ASYMPTOTIC SOLUTION FOR THE STATIONARY STATE OF TWO-PHASE FLOWS IN PIPELINE-RISER SYSTEMS.

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Abstract.
Intermittent flow regimes, like severe slugging, may appear in offshore petroleum production systems for low liquid and gas mass flow rates when a pipeline section with a downward inclination angle is followed by a section with upward inclination or by the riser. It would be useful to know the region of the pipeline-riser system parameter space for which the two-phase flow has a steady state. A linear stability analysis of an adequate model for the two-phase flow in pipeline-riser systems should provide a stability criteria to decide about the stability of the two-phase flow stationary states.
The non-linear governing equations for the stationary states are obtained from the model used here, and do not have closed form solution for risers of general geometry. These equations in non-dimensional form present non-dimensional numbers which are small or large for usual values of the system parameters. We take advantage of this fact to use a perturbation technique to construct asymptotic solutions for the stationary state governing equations. Analytic asymptotic approximations for the stationary sate are a necessary step to obtain a closed form expressions representing the stability criteria. The asymptotic solution for the stationary state governing equations are compared with the closed form solution available for a vertical straight riser with only gravity dominated flow.

Keywords: *two-phase flows, pipeline-riser system, hydrodynamic instability, stability analysis, perturbation technique*

1. INTRODUCTION

Under certain conditions, a steady two-phase flow with constant mass and liquid flow rates does not exists in pipeline-riser systems, and intermittent flow regimes are observed. Whenever a sub-sea pipeline with a downward section connects to a vertical riser, liquid may accumulate at the end of the downward pipeline section and stop gas motion. When this happens, the upstream gas is being compressed and its pressure rises while the liquid accumulates at both riser and pipeline. This situation continues until the gas pressure is large enough to push the liquid slug out of the pipeline and to start gas penetration into the riser. With gas penetration into the riser, the gas inventory in the pipeline decreases steadily with a corresponding rapid decrease in pipeline pressure. After a while the gas velocity in the riser becomes insufficient to support liquid on the riser wall, and liquid starts to fall back and to accumulate at the end of the downward pipeline section, blocking again the gas passage, and the repetitive cycle starts again. This cyclic phenomenon has tremendous impact in oil production systems when it happens. It causes reservoir flow oscillations, high average back pressure at the well head, high instantaneous flow rates, which are difficult to control, and eventually causes the oil production facility to shutdown.

Linear stability analysis of an adequate model for two-phase flow in pipeline-riser system allows us to determine the range of the system parameters and gas and liquid mass flow rates for which steady two-phase flow exists for a given pipeline-riser system. The linear stability analysis of a dynamic system can be split in two parts. First, we need to obtain the stationary states. Second, we need to study the stability of the stationary states under small perturbations. To study the stability of a stationary state, we write the dependent variables as their stationary state value plus a perturbation, and then substitute them into the system governing equations. We linearize the resulting equations to obtain the governing equations for the stationary state perturbations. If the solution of this set of equations grows with time, the stationary state is unstable, but if the solution decays with time, the stationary state is stable and it represents a steady state of the dynamic system.

Even for relatively simple models for two-phase flows in pipeline-riser systems, the linear stability analysis requires the solution of linear partial-differential equations with variable coefficients. The usual approach to solve this type of equations is to discretize them in space using finite difference or finite elements, reducing the partial differential equations to a system of ordinary differential equations. The time dependence can be eliminated by using Laplace transform. The system of ordinary differential equations is now reduced to a system of algebraic equations, which is solved numerically. Inverse Laplace transform of the solution of the resulting system of algebraic equations gives the time evolution of a perturbation of the stationary state, and tell us that the exponential growth rate of a perturbation of the stationary state is given by the eigenvalues of the resulting algebraic system of equations. The stationary state is stable if all the eigenvalues

