually presens limit points. To obtain these curves, a methodology for the solution of non-linear equations based on continuation and arc-length procedures is derived. The results are compared with analitical solutions found in the literature.*

Keywords*: nonlinear vibration, nonlinear modes, beams, frames, continuation techniques*

1. INTRODUCTION

According to Sathyamoorthy (1997) linear theories are, at best, first order approximations of the solution of engineering problems. In vibration analysis, linearized formulations are insufficient to explain many phenomena such as multivalued equilibrium solutions, amplitude-dependent frequency, jump phenomena, bifurcation points and subharmonic oscillations. For this reason, a formulation containing the most important sources of nonlinearities must be used in the analysis. Nowadays, the most common way to perform a nonlinear analysis of structural systems is to use a nonlinear finite element software. However the use of FE packages to perform a parametric non-linear analysis for the understanding of the nonlinear behavior of a given structure is not feasible. An alternative to this time-consuming procedure is to derive a consistent low dimensional model capable of capturing the main nonlinear characteristics of the structure (Rega and Troger, 2005).

Experiments, numerical simulations and rigorous mathematical analysis show that for some systems, an accurate description of their asymptotic behavior should still be possible by reducing the originally high dimensional (infinite in the case of a continuous structure) space to one of much smaller dimension (Rega and Troger, 2005). This approach simplifies the nonlinear vibration analysis and reduces the time consumed by a nonlinear finite element procedure. Various numerical and analytical asymptotic approaches have been used for the determination of periodic solutions of nonlinear continuous structures. An overview of these methods can be found in the paper written by Rega and Troger (2005).

Among the most efficient approximation methods in nonlinear dynamics is the Ritz Method used together with the Harmonic Balance methods to approximate the continuous structure behavior in the space and time domains (Lewandowski, 1987). The major difficulty in applying this procedure is to find appropriate shape functions to approximate the nonlinear vibration modes in the Ritz procedure. Generally the linear modes are used for this purpose (Lewandowski, 1987). However for many common engineering structures like portal frames there are no exact analytical solutions for the linearized problem. The polynomial interpolation of the vibration modes obtained with the finite element method is presented here as a feasible tool to overcome these difficulties.

By applying the Ritz and Harmonic Balance methods, a system of algebraic nonlinear equations is obtained whose solution gives the frequency-response curve of the problem. Iterative methods like Newton-Rapshon have been successfully applied for this purpose (Ferreira and Serpa, 2005). However there are situations where this method does not converge due to the existence of complex paths with, for example, limit points. Ferreira and Serpa (2005) show that the arc-length method can be applied with success to overcome this difficulty.

2. GENERAL FORMULATION

This section considers the variational approach to the system leading to a partial differential equation that governs the beam motion and a brief discussion of the approximated methodology employed by the nonlinear analysis.

2.1. Partial differential equation of motion

The equation of motion of a continuous body can be derived directly by applying the generalized Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T - \Pi)\, dt + \int_{t_1}^{t_2} \delta W_{nc}\, dt = 0, \tag{1}$$

where T is the kinetic energy, Π is the total potential energy and W_{nc} is the total work done by the nonconservative forces acting upon the system. The kinetic energy of a homogeneous and isotropic beam is given by

$$T = \frac{1}{2}\rho A \int_0^L \left(\frac{\partial w}{\partial t}\right)^2 dx, \tag{2}$$

where ρ, A and L are the specific mass, cross-sectional area and the length of beam, respectively. In addition w denotes the transverse displacement which depends only on time, t, and the axial coordinate, x. The total potential energy is given by:

$$\Pi = U + V, \tag{3}$$

here U is the elastic strain energy and V is the energy due the work done by the conservative external forces acting on the structure. For an elastic geometrically nonlinear inextensional beam, Sampaio (2004) proposes the following nonlinear approximation for the strain energy:

$$U \cong \frac{1}{2} EI \int_0^L \left(w_{,xx}^2 + w_{,xx}^2 w_{,x}^2 + \frac{1}{4} w_{,xx}^2 w_{,x}^4 \right) dx, \tag{4}$$

