ENHANCED CONVERGENCE OF INTEGRAL TRANSFORM SOLUTION OF ABLATION PROBLEMS

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Abstract. *This work is aimed at further advancing a computational procedure for the design of thermal protection systems of space vehicles during atmospheric reentry. The Generalized Integral Transform Technique is thus employed in obtaining a hybrid numerical-analytical solution with computational performance and robustness for incorporation into an optimization engineering code that works towards weight minimization in the TPS design. For this purpose, an integral balance approach is employed in the convergence acceleration of eigenfunction expansions for transient heat conduction with an ablative moving boundary. The proposed approach is first demonstrated for a previously studied benchmark case, and then illustrated for a more realistic situation of aerodynamic heating in balistic reentry flights, using typical thermophysical properties of common thermal protection materials.*

Keywords*: Integral transforms, Hybrid methods, Ablation, Heat conduction, Moving boundary, Thermal protection*

1. INTRODUCTION

The design of recoverable orbital platforms requires the detailed analysis of the heat transfer problem during the atmospheric reentry phase of the flight, involving the aerodynamic heating delivered to the vehicle surface and its interaction with the thermal protection system (TPS), either of the ablative or rejection types (Tauber, 1989, Bouilly *et al*., 1998, Amundsen *et al*., 2000). Such analysis is in general aimed at optimizing the TPS weight while warranting the integrity of the satellite structure and of the payload thermal environment restrictions (Chen & Milos, 1999). The Institute for Aeronautics and Space, IAE/CTA, has been leading the design and construction of an orbital platform for microgravity experiments along the last ten years, the vehicle SARA (Moraes, 1998), and has devoted substantial effort towards the comprehension of the involved thermal phenomena, adequate selection of thermal protection materials, and completion of this technological task in a sensible area (Pessoa Filho, 1997, Costa, 2000). Besides the accurate characterization of thermophysical properties of candidate materials, there is also the need of constructing a computer simulation tool that is sufficiently robust and computationally feasible, in order to reproduce the complex heat transfer process that takes place within such severe environmental conditions and surface heat flux excitations, but also permitting the recurrent solution of the problem to achieve an optimized form of the thermal protection system.

In the realm of the present long term project, a few earlier contributions were advanced towards the thermal analysis of ablative TPS even before the initiation of the SARA project, always in collaboration with IAE/CTA, including integral transform analysis (Diniz *et al*., 1990, Ruperti Jr. & Cotta, 1991) and improved lumped approaches (Cotta *et al*., 1992) for ablation problems, also reviewed in (Cotta, 1993, Cotta & Mikhailov, 1997, Ruperti *et al*., 1998, and Cotta, 1998). More recently, within the context of the Uniespaço program sponsored by the Brazilian Space Agency, AEB, further studies on improved lumped-differential formulations of heat conduction problems with ablation (Ruperti Jr. & Cotta, 2000), allowed for the construction of an engineering-type code that permits the minimization of TPS thicknesses in a typical reentry flight simulation, including the simultaneous computation of the net heat flux at the vehicle surface along the flight (Cotta *et al*., 2001, Cotta *et al*., 2004). This so-called TPS Nose computer code was constructed in the *Mathematica* system platform, employing mixed symbolic-numerical computations (Wolfram, 1999), offering a fast and simple to use approach for pre-design purposes. Nevertheless, research efforts have been progressing towards the more accurate and/or more complex modeling of ablative thermal protections, including pyrolysis, different geometries, interaction with the structure, variable thermophysical properties, etc. (Rey Silva & Orlande, 2002, Sias *et al*., 2005, Gomes *et al.*, 2006, Machado Jr., 2006), and this trend has also been followed in the context of the present project, aiming at validating and extending the TPS Nose code (Cotta *et al*., 2006).

The present work reports one such attempt of providing a more accurate but sufficiently computational effective approach, based on integral transforms, for the hybrid numerical-analytical solution of heat conduction problems with ablative surfaces and arbitrarily varying applied heat flux. Convergence acceleration of the related eigenfunction expansion is proposed, through the application of the technique known as integral balance approach (Scofano Neto *et al*., 1990, Leiroz & Cotta, 1990, Cotta, 1993) that provides alternative series expansions for the potential field and its derivative with respect to space variables. Besides, the initial value problem solver available in the

Mathematica system (Wolfram, 1999) is critically evaluated, under different parametric control, so as to explore different implemented numerical schemes and further enhance the performance of the constructed *Mathematica* code. Such aspects are illustrated via an example previously considered as a benchmark case (Diniz *et al*., 1990) and then employed in the solution of a more realistic situation, by considering the time variable wall heat flux in (Ruperti and Cotta, 2000).