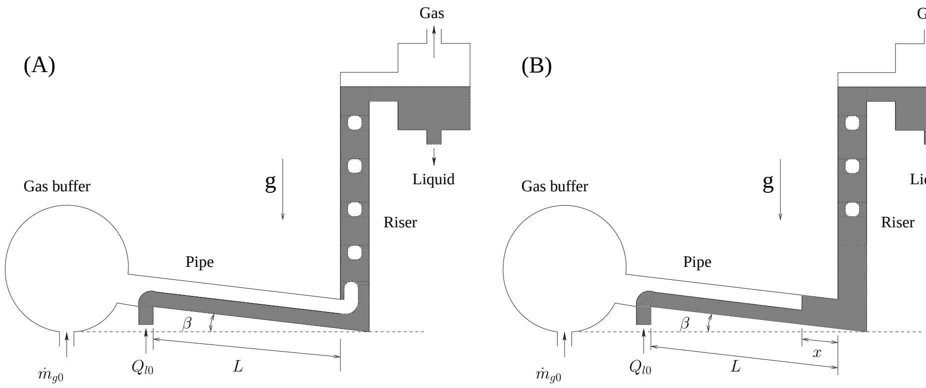


Figure 1. Part (A) - First configuration: $x = 0$. Part (B) - Second configuration: $x > 0$.

have negative real part. This approach to the linear stability analysis is numerically intensive, but can handle risers with general geometry.

There is another way to look to the linear stability analysis. The governing equations for the stationary state and perturbations have non-dimensional parameters which are small for pipeline-riser systems presented in the literature (see for example, Schmidt et al., 1980), and an asymptotic approach could be applied. We assume series expansion for the independent variables in terms of one of the small non-dimensional parameter. The governing equations for the stationary state are now reduced to a sequence of non-homogeneous linear problems which can be solved in closed form for risers of general geometry. This is not the case for the perturbations governing equations, and further simplifying assumptions need to be made to obtain any sort of analytical result.

The objective of this work is to describe a perturbation method to obtain an asymptotic approximation for the stationary states of the model for two-phase flows presented in the next section, which is the first step in the the linear stability analysis of a dynamic system.

The next section describes the one-dimensional model used to analyse two-phase flows in pipeline-riser systems. In the third section, we present the first step in the linear stability analysis for the two-phase flow model discussed in the previous section. We derive the governing equations for the stationary states and describe the perturbation approach to obtain an asymptotic solution for the stationary state which is valid for risers of general geometry. We also obtain the closed form solution of the stationary state governing equations for a straight vertical riser under the assumption of no friction. In the fourth section, we compare the closed form solution and the asymptotic solution for the stationary state for a straight vertical riser under the assumption of no friction. We consider the two-phase flow in the straight vertical riser given in Schmidt et al. (1980). Section five has a discussion and conclusion about the work done here.

2. Two-Phase Flow Model.

The pipeline-riser system is composed basically of two parts. The pipeline plus a gas buffer and the riser (see Fig. 1). The pipeline and riser are connected at the bottom of the riser. The pressure at the top of the riser is assumed to be the atmospheric pressure and we have liquid and gas mass flowing into the pipeline.

The gas-liquid flow in the pipeline is assumed as always stratified. This flow behavior extends either to the whole pipeline (see part (A) of Fig. 1) or it extends until the liquid penetration position in the pipeline (see part (B) of Fig. 1). The configuration illustrated in part (A) of Fig. 1 corresponds to continuous gas flow from the pipeline into the riser and the configuration illustrated in part (B) of Fig. 1 corresponds to no gas flow from the pipeline into the riser and partial liquid flooding of the pipeline. Variables Q_{l0}, $\dot{m}_{g0}$, β, L, g and x illustrated in Fig. 1 represents, respectively, the volumetric flow rate of liquid into the pipeline, the gas mass flow rate into the pipeline, the pipeline inclination angle, the distance of the liquid inlet from the bottom of the riser, the gravity acceleration constant and the pipeline liquid flooding distance from the bottom of the riser (parts (B) of Fig. 1).

We consider an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser.

In summary, we consider a set of two different configurations. The first one is illustrated in part (A) of Fig. 1. In this configuration we have stratified flow in the pipeline and continuous gas penetration from the pipeline into the riser. The second configuration is illustrated in part (B) of Fig. 1, where we have stratified flow in part of the pipeline with liquid flooding until a distance x from the bottom of the riser.

The set of governing equations is not the same for the two different configurations represented in Fig. 1. Below we give governing equations for the different configurations illustrated in Fig. 1.