where E is the Young's Modulus and I, the moment of inertia of the cross-section. The forced vibration is considered by the consideration of a harmonic force with amplitude γ_0 and frequency ω distributed over the whole length of the beam, so this force can be simply expressed as,

$$P(t) = \gamma_0 \sin(\omega t) \tag{5}$$

The work done by this force is consequently:

$$V = \int_0^L P(t) w\, dx \tag{6}$$

This paper takes in consideration the effects of a viscous damping force, assuming that the damping force is proportional to velocity by the factor c - called damping constant. The work done by this dissipative force can be written as,

$$W_{nc} = \frac{1}{2} c \int_0^L \left(\frac{\partial w}{\partial t} \right)^2 dx \qquad (7)$$

Substituting Eq. (2), (4), (6) and (7) into Eq. (1) and performing the appropriated variational techniques, yields

$$\rho A w_{,tt} + c w_{,t} + EI w_{,xxxx} + EI \Big(w_{,xx}^3 + w_{,x}^2 w_{,xxxx} + 4 w_{,x} w_{,xx} w_{,xxx} + \frac{3}{2} w_{,x}^2 w_{,xx}^3 + \frac{1}{4} w_{,x}^4 w_{,xxxx} + 2 w_{,x}^3 w_{,xx} w_{,xxx} \Big) = \gamma_0 \sin(\omega t) \qquad (8)$$

Equation (8) describes the transversal forced vibrations of damped beams, where the commas denote the derivative which respect to the variable.

2.2. Solution

In general Eq. (8) only has feasible solutions if solved by an approximate approach. As a first order simplification, the problem can be considered linear. If the effects of damping and external excitation are not considered, the following free vibration problem is obtained,

$$w_{,tt} + \frac{EI}{\rho A} w_{,xxxx} = 0 \qquad (9)$$

Equation (11) can be solved by separation of variables. Since the concern here is to find the periodic solutions of the problem, the solution is assumed to have the form:

$$w(x,t) = a \sin(\omega_0 t) \varphi(x) \qquad (10)$$

where *a* is a constant and ω_0, the natural frequency of the system. Substituting Eq. (12) into Eq. (11) and solving the resulting ordinary differential equation in *x*, one obtains the following solution:

$$\varphi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x) + c_3 \cosh(\alpha x) + c_4 \sinh(\alpha x), \qquad (11)$$

where c_1, c_2, c_3 and c_4 are constants obtained by applying the problem's boundary conditions and $\alpha^4 = \omega_0^2 \rho A / EI$.

If the geometric nonlinearities are taken into account, the Ritz method can be used to obtain an approximate solution of the problem. Here the solution of Eq. (8) is assumed as:

$$w(x,t) = \sum_{i=1}^{n} q_i(t) \varphi_i(x), \qquad (12)$$

where ϕ_i are the interpolating functions which must satisfy the cinematic boundary conditions, $q_i(t)$ are the time-dependent generalized coordinates and *n* is the appropriate number of these coordinates necessary to describe de displacement field properly.

Generally the functions ϕ_i are taken as the linear modes obtained by from the free-vibration analysis of the problem, since these functions satisfy all the problem boundary conditions. For generic sets of boundary conditions, the use of Eq. (11) leads to terms involving powers and products of hyperbolic and trigonometric functions, causing high of processing costs. Furthermore the extension of this analysis to a framed structure is rather difficult, since solutions like Eq. (11) are very hard to obtain. A way to overcome these difficulties is to use simpler functions such as polynomials. In this work these polynomial functions are taken from the eigenvectors obtained in the finite element eigenvalue problem. Beam elements with 2 nodes and 2 degrees of freedom per node (1 rotation and 1 displacement) are used here to obtain a discretization of Eq. (9), resulting in the system,

$$[M]\{q_{,tt}\} + [K]\{q\} = 0 \qquad (13)$$

where *M* is the mass matrix and *K* is the stiffness matrix. Considering that q has a periodic form, Eq. (13) results in an eigenvalue problem whose solution gives the natural frequencies and the correspondent linear modes of the system. These linear modes are interpolated by normalized polynomial functions.