2. ANALYSIS

We consider transient one-dimensional heat conduction in a slab, representing the thermal protection material layer applied over the surface of a reentry vehicle, with characteristic predominance of temperature gradients across its thickness. As the external surface applied heat flux increases, the slab wall reaches the phase change temperature and the material enters the so-called ablation period, while the wall recession rate is formulated from the energy balance at this interface. A major concern is the determination of the ablative boundary evolution, but under thermal stresses restrictions and internal temperature limits. A simplified formulation of the TPS design problem is here considered, so as to demonstrate the solution methodology enhancement proposed. For this purpose, we consider that the thermophysical properties are temperature independent and assume that the material does not undergo pirolysis, also neglecting the thermal interaction with the vehicle structural material, with radiative losses already accounted for in the net heat flux at the heated surface (Diniz *et al*., 1990, Ruperti Jr. & Cotta, 2000).

The solution temporal domain is split in two periods, a pre-ablation period that allows for an exact solution of the related linear heat conduction problem based on the Classical Integral Transform Method, and the ablation period itself, when the problem formulation becomes nonlinear due to presence of the unknown moving boundary, here handled by the Generalized Integral Transform Technique, GITT (Cotta, 1993, Cotta & Mikhailov, 1997, Cotta, 1998). The problem formulation in dimensionless form is then given as:

Pre-ablation period

$$\frac{\partial \Theta(\eta,\tau)}{\partial \tau} = \frac{\partial^2 \Theta(\eta,\tau)}{\partial \eta^2} \qquad 0 < \eta < 1;\ 0 < \tau < \tau_{ab} \tag{1.a}$$

$$\Theta(\eta,0) = 0 \qquad 0 \le \eta \le 1 \tag{1.b}$$

$$-\left.\frac{\partial \Theta(\eta,\tau)}{\partial \eta}\right|_{\eta=0} = Q(\tau) \qquad \tau > 0 \tag{1.c}$$

$$\left.\frac{\partial \Theta(\eta,\tau)}{\partial \eta}\right|_{\eta=1} = 0 \qquad \tau > 0 \tag{1.d}$$

Ablation Period

$$\frac{\partial \Theta(\eta,\tau)}{\partial \tau} = \frac{\partial^2 \Theta(\eta,\tau)}{\partial \eta^2} \qquad S(\tau) < \eta < 1;\ \tau > \tau_{ab} \tag{2.a}$$

$$\Theta(\eta,\tau_{ab}) = \Theta_{ab}(\eta) \qquad 0 \le \eta \le 1 \tag{2.b}$$

$$\Theta(S(\tau),\tau) = 1 \qquad \tau > \tau_{ab} \tag{2.c}$$

$$\left.\frac{\partial \Theta(\eta,\tau)}{\partial \eta}\right|_{\eta=1} = 0 \qquad \tau > \tau_{ab} \tag{2.d}$$

Wall Recession Equation

$$\nu \frac{dS(\tau)}{d\tau} = Q(\tau) + \left.\frac{\partial \Theta(\eta,\tau)}{\partial \eta}\right|_{\eta=S(\tau)} \qquad \tau > \tau_{ab} \tag{3.a}$$

$$S(\tau_{ab}) = 0 \tag{3.b}$$

The employed dimensionless groups are:

$$\Theta(\eta,\tau) = \frac{T(x,t)-T_0}{T_{ab}-T_0};\ \eta = \frac{x}{L};\ \tau = \frac{\alpha t}{L^2};\ Q(\tau) = \frac{L\,q(t)}{k(T_{ab}-T_0)};\ S(\tau) = \frac{s(t)}{L};\ \nu = \frac{H}{C_p(T_{ab}-T_0)} \tag{4}$$

where T_{ab} is the material ablation temperature, L is the slab thickness, α is thermal diffusivity, $q(t)$ is the applied wall heat flux, k is thermal conductivity, $s(t)$ is the moving boundary position, c_p is specific heat, T_0 is the initial wall temperature, H is the material heat of ablation, and ν is the inverse of the Stefan number.