2.1 Governing Equations for the Two-Phase Flow.

We give the governing equations for the two-phase flow in pipeline-riser system in non-dimensional form. We define the following non-dimensional variables according to the set of equations below.

$$x^* = \frac{x}{L_r}, \quad (1) \qquad P^* = \frac{P}{\rho_l R_g T_g}, \quad (3) \qquad t^* = t\frac{Q_{l0}}{AL_r}, \quad (5)$$

$$s^* = \frac{s}{L_r}, \quad (2) \qquad j^* = j\frac{A}{Q_{l0}}, \quad (4) \qquad \dot{m}^* = \frac{\dot{m}}{\rho_l Q_{l0}}, \quad (6)$$

where L_r is the riser length, A is the cross-sectional area of the pipeline and riser, s is the space parameterization along the riser length, T_g is the absolute temperature of the gas, ρ_l is the liquid phase density, R_g is the gas constant, j stands for superficial velocity, $\dot{m}$ stands for mass flow rate, P stands for pressure and t stands for time. The variables with * as a superscript are non-dimensional variables.

2.1.1 Pipeline Governing Equations.

We first give the non-dimensional governing equation for the pipeline. We consider the gas in the pipeline behaving as a pressure cavity at non-dimensional pressure P_g^*, constant in position and evolving isothermically as a perfect gas. We consider a fixed control volume with the pipeline and gas buffer contours as the control volume surface. For this control volume, we obtain the mass conservation equation for each of the two phases. We have to consider two different situation at the pipeline. We have either continuous gas penetration from the pipeline into the riser ($x^* = 0$, see part (A) of Fig. 1) or partial liquid flooding of the pipeline ($x^* > 0$, see part (B) of Fig. 1).

Below follows the governing equations for the pipeline for the conditions $x^* > 0$ and $x^* = 0$. We start with the equations for the case where $x^* > 0$. The mass conservation equation for the liquid phase is

$$-(\delta - x^*)\frac{d\alpha_p}{dt^*} + \alpha_p\frac{dx^*}{dt^*} + j_{lb}^* - 1 = 0, \tag{7}$$

where $\delta = L/L_r$. α_p is the pipeline void fraction and j_{lb}^* is the non-dimensional liquid phase superficial velocity at the bottom of the riser. The mass conservation equation for the gas phase is

$$[(\delta - x^*)\alpha_p + \delta_b]\frac{dP_g^*}{dt^*} + P_g^*(\delta - x^*)\frac{d\alpha_p}{dt^*} - \alpha_p P_g^*\frac{dx^*}{dt^*} - \dot{m}_{g0}^* = 0, \tag{8}$$

where we used the perfect gas relation $P_g = \rho_g R_g T_g$. $\delta_b = V_b/(AL_r)$ is the non-dimensional length equivalent to the gas buffer volume V_b divided by the product of the pipeline cross sectional area A by the riser length. We consider variations of pressure in the pipeline only due to hydrostatic effects. Then, the momentum equation is

$$P_g^* = P_b^* + \Pi_L x^* \sin(\beta), \tag{9}$$

where P_b^* is the non-dimensional pressure at the bottom of the riser and the non-dimensional number Π_L is given by the equation

$$\Pi_L = \frac{gL_r}{R_g T_g}. \tag{10}$$

This non-dimensional number is the ratio between the hydrostatic pressure at the bottom of the riser when it is filled completely with liquid and the gas pressure times the ratio between the gas and liquid densities.

We can eliminate the gas non-dimensional pressure P_g^* in favor of the riser bottom non-dimensional pressure P_b^*, by using the equation (9). Then the liquid phase mass conservation equation is not affected, but the gas phase mass conservation equation assumes the form

$$[(\delta - x^*)\alpha_p + \delta_b]\left(\frac{dP_b^*}{dt^*} + \Pi_L\sin(\beta)\frac{dx^*}{dt^*}\right) + (P_b^* + \Pi_L x^*\sin(\beta))\left[(\delta - x^*)\frac{d\alpha_p}{dt^*} - \alpha_p\frac{dx^*}{dt^*}\right] - \dot{m}_{g0}^* = 0. \tag{11}$$

Next, we present the equations for the case $x^* = 0$. The liquid phase mass conservation equation is

$$-\delta\frac{d\alpha_p}{dt^*} + j_{lb}^* - 1 = 0. \tag{12}$$

Notice that in this case, the gas non-dimensional pressure P_g^* is equal to the non-dimensional pressure at the bottom of the riser. Then, we use the riser bottom non-dimensional pressure P_b^* instead of the gas non-dimensional pressure P_g^* in the gas phase mass conservation equation, which is

$$(\delta\alpha_p + \delta_b)\frac{dP_b^*}{dt^*} + \delta P_b^* \frac{d\alpha_p}{dt^*} + P_b^* j_{gb}^* - \dot{m}_{g0}^* = 0, \tag{13}$$

where j_{gb}^* is the gas non-dimensional superficial velocity at the bottom of the riser.