The modal approximation (12) is substituted into the beam functional and the Ritz method is applied, resulting in a system of *n* ordinary nonlinear differential equations of motion in terms of the generalized coordinates $q_i(t)$. This system can be solved by the application of the harmonic balance method. The coordinates $q_i(t)$ are assumed to be periodic, that is

$$q_i(\tau) = X_{i,1}\sin(\tau) + X_{i,2}\cos(\tau), \tag{14}$$

here $X_{i,1}$ and $X_{i,2}$ are the amplitudes. By substituting Eq. (14) into the system of ordinary nonlinear differential equations, and using trigonometric identities and collecting the terms in sine and cosine, a homogeneous system of *2n* nonlinear algebraic equations is obtained, which can be written as:

$$\left[\Psi\left(\{X\},\omega\right)\right] = 0 \tag{15}$$

The system (15) can be solved iteratively by the arc-length method. Basically, the arc-length considers a frequency factor λ as a new variable, so the system can be rewritten as,

$$\left[\Psi\left(\{X\},\lambda\omega\right)\right] = 0 \tag{16}$$

The new system described by Eq. (16) contains 2n + 1 variables and an extra constraint equation is added to define unequivocally the next equilibrium point in the problem's solution. Crisfield (1997) proposed a constraint equation for static problems. This general format can be adapted in the dynamic case, as shown by Ferreira and Serpa (2005),

$$\Theta\left(\{\Delta X\},\Delta\lambda\right) = \left(\{\Delta X\}^t\{\Delta X\} + \Delta\lambda^2\psi^2\omega^2\right) - \Delta l^2 = 0, \tag{17}$$

where $\{\Delta X\}$, $\Delta\lambda$ and Δl are respectively the incremental displacement vector, the incremental frequency factor and the fixed radius of the desired intersection, while the scalar ψ is a scale parameter. The system (16) plus (17) is solved by the incremental-iterative Newton-Raphson Method. With this procedure one can find the stable and unstable branches of the nonlinear resonance curves for the damped and undamped case as well as the frequency-amplitude relation.

3. EXAMPLES

First, in this section, the results for a simply supported beam are obtained by the Ritz method using trigonometric and polynomial functions. Then, the approach using polynomial functions as an approximation to the linear vibration mode is extended for beams with other boundary conditions.

3.1. Simply supported beam

Applying the appropriated boundary conditions to Eq. (11), the linear solution for a simply supported beam is given by:

$$\varphi(x) = c_2\sin(\alpha x) \;\rightarrow\; \alpha = \frac{j\pi}{L},\; j = 1,2... \tag{18}$$

Making *j* equal to 1 in Eq. (18), one obtains the first linear vibration mode for a simply supported beam and the first natural frequency of the system, which is given by,

$$\omega_0 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho A}} \tag{19}$$

Substituting Eq. (18) into Eq. (12) with n=1 and applying the Ritz method, the continuous system is reduced to a one-degree-of-freedom system. By applying the harmonic balance method, a system of two algebraic equations is obtained, which is solved by the arc-length method with ψ = 0.001.