The pre-ablation period is readily solved through classical integral transforms (Cotta, 1993), by taking the following integral transform pair

$$\bar{\theta}_i(\tau) = \int_0^1 \Psi_i(\eta)\,\Theta(\eta,\tau)\,d\eta \qquad \textit{Transform} \tag{5.a}$$

$$\Theta(\eta,\tau) = \sum_{i=0}^{\infty} \frac{\Psi_i(\eta)}{N_i}\,\bar{\theta}_i(\tau) \qquad \textit{Inverse} \tag{5.b}$$

where $\Psi_i(\eta) = \cos(\beta_i \eta)$ are the eigenfunctions obtained in separating variables of the homogeneous version of the proposed problem, $N_i = \frac{1}{2}$ is the associated norm for i>0, $\bar{\theta}_i(\tau)$ are the transformed potentials to be obtained, and the index i starts from zero so as to account for the first eigenvalue with zero value due to the second type boundary conditions on both walls. The pre-ablation problem solution was here considered in its simplest form without the aid of a filtering solution, in general employed to enhance convergence behavior. The integral balance procedure here recalled also works towards accelerating the expansions convergence, and the combined use of these two techniques provides even further enhancement (Cotta & Mikhailov, 1997).

Upon integral transformation of the pre-ablation problem, the transformed temperature field is readily obtained in explicit form, and the final temperature field is composed as:

$$\Theta(\eta,\tau) = \theta_{av}(\tau) + 2\sum_{i=1}^{\infty} \cos(\beta_i \eta) \int_0^{\tau} Q(\tau')\, e^{-\beta_i^2(\tau-\tau')}\, d\tau' \tag{6}$$

where β_i are the eigenvalues, and the contribution of the first eigenvalue, β_0=0, has already been separated as the average temperature evolution, $\theta_{av}(\tau)$, given by:

$$\theta_{av}(\tau) = \int_0^{\tau} Q(\tau')\, d\tau' \tag{7}$$

To determine the time required for the onset of ablation, τ_{ab}, we let $\eta = 0$ in Eq.(6), making the result equal to 1, which corresponds to letting $T(x,t) = T_{ab}$, and then τ_{ab} is computed from the following transcendental equation:

$$\Theta(0,\tau_{ab}) = 1 = \theta_{av}(\tau_{ab}) + 2\sum_{i=1}^{\infty} \int_0^{\tau_{ab}} Q(\tau')\, e^{-\beta_i^2(\tau_{ab}-\tau')}\, d\tau' \tag{8}$$

Once the onset of ablation is identified and the corresponding temperature distribution is obtained from Eq.(6), the ablation problem needs to be handled. Following previous works notation, the following change of variables is introduced:

$$\Theta^*\left(\eta^*,\tau\right) = \Theta\left(\eta,\tau\right) - 1;\ \eta^* = 1 - \eta;\ \eta_b(\tau) = 1 - S(\tau) \tag{9}$$

and Eqs.(2,3) are rewritten as:

Ablation Period

$$\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \tau} = \frac{\partial^2 \Theta^*(\eta^*,\tau)}{\partial \eta^{*2}} \qquad 0 < \eta^* < \eta_b(\tau);\ \tau > \tau_{ab} \tag{10.a}$$

$$\Theta^*(\eta^*,\tau_{ab}) = \Theta_{ab}(\eta^*) - 1 \qquad 0 \le \eta^* \le 1 \tag{10.b}$$

$$\left.\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=0} = 0 \qquad \tau > \tau_{ab} \tag{10.c}$$

$$\Theta^*(\eta_b(\tau),\tau) = 0 \qquad \tau > \tau_{ab} \tag{10.d}$$

Wall Recession Equation

$$\frac{d\eta_b(\tau)}{d\tau} = -\frac{1}{\nu}\left[Q(\tau) - \left.\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=\eta_b(\tau)} \right] \qquad \tau > \tau_{ab} \tag{11.a}$$

$$\eta_b(\tau_{ab}) = 1 \tag{11.b}$$

For this nonlinear formulation, we need to recall the Generalized Integral Transform Technique (Cotta, 1993, Cotta & Mikhailov, 1997, Cotta, 1998), since the unknown boundary movement generates a time-dependent eigenvalue problem, which does not allow for an exact decoupled integral transformation process as in the previous pre-ablation problem. Thus, the integral transform pair is now given by:

$$\overline{\theta}_i^*(\tau) = \int_0^{\eta_b(\tau)} \tilde{\Psi}_i(\eta^*,\tau)\,\Theta^*(\eta^*,\tau)\,d\eta \qquad \textit{Transform} \tag{12.a}$$

$$\Theta^*(\eta^*,\tau) = \sum_{i=1}^{\infty} \tilde{\Psi}_i(\eta^*,\tau)\,\overline{\theta}_i^*(\tau) \qquad \textit{Inverse} \tag{12.b}$$

where a symmetric kernel was adopted, with the normalized eigenfunctions, $\tilde{\Psi}_i(\eta,\tau)$, written as:

$$\tilde{\Psi}_i(\eta^*,\tau) = \frac{\Psi_i^*(\eta^*,\tau)}{\sqrt{N_i(\tau)}} \tag{13}$$

where $\Psi_i^*(\eta^*,\tau) = \cos(\beta_i^*(\tau)\,\eta^*)$ and $N_i(\tau)$ is the corresponding norm.

The time-dependent eigenvalues, due to the moving boundary, are explicitly given as:

$$\beta_i^*(\tau) = \frac{(2i-1)\pi}{2\eta_b(\tau)} \tag{14}$$

The integral transformation process as applied to Eqs.(10) above now results in a coupled system of ordinary differential equations for the transformed potentials, in the form:

$$\frac{d\overline{\theta}_i^*(\tau)}{d\tau} + \sum_{j=1}^{\infty} A_{ij}(\tau)\,\overline{\theta}_j^*(\tau) = 0 \qquad \tau > \tau_{ab} \tag{15.a}$$

$$\overline{\theta}_i^*(\tau_{ab}) = \overline{f}_i^*(\tau_{ab}) \tag{15.b}$$

which is to be solved jointly with the ablative wall position equation, Eq.(11), after substitution of the wall temperature derivative:

$$\nu\frac{d\eta_b(\tau)}{d\tau} - \sum_{j=1}^{\infty} \left.\frac{\partial \tilde{\Psi}_j(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=\eta_b(\tau)} \overline{\theta}_j^*(\tau) = -Q(\tau) \qquad \tau > \tau_{ab} \tag{16.a}$$

$$\eta_b(\tau_{ab}) = 1 \tag{16.b}$$

The transformed system coefficients and transformed initial conditions are analytically obtained from:

$$A_{ij}(\tau) = \beta_i^{*2}\delta_{ij} + \int_0^{\eta_b(\tau)} \tilde{\Psi}_i(\eta^*,\tau)\frac{\partial \tilde{\Psi}_j(\eta^*,\tau)}{\partial \tau} d\eta^* \tag{17.a}$$

$$\overline{f}_i^* = \int_0^{\eta_b(\tau)} \tilde{\Psi}_i(\eta^*,\tau)(\Theta_{ab}(\eta^*) - 1) d\eta^* \tag{17.b}$$

The above formal solution is however not the most appropriate one from the computational point of view, as we shall examine below in the results section. Substantial convergence acceleration can be achieved by employing one of the techniques discussed in (Cotta and Mikhailov, 1997). Here we have chosen to adopt the integral balance approach to avoid the use of the inverse formula in the temperature derivative at the ablative wall, Eq.(16a), that couples the transformed system with the wall recession equation. As written above, the derivative of the eigenfunction would unavoidably introduce the eigenvalue in the numerator of the series expansion, markedly retarding convergence of the whole solution. On the other hand, one may seek an alternative expression for this boundary temperature derivative by integrating the original energy balance, Eq.(10a), over the whole spatial domain, as follows:

$$\int_0^{\eta_b(\tau)} \frac{\partial \Theta^*(\eta^*,\tau)}{\partial \tau} d\eta^* = \left.\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=\eta_b(\tau)} \tag{18.a}$$

Applying Leibnitz rule for the differentiation of an integral, we obtain

$$\frac{d}{d\tau}\left(\int_0^{\eta_b(\tau)} \Theta^*(\eta^*,\tau) d\eta^*\right) - \frac{d\eta_b(\tau)}{d\tau}.\Theta^*(\eta_b(\tau),\tau) = \left.\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=\eta_b(\tau)}$$

And employing boundary condition Eq.(10d) :

$$\frac{d}{d\tau}\left(\int_0^{\eta_b(\tau)} \Theta^*(\eta^*,\tau) d\eta^*\right) = \left.\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=\eta_b(\tau)} \tag{18.b}$$