To close the model for the pipeline, we use an implicit algebraic relation for the pipeline void fraction α_p which relates it with the non-dimensional gas superficial velocity at the bottom of the riser j_{gb}^*, with the non-dimensional liquid superficial velocity at the bottom of the riser j_{lb}^* and with the non-dimensional gas pressure P_g^*, and is derived from local momentum equilibrium for each phase of a stratified flow in a pipeline (Yemada and Dukler 1976, Kokal and Stanislav 1989 and others). For the case $x^* = 0$ we write

$$A_p(\alpha_p, j_{lb}^*, j_{gb}^*, P_b^*) = 0, \tag{14}$$

since in this case $P_b^* = P_g^*$. For the condition $x^* > 0$ we write the algebraic relation for α_p as

$$A_p(\alpha_p, j_{lb}^*, x^*, P_b^*, \frac{dx^*}{dt^*}) = 0. \tag{15}$$

To derive these algebraic relations we assume stratified flow in the pipeline. We consider local momentum equilibrium for each phase and assume that the pressure gradient is the same for both phases. Then we eliminate the pressure gradient and end up with an algebraic relation among the quantities mentioned in the above paragraph. This procedure leads to an algebraic relation similar to Eq. (3) of Yemada and Dukler (1976).

2.1.2 Equations for the Riser.

For the riser, non-dimensional equations are derived from an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser. The mass conservation equation for the liquid phase is

$$-\frac{\partial\alpha_r}{\partial t^*} + \frac{\partial j_l^*}{\partial s^*} = 0, \tag{16}$$

where j_l^* is the non-dimensional liquid superficial velocity along the riser and α_r is the void fraction along the riser. The mass conservation equation for the gas phase is

$$\frac{\partial}{\partial t^*}(P^*\alpha_r) + \frac{\partial}{\partial s^*}(P^* j_g^*) = 0, \tag{17}$$

where P^* and j_g^* are, respectively, the non-dimensional pressure and the non-dimensional gas superficial velocity along the riser.

We assume the inertia forces small and neglect them. We consider pressure variation due to the hydrostatic force and friction. The shear stress at the riser wall was modeled using a homogeneous two-phase flow model (Kokal & Stanislav 1989) for the fluid and a Fanning friction coefficient f_m. Then, the linear momentum equation is

$$\frac{\partial P^*}{\partial s^*} = -\Pi_L[1 - \alpha_r + P^*\alpha_r]\left(\sin(\theta(s^*)) + \frac{4}{\Pi_D} f_m j^* |j^*|\right), \tag{18}$$

where $\theta(s^*)$ is the local riser inclination angle at position s^* along the riser arc length, j^* is the sum of the liquid and gas superficial velocities and $f_m = f_m(R_{e,m}, \epsilon_r/D)$. The quantity ϵ_r represent the riser wall roughness, D represents the riser diameter and $R_{e,m}$ is the liquid-gas mixture Reynolds number given by

$$R_{e,m} = \frac{Q_{l0}D}{A\nu_l}\frac{(1 - \alpha_r + P\alpha_r)|j^*|}{1 - \alpha_r + \delta_\mu\alpha_r}, \tag{19}$$

where δ_μ is the ratio between the gas and liquid dynamic viscosities. The non-dimensional number Π_L is already defined by Eq. (10) and the non-dimensional number Π_D is defined as

$$\Pi_D = \frac{2gDA^2}{Q_{l0}^2}. \tag{20}$$

We consider the constitutive law corresponding to the drift flux model (Zuber and Findlay 1965) to relate the void fraction along the riser with the local values of the gas and liquid non-dimensional superficial velocities. At the bottom of the riser we have the relation

$$j_g^* = \alpha_r[C_d(j_l^* + j_g^*) + U_d^*]. \tag{21}$$

For the drift flux coefficients C_d and U_d^* we use the following correlation based on experimental data (Bendiksen 1984)