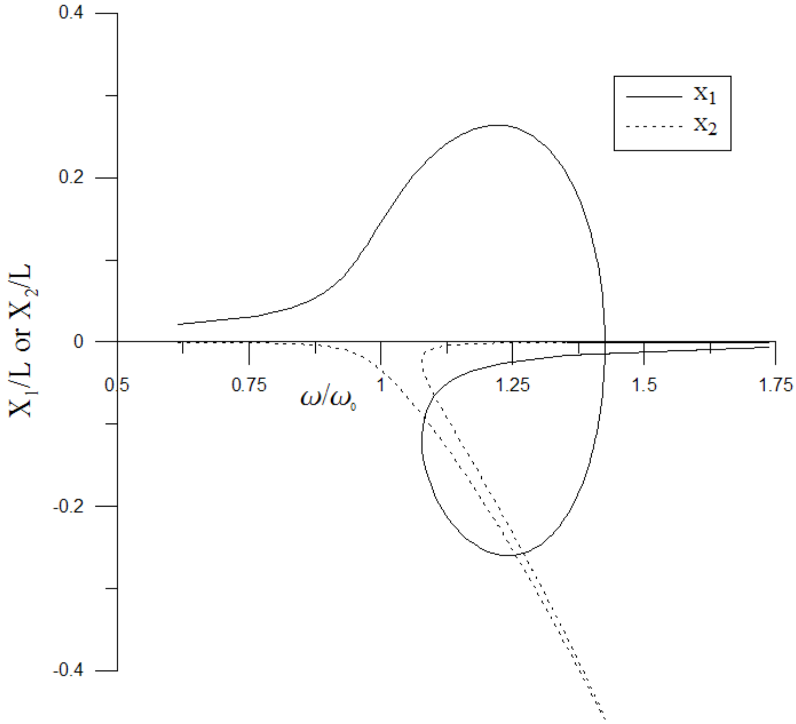

Figure 1. Amplitude-dependent frequency phenomena

The variation of the normalized modal amplitudes X_1/L and X_2/L as a function of the frequency relation ω/ω_0 is shown in Fig. 2. The results are obtained, without loss of generality, by considering unit values for geometric and material beam parameters as well as for the periodic force amplitude, and the damping constant is taken as 0.1. Typical nonlinear phenomena, such as frequency dependence on the vibration amplitude and the multiplicity of solutions for a given value of ω/ω_0, are observed.

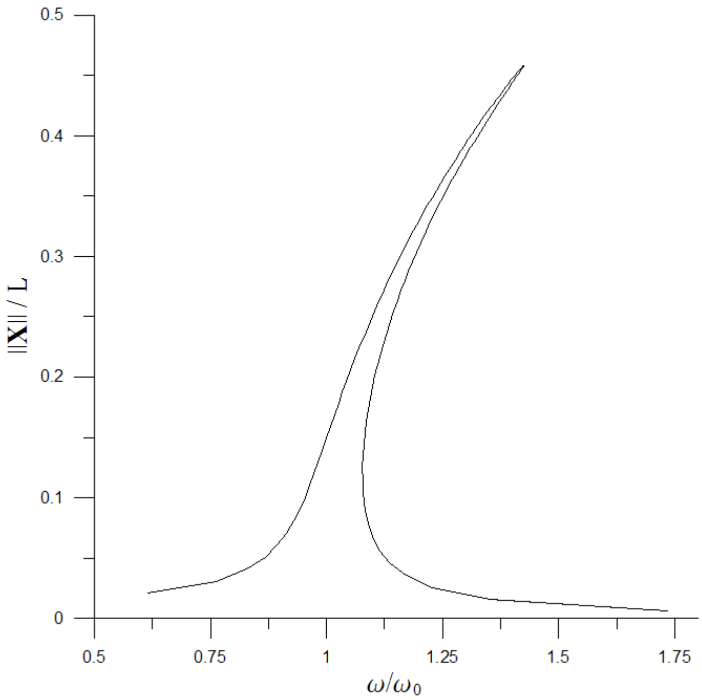

Figure 2. Frequency-response curve for a geometrically nonlinear simply supported beam

The resonance curve can be better understood by plotting the magnitude of the vibration amplitude as a function of the frequency parameter ω / ω_0. This is illustrated in Fig. 2 where a resonance curve with hardening type nonlinearity is observed, which is typical for this class of structures (Nayfeh and Mook, 1979; Sathyamoorthy, 1997). Indeed, the frequency-response curve is multivalued and the amplitude peak is finite. This multivaluedness of the response curve leads to a jump phenomena, which has usually a deleterious effect on the system response leading frequently to high stresses and strains.

Next, a polynomial function of 4th degree interpolated from the finite element results is used, instead of the trigonometric exact solution for the eigenfunction, to perform the same analysis. Two beam finite elements were used, and other higher polynomial degrees were employed, but the one of 4th degree was found to be sufficient to obtain a result similar to that obtained using the trigonometric function. These results are shown in Fig. 3.