Substitution of the inverse formula in the lhs term only, leads to the alternative series expansion for the wall temperature derivative:

$$\left.\frac{\partial \Theta^*(\eta^*,\tau)}{\partial \eta^*}\right|_{\eta^*=\eta_b(\tau)} = \sum_{j=1}^{\infty} \frac{d}{d\tau}\left(\int_0^{\eta_b(\tau)} \tilde{\Psi}_j(\eta^*,\tau) d\eta^* \, \overline{\theta}_j^*(\tau)\right) \tag{18.c}$$

which is then applied to the wall recession equation to yield the new working relation:

$$\frac{d\eta_b(\tau)}{d\tau} = -\frac{1}{\nu}\left[Q(\tau) - \sum_{j=1}^{\infty} \frac{d}{d\tau}\left(\int_0^{\eta_b(\tau)} \tilde{\Psi}_j(\eta^*,\tau) d\eta^* \, \overline{\theta}_j^*(\tau)\right)\right] \qquad \tau > \tau_{ab} \tag{19}$$

3. RESULTS AND DISCUSSION

The presentation and discussion of results starts with the covalidation and demonstration of the constructed mixed symbolic-numerical code and of the proposed approach with enhanced convergence. For such purposes, we take a test case previously considered by (Diniz *et al.*, 1990) with ν=1 and a prescribed wall heat flux in the form $Q(\tau)=10\ \tau$. The test case corresponds to a single finite slab thermal protection with constant thermophysical properties and arbitrary variation with time of the applied heat flux on one wall and insulated boundary at the opposite wall, such as eqs.(1-3).

The numerical results to be presented were obtained throughout with an absolute error target (parameter AccuracyGoal) equal to infinity and a relative error target (parameter PrecisionGoal) equal to 5 (five significant digits) in the numerical solution of the transformed ODE system with the function NDSolve of the *Mathematica* v5.2 system. For ordinary differential equations, NDSolve by default uses an LSODA approach (Hindmarsh, 1983), switching between a non-stiff Adams method (predictor-corrector with orders 1 through 12) and a stiff Gear backward differentiation formula method (Wolfram, 1999). Another parameter in the NDSolve function allows one to choose the numerical integration scheme for the initial value problem, and in particular for stiff ODE systems such as those in general produced by eigenfunction expansion approaches, at least two options are of interest in the comparative performance of this advanced solver. The option “BDF” forces the use of implicit backward differentiation formulas with orders 1 through 5, which are well-known in the robust solution of stiff systems and are also employed by the

default use of the routine once a sufficient degree of stiffness has been identified by the algorithm. On the other hand, the option "StiffnessSwitching" automatically switches from explicit to implicit schemes once stiffness has been detected, since stiff systems are more adequately handled by specialized implicit schemes. The basic idea behind the "StiffnessSwitching" method is to provide an automatic means of switching between a nonstiff and a stiff solver, but controlling the sub-methods to be employed in each case. Therefore, before obtaining numerical results and inspecting for the convergence behavior of the proposed eigenfunction expansion, we have examined the comparative performance of these three options of numerical scheme in the NDSolve function, for both the simpler case without the integral balance and the here proposed algorithm with convergence enhancement. In synthesis, for the lower system sizes, when a sufficiently high stiffness ratio is not yet achieved, the performance of the three approaches is comparable. However, as stiffness builds up, the dedicated routines demonstrate their usefulness, offering a significant reduction in computational effort. Although the default use of NDSolve itself allows for the switching on stiffness to the BDF approach, the use of the Adams method while stiffness was not detected, markedly increases the overall computational effort. The two methods specifically designed to work with stiff systems perform quite comparably, with a slightly better behavior of the BDF approach for the present application, which was therefore the preferred option in the continuation of the computations that are reported below.