$$C_d = \begin{cases} 1,05 + 0,15\sin(\theta(s^*)) & \text{for } |j^*| < 3,5\frac{\sqrt{gDA}}{Q_{l0}} \\ 1,2 & \text{for } |j^*| \geq 3,5\frac{\sqrt{gDA}}{Q_{l0}} \end{cases} \tag{22}$$

$$U_d^* = \begin{cases} \frac{\sqrt{gDA}}{Q_{l0}}(0,35\sin(\theta(s^*)) + 0,54\cos(\theta(s^*))) & \text{for } |j^*| < 3,5\frac{\sqrt{gDA}}{Q_{l0}} \\ 0,35\frac{\sqrt{gDA}}{Q_{l0}}\sin(\theta(s^*)) & \text{for } |j^*| \geq 3,5\frac{\sqrt{gDA}}{Q_{l0}} \end{cases} \tag{23}$$

Not all equations above are valid for the two configurations defined previously and illustrated in Fig. 1. In Tab 1 we define which equations are the governing equations for each configuration and which dependent variables are used.

Table 1. Necessary governing equations and variables for each configuration defined in Fig. 1.

Configuration	Governing Equations	Dependent variables
1st	(12)-(14), (16)-(23)	$\alpha_p, j_{lb}^*, j_{gb}^*, P_b^*, \alpha_r, j_l^*, j_g^*, P^*$
2nd	(7), (11), (15), (16)-(23)	$\alpha_p, x^*, j_{lb}^*, j_{gb}^* = 0, P_b^*, j_l^*, j_g^*, P^*$

Next, we have to describe when we switch from one configuration to another, or from one set of equations to another. Table 2 illustrate the conditions characterizing each configuration and the conditions to switch from the current configuration.

Table 2. Characterization and switching conditions between configurations and correspondent set of equations.

Configuration	Characterized by	Switch from when
1st	$x^* = 0, j_{gb}^* \neq 0$	$j_{gb}^* \to 0, x^* > 0$
2nd	$x^* > 0, j_{gb}^* = 0$	$j_{gb}^* > 0, x^* \to 0$

The boundary conditions are the pressure P_t at the top of the riser which is the atmospheric pressure, the gas mass flow rate $\dot{m}_{go}$ and the liquid volumetric flow rate Q_{l0} (see Fig. 1 for details). The boundary condition at the top of the riser in non-dimensional form is $P_t^* = P_t/(\rho_l R_g T_g)$.

Since we are working only with non-dimensional variables, and for simplicity, from now on we omit the superscript * from the equations.

3. Linear Stability Analysis.

The description of the model for two-phase flows in pipeline-riser systems is complete. The objectives of the linear stability analysis of a dynamic system are to find the stationary states of the system and to determine if the stationary states are stable or unstable under small perturbations. Here we just consider the first step of the linear stability analysis. We discuss how to obtain the stationary states of the model for two-phase flows presented above. We give the governing equations for the stationary states and obtain a closed form solution for the case of a vertical straight riser under the assumption of no friction. For risers of general geometry, we obtain an asymptotic closed form solution for the stationary state. The stability analysis of the stationary states is left as future work.

3.1 Stationary State.

Only the first configuration ($x = 0$ for the pipeline) of the model for two-phase flows in pipeline-riser systems presented in the previous section has a stationary state. The equations for the stationary state are given by the Eqs.

(12)-(14) and by the riser governing equations with the time partial derivatives set to zero ($\partial/\partial t = 0$). Liquid mass conservation equation for the pipeline ($x = 0$) reduces to

$$j_{lb} = 1. \tag{24}$$

Gas mass conservation equation for the pipeline ($x = 0$) reduces to

$$P_b j_g = \dot{m}_{g0}, \tag{25}$$

and the pipeline void fraction α_p is given by Eq. (14). Liquid mass conservation for the riser reduces to

$$\frac{\partial j_l}{\partial s} = 0 \rightarrow j_l = 1, \tag{26}$$

since $j_l(s = 0) = j_{lb} = 1$ (continuity condition between pipeline and riser bottom ($s = 0$) variables) according to Eq. (24). Gas mass conservation equation for the riser reduces to

$$\frac{\partial}{\partial s}(P j_g) = 0 \rightarrow P j_g = \dot{m}_{g0}, \tag{27}$$

since $P(s = 0)j_g(s = 0) = P_b j_{gb} = \dot{m}_{g0}$ (continuity between pipeline and riser bottom ($s = 0$) variables) according to Eq. (25). The linear momentum equation for the riser used to obtain the stationary state is Eq. (18). The constitutive law corresponding to the drift flux model used to determine the stationary state is given by Eq. (21).