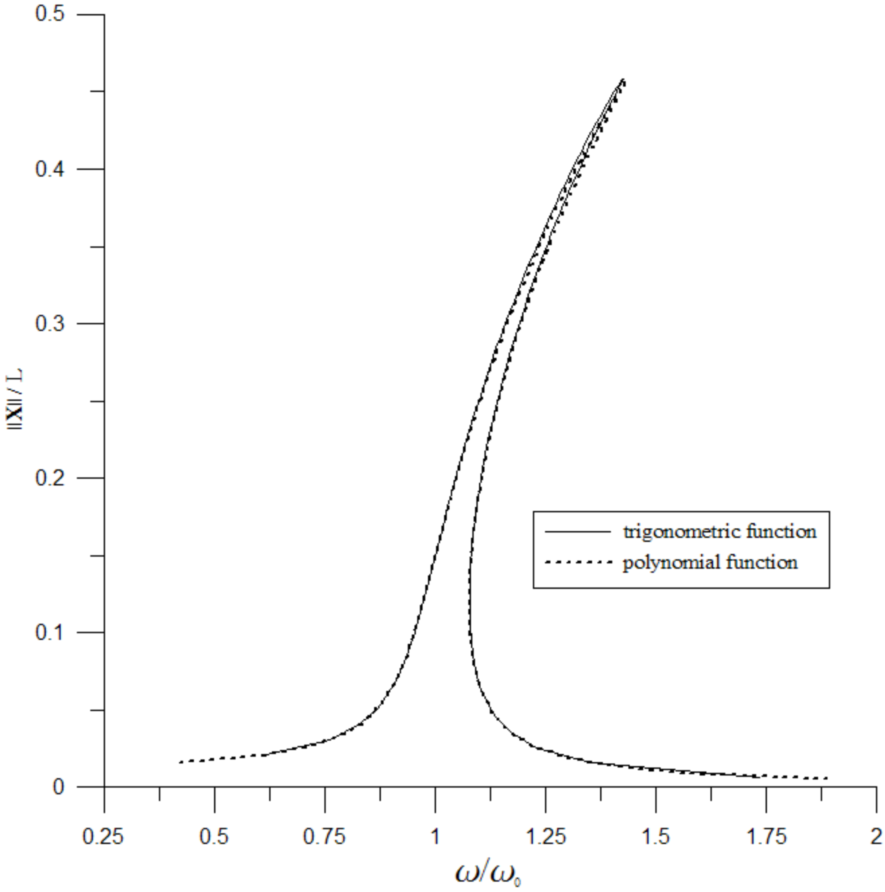

Figure 3. Comparison of frequency-response curve obtained by trigonometric and polynomial approximations of the nonlinear vibration mode

The comparison made in Fig. 3 shows a minimal difference between the two curves, confirming the quality of the proposed methodology.

3.2. Beams with other boundary conditions

The same analysis was conduced for beams with other boundary conditions. For comparison purposes, the beam parameters are considered to be the same as in the previous analysis. The same polynomial degree and finite element discretization used for the simply supported beam are found to be sufficient to describe correctly the frequency-response curves for these examples.