Tables 1.a, b illustrate the difference in convergence behavior for the two GITT solution alternatives, respectively, for the formal solution without convergence acceleration and for the proposed solution with integral balance application. The parameter chosen to illustrate this comparison is the ablating boundary position, $S(\tau)$, due to its importance to the physical application pursued, at different times along the ablation process and for increasing truncation orders. Also, the results from the earlier contribution of (Diniz *et al.*, 1990), with smaller truncation orders, are presented within the same tables for comparison purposes. From Tables 1.a,b one may readily conclude that the integral balance scheme introduces a marked acceleration in the eigenfunction expansion convergence, here indirectly inspected via the analysis of the recessing boundary position results. It can be observed that the results with the integral balance strategy already present three converged significant digits with truncation orders as low as 20, and four converged significant digits were observed in the column related to just 40 terms in Table 1.b. On the other hand, the formal solution achieves convergence to the fourth significant digit only at the higher truncation orders, the two last columns in Table 1.a. Therefore, a more fair comparison of computational effort between the two strategies should account for this marked difference, and put side by side the CPU time required for the integral balance approach with few terms and the much higher computer time required for the formal solution with a large number of terms. Undoubtedly, the formal GITT solution would not be a reasonable choice for implementation within a computer code for optimization purposes, which would require several runs under different parametric configurations.

Table 1.a Convergence of GITT solution for the ablating boundary position, $S(\tau)$, with different truncation orders of the eigenfunction expansion (without integral balance).

Time	**Truncation order** N							
$\tau - \tau_{ab}$	**Diniz *et al.*** 1990	**20**	**40**	**60**	**80**	**100**	**120**	**140**
0.00854	0.00348	0.00333	0.00324	0.00321	0.00321	0.00320	0.00320	0.00320
0.05122	0.04340	0.04267	0.04229	0.04217	0.04212	0.04208	0.04206	0.04204
0.10244	0.12000	0.11864	0.11779	0.11751	0.11738	0.11730	0.11724	0.11720
0.15366	0.22120	0.21922	0.21783	0.21737	0.21714	0.21701	0.21692	0.21685
0.20488	0.34610	0.34339	0.34141	0.34076	0.34043	0.34024	0.34011	0.34002
0.25610	0.49670	0.49326	0.49067	0.48981	0.48939	0.48914	0.48897	0.48885
0.30732	0.68090	0.67650	0.67327	0.67214	0.67161	0.67143	0.67108	0.67098
0.35000	0.88220	0.87622	0.87190	0.87040	0.86969	0.86944	0.86898	0.86885

Table 1.b Convergence of GITT solution for the ablating boundary position, $S(\tau)$, with different truncation orders of the eigenfunction expansion (with integral balance).

Time	**Truncation order** N							
$\tau - \tau_{ab}$	**Diniz *et al.*** 1990	**20**	**40**	**60**	**80**	**100**	**120**	**140**
0.00854	0.00348	0.00326	0.00320	0.00319	0.00319	0.00319	0.00319	0.00319
0.05122	0.04340	0.04202	0.04196	0.04196	0.04195	0.04195	0.04195	0.04195
0.10244	0.12000	0.11704	0.11699	0.11698	0.11698	0.11698	0.11698	0.11698
0.15366	0.22120	0.21653	0.21649	0.21648	0.21647	0.21647	0.21647	0.21647
0.20488	0.34610	0.33953	0.33948	0.33948	0.33947	0.33947	0.33947	0.33947
0.25610	0.49670	0.48818	0.48814	0.48814	0.48813	0.48813	0.48813	0.48812
0.30732	0.68090	0.67006	0.67001	0.67006	0.67004	0.67000	0.67000	0.67000
0.35000	0.88220	0.86763	0.86758	0.86762	0.86760	0.86756	0.86756	0.86755

Next, the application of the constructed code to a more realistic situation was pursued, taking the situation considered in the validation of the Coupled Integral Equations Approach, which is the approximate reformulation technique employed in the TPS Nose code (Ruperti and Cotta, 2000), which corresponds to a test case of a Teflon ablator as applied to thermal protection in a typical ballistic reentry flight. Although Teflon would not be the natural choice as ablator for this specific situation, it offers a reliable test case once pirolysis is not present for this material and the thermophysical properties are fairly temperature independent. The pertinent data for this simulation are as follows:

$$k = 0.22\ \frac{\mathrm{W}}{\mathrm{mK}},\ \rho = 1922\ \frac{\mathrm{kg}}{\mathrm{m}^3},\ Cp = 1256\ \frac{J}{kgK},\ H = 2326x10^3\ \frac{J}{kg},\ T_{ab} = 833\ \mathrm{K},\ T_0 = 416\ \mathrm{K}\ \text{ e }\ L = 0.0065\ m\ .$$

The situation to be illustrated is that associated with the thermal protection of a space vehicle subjected to a value of the parameter ν=4.4 and a net wall heat flux, already discounting the radiative losses from the aerodynamic heating, shown in Figure 1 below in dimensionless form, much higher than in the previous test case presented.