The main difficulty to solve the governing equations for the stationary state results from Eq. (18) which is non-linear. For general riser geometries and no further simplifying assumptions, this set of equations has no closed form solution, but for a straight vertical riser ($\theta(s) = \pi/2$) under the assumption of no friction ($f_m = 0$), the governing equations for the stationary state have closed form solution.

3.1.1 Vertical Straight Riser Under no Friction.

For a straight vertical riser ($\theta(s) = \pi/2$), under no friction ($f_m = 0$), we can integrate the governing equations for the stationary state along the riser to obtain the pressure along the riser as an implicit function of the riser length parameterization, given by the equation

$$\frac{C_{11}}{C_{21}}(P - P_t) + \frac{C_{12}C_{21} - C_{22}C_{11}}{C_{21}^2} \ln\left(\frac{C_{21}P + C_{22}}{C_{21}P_t + C_{22}}\right) = -\Pi_L(s - 1), \tag{28}$$

where the constants $C_{ij}, i, j = 1, 2$ are defined in terms of $\dot{m}_{g0}$ and in terms of the drift flux coefficients C_d and U_d according to

$$C_{11} = C_d + U_d, \tag{29}$$
$$C_{12} = \dot{m}_{g0} C_d, \tag{30}$$
$$C_{21} = C_d + U_d^* + \dot{m}_{g0} = C_{11} + \dot{m}_{g0}, \tag{31}$$
$$C_{22} = \dot{m}_{g0}(C_d - 1) = C_{12} + \dot{m}_{g0}. \tag{32}$$

The liquid superficial velocity j_l is already given by Eq. (24) and the gas superficial velocity j_g along the riser is given by

$$j_g = \frac{\dot{m}_{g0}}{P}, \tag{33}$$

where the pressure P for a given value of s is given implicitly by Eq. (28). The void fraction along the riser is obtained by using Eqs. (33) and (21). We have that

$$\alpha_r = \frac{\frac{\dot{m}_{g0}}{P}}{C_d\left[1 + \frac{\dot{m}_{g0}}{P}\right] + U_d^*}. \tag{34}$$

We were able to integrate the governing equations for the steady state in the case of a straight vertical riser because the drift flux coefficients C_d and U_d in this case are constants, as we can see from Eqs. (22) and (23).

4. Asymptotic Approximation for the Stationary State.

For risers of more general geometry, closed form solutions are not possible, but we may obtain asymptotic closed form solutions of the governing equations for the stationary state. We assume the non-dimensional number $\Pi_L << 1$, and use a perturbation approach to solve the set of Eqs. (24)-(27), (18) and (21) asymptotically. The assumption regarding the non-dimensional number Π_L is true for experiments with two-phase flow in pipeline-riser systems reported in the literature (see Schmidt et al 1980, for example). To apply the perturbation approach mentioned above, we set $\Pi_L = \epsilon$ in Eq. (18) and assume series expansion in the parameter ϵ for the dependent variables j_g, α_r and P. For example, we write for $j_g(s)$ the series expansion

$$j_g(s) = \sum_{n=0}^{\infty} \epsilon^n j_{g,n}(s). \tag{35}$$

The Fanning friction coefficient f_m in Eq. (18) is a function of the local Reynolds number $R_{e,m}$, given by Eq. (19), which is for a given value of s function of the variables $\alpha_r(s), j(s)$ and $P(s)$. Therefore, if we substitute the series expansion in ϵ for these variables in Eq. (19), we obtain a series expansion for the Reynolds number $R_{e,m}$ in terms of the elements of the series expansion for the dependent variables j_g, α_r and P. This will imply also in a series expansion for the Fanning coefficient in terms of the parameter ϵ. To avoid too complicate expressions, we write for the Reynolds number $R_{e,m}$ the series expansion