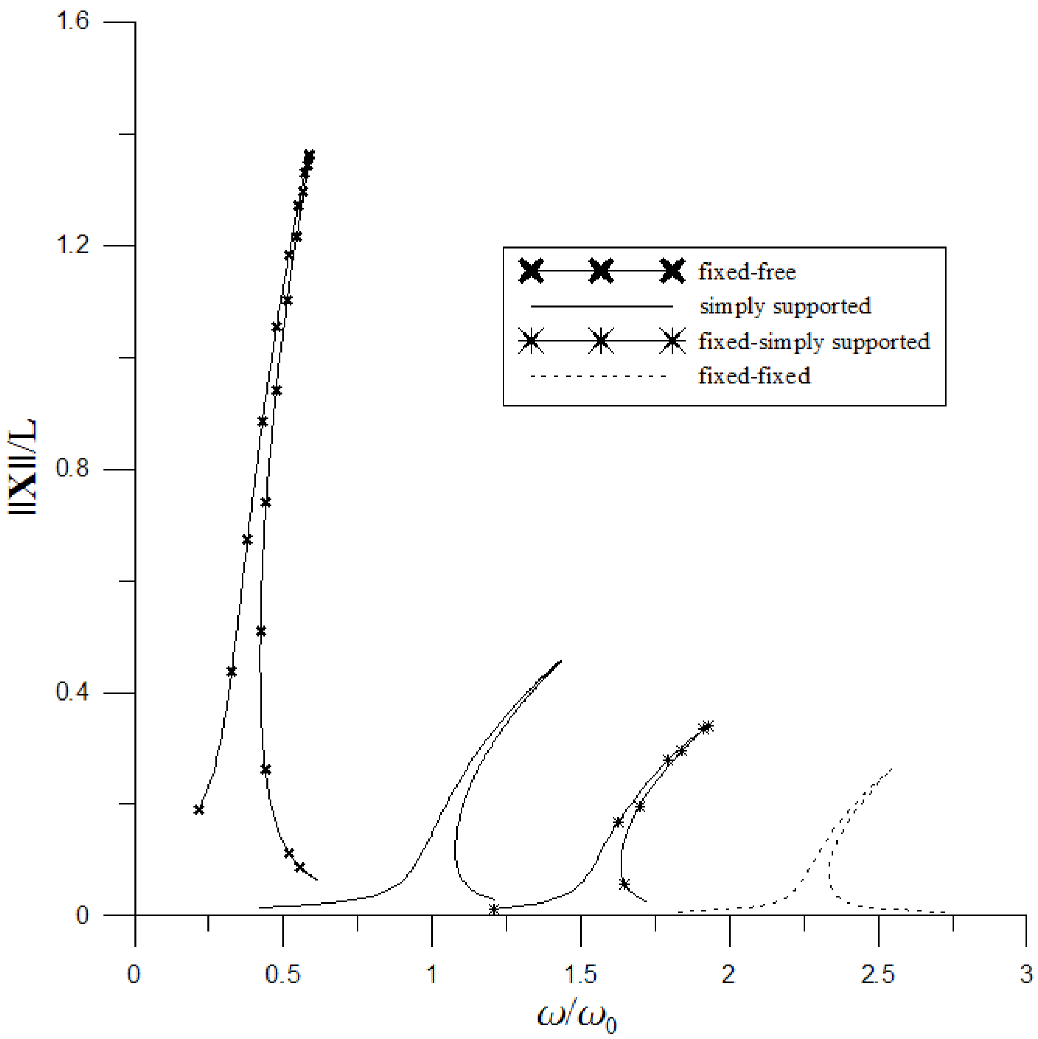

Figure 4. Frequency-response curves for various boundary conditions

Figure 4 shows the nonlinear resonance curves for four different sets of boundary conditions. It is clear from the results the high influence of the supports' stiffness on the natural frequency and on the nonlinearity of the response. However, the hardening character of the response remains unaltered, as expected (Sathyamoorthy, 1997). For the same amount of damping, one can observe a marked decrease in the peak response and, consequently, on the magnitude of the jumps as the stiffness of the system increases due to the boundary conditions. On the other hand, the variation of the frequency parameter with the vibration amplitude increases significantly with the effective stiffness of the beam.

4. CONCLUSIONS

The Ritz and harmonic balance methods were used in this study to obtain a consistent low dimensional model to describe the geometrically nonlinear free and forced vibrations of a slender beam with various sets of boundary conditions. The use of relatively simple polynomial functions produced satisfactory frequency-response curves for the structures studied is this paper, comparing well with the more refined trigonometric solution developed for the simply-supported beam. This methodology allows the use of traditional finite element software for the linear analysis of structures as a basis for the derivation of a set of efficient interpolation functions for the Ritz method. The harmonic balance method is shown to be an efficient method, which can be easily implemented for multi-degree-of-freedom systems, to reduce the discretized nonlinear equations of motion into a system of nonlinear algebraic equations in the frequency domain. The application of the arc-length method allows one to obtain the nonlinear amplitude-frequency relation for this class of problems, in spite of the two sharp limit points observed in the resonance curves of the damped system. These results indicate that this methodology can be easily extended to the nonlinear vibration analysis of portal frames. This part of the present research project is being implemented at the moment, together with the derivation of the nonlinear part of the vibration modes.

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