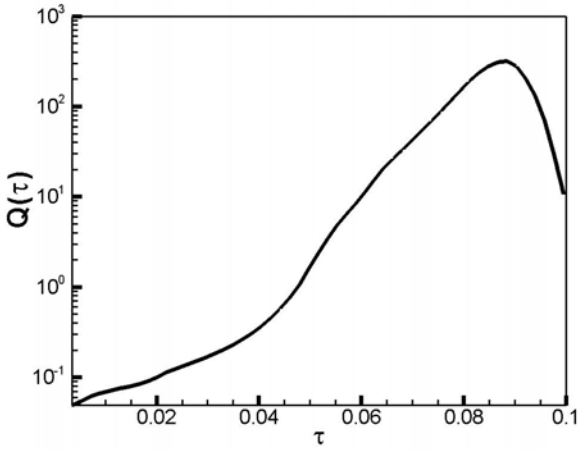

Figure 1. Net applied wall heat flux, Q(τ), employed in the TPS Nose code validation (Ruperti & Cotta, 2000).

We first illustrate the convergence behavior of the recessing boundary position for this realistic situation with a very intense applied wall heat flux, as shown in Table 2 below, for increasing truncations orders and within the ablative period.

Table 2 - Convergence of GITT solution without and with integral balance for the ablating boundary position, *S(τ)*, with wall heat flux from (Ruperti & Cotta, 2000).

Time	**GITT without IB - Truncation order** N				
τ	**50**	**60**	**70**	**80**	**90**
0.065810	0.010257	0.010247	0.010240	0.010235	0.010231
0.069618	0.031775	0.031743	0.031721	0.031704	0.031691
0.073427	0.070033	0.069959	0.069907	0.069867	0.069836
0.077235	0.135136	0.134971	0.134853	0.134764	0.134694
0.081044	0.244607	0.244221	0.243942	0.243732	0.243568
0.084852	0.418244	0.417357	0.416717	0.416233	0.415855
0.088661	0.638685	0.637163	0.636062	0.635230	0.634579
0.092469	0.822315	0.820501	0.819191	0.818199	0.817423
0.096278	0.901669	0.899815	0.898475	0.897462	0.896669
Time	**GITT with IB - Truncation order** N				
τ	**10**	**20**	**30**	**40**	**50**
0.065810	0.010325	0.010195	0.010187	0.010188	0.010189
0.069618	0.031696	0.031581	0.031575	0.031575	0.031576
0.073427	0.069682	0.069582	0.069577	0.069578	0.069579
0.077235	0.134223	0.134127	0.134122	0.134123	0.134125
0.081044	0.242354	0.242233	0.242226	0.242226	0.242227
0.084852	0.412942	0.412774	0.412761	0.412760	0.412761
0.088661	0.629433	0.629267	0.629254	0.629253	0.629253
0.092469	0.811228	0.811079	0.811068	0.811067	0.811068
0.096278	0.890326	0.890182	0.890172	0.890171	0.890172

It has been observed that convergence to at least the fifth significant digit is achieved up to N=50 with the GITT with integral balance. On the other hand, the formal GITT solution, even with N=90 terms, may still experience some variations in the third significant digit. Most interesting, the proposed enhanced approach offers convergence to the fourth significant digit with much lower truncation orders, N<30. With such behavior, very low truncation orders may be employed in the optimization procedure for minimization of the thermal protection thickness over the whole vehicle body.

The TPS Nose code (Cotta *et al*., 2001, Cotta *et al*., 2004), to achieve the goal of becoming an engineering design and optimization tool, implemented an improved lumped-differential formulation for the energy equation in the ablative material, by making use of the so-called Coupled Integral Equations Approach (CIEA), (Cotta & Mikhailov, 1997, Correa & Cotta, 1998, Ruperti Jr. & Cotta, 2000). This approach is based on the approximation of the average temperature and heat flux by Hermite integration formulae, which takes into consideration the information on the boundaries in the averaging process related to the lumped formulation. Thus, this improved lumped approach is able of offer more accurate formulations than the classical lumped system analysis for the same problem, while maintaining the same degree of simplicity in the resultant mathematical formulation. The classical lumped system analysis was observed to be markedly conservative in the solution of such class of ablation problems, since the average ablating slab temperature needs to approximate the wall temperature evolution as well. The CIEA reformulation in the TPS Nose code then allowed for a significant improvement on such estimates of the ablating boundary movement, in particular in conjunction with the penetration depth concept (Ruperti Jr. & Cotta, 2000). Therefore, it is now of interest to critically compare this simpler lumped-differential formulation results with the present local error-controlled solution of the same ablation problem, as done in (Ruperti & Cotta, 2000). The classical lumped analysis could not certainly be recommended to the present situation with intense heating and rapid boundary recession, in light of the very high temperature gradients that are achieved, but the improved lumped approach was shown to be fairly accurate in (Ruperti Jr. & Cotta, 2000), which is now confirmed for the present example. Figure 2 then illustrates the dimensionless moving boundary behavior along the ablative period, $S(\tau)$, as promoted by the above applied wall heat flux, for both the CIEA (TPS Nose code) and GITT without and with integral balance (present) approaches. Clearly, the present GITT solution, with and without integral balance (dotted lines with symbols and dashed lines), and the CIEA solution with penetration depth (Ruperti & Cotta, 2000), have an excellent agreement, practically coincident to the graph scale.

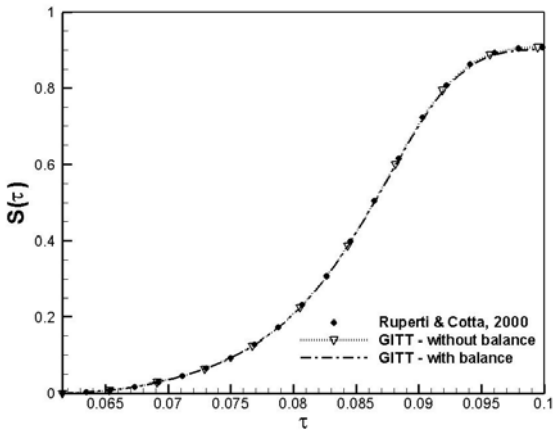

Figure 2. Comparison of the ablating boundary position as obtained with the Coupled Integral Equations Approach (Ruperti &Cotta, 2000), and with the GITT method without and with integral balance.

Figure 3 below illustrates the evolution of the temperature distributions along the thermal protection as obtained by the GITT without integral balance and N=120 terms in the eigenfunction expansions (dotted lines with symbols) and with integral balance and N=50 terms in the expansions (dashed lines). Clearly, the only noticeable deviations occur at the largest values of time, at the end of the ablation period. There, the solution without convergence enhancement requires an even larger truncation order for full agreement with the solution here proposed. Nevertheless, both solutions can reproduce the very steep behavior that occurs within the ablation period, with the very large gradients at the ablating wall.

Finally, it can be said that the proposed integral transform method with convergence enhancement has been demonstrated as an accurate, robust and cost-effective alternative for handling ablation-type problems, and should be particularly suitable for the local solution of temperature distributions as required for the thermo-mechanical analysis of thermal protection systems (Cotta *et al*., 2006). In addition, its incorporation into engineering optimization codes such as the TPS Nose code (Cotta *et al*., 2004), should offer more refined optimized configurations for TPS systems.

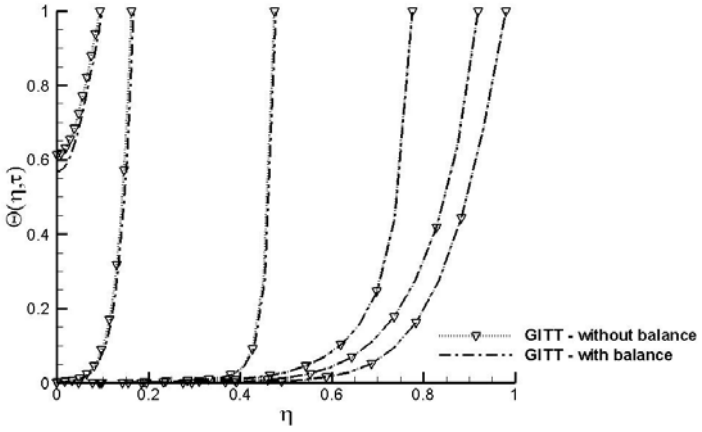

Figure 3. Comparison of the temperature distributions during ablating period as obtained with the GITT method with integral balance, N=50, and without integral balance, N=120.

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