$$R_{e,m} = \sum_{n=0}^{\infty} \epsilon^n R_{e,m,n}, \tag{36}$$

where the coefficients $R_{e,m,n}$ are obtained by substituting the series expansions in ϵ for the dependent variables j_g, α_r and P in Eq. (19) and expanding the resulting expression in terms of ϵ. This can be easily accomplished using software for symbolic computation, like MAPLE or MATHEMATICA. The expansion for the Fanning coefficient f_m in terms of the parameter ϵ is

$$f_m(R_{e,m}, \epsilon_r/D) = f_m(R_{e,m,0}, \epsilon_r/D) + \sum_{n=1}^{\infty} \epsilon^n \left\{ \sum_{l=1}^{n} \frac{1}{l!} D^l f_m(R_{e,m,0}, \epsilon_r/D) \sum_{j_1+\ldots+j_l=n} R_{e,m,j_1} \ldots R_{e,m,j_l} \right\}, \tag{37}$$

where $D^l f_m$ stands for the l-th order derivative of the expression for f_m with respect to the Reynolds number $R_{e,m}$. Here we assume that this derivatives exists, which is the case for the expression for f_m to be considered and given in Chen (1979). The term $j|j|$ appearing in Eq. (18) has expansion in terms of the parameter ϵ given by

$$j|j| = (1 + j_g)|1 + j_g| = (1 + j_{g,0})|1 + j_{g,0}| + \sum_{n=1}^{\infty} \epsilon^n \left\{ \sum_{l=1}^{n} \frac{1}{l!} D^l g(j_{g,0}) \sum_{j_1+\ldots+j_l=n} j_{g,j_1} \cdots j_{g,j_l} \right\}, \tag{38}$$

where $g(x) = (1 + x)|1 + x|$, $Dg(x) = 2 + 2x$ for $x \geq -1$, $Dg(x) = -2 - 2x$ for $x < -1$, $D^2 g(x) = 2$ for $x > -1$, $D^2 g(x) = -2$ for $x < -1$ and $D^n g(x) = 0$ for $n > 2$. Next, we substitute the expansions for the variables $j_g(s), \alpha_r(s)$ and $P(s)$ in terms of the parameter ϵ into the Eqs. (26), (18) and (21), and we take into account Eqs. (37) and (38). Then, we collect terms of the same order in ϵ and obtain a set of linear algebraic-differential equations. For the problem of $O(\epsilon^n)$, we first integrate the equation which comes from Eq. (18) to obtain $P_n(s)$. Next, we obtain $j_{g,n}(s)$ from the algebraic equation which comes from Eq. (26), and finally we obtain $\alpha_{r,n}(s)$ from the algebraic equation which comes from Eq. (21). $P_n(s), j_{g,n}(s)$ and $\alpha_{r,n}(s)$ are obtained in terms of the variables $P_l(s), j_{g,l}(s), \alpha_{r,l}(s)$ and $R_{e,m,l}(s)$ for $l = 0, 1, \ldots, n-1$. The solution of the problem of $O(1)$ is given by

$$P_0(s) = P_t(\text{constant}), \tag{39}$$

$$j_{g,0}(s) = \frac{\dot{m}_{g0}}{P_t}, \tag{40}$$

$$\alpha_{r,0}(s) = \frac{\frac{\dot{m}_{g0}}{P_t}}{C_d(s)\left[1 + \frac{\dot{m}_{g0}}{P_t}\right] + U_d(s)}. \tag{41}$$

where P_t is the non-dimensional pressure at the top of the riser, which is one of the boundary conditions. The solution of the problem of $O(\epsilon)$ is given by

$$P_1(s) = -\int_s^1 (1-\alpha_0(z)+P_0(z)\alpha_0(z))\left(\sin(\theta(z)) - \frac{2}{\Pi_d} f_m(R_{e,m,0})(1+j_{g,0}(z))|1+j_{g,0}(z)|\right) dz, \tag{42}$$

$$j_{g,1}(s) = \frac{\dot{m}_{g0}}{P_1(s)}, \tag{43}$$

$$\alpha_{r,1}(s) = \frac{j_{g,1}(s)}{C_d(s)\,[1+j_{g,1}(s)] + U_d(s)}. \tag{44}$$

The solution of the problem of $O(\epsilon^{n+1})$ is given by

$$\begin{aligned} P_{n+1}(s) = &-\int_s^1 \Bigg\{ (-\alpha_{r,n}(z) + \sum_{n_1=0}^{n} P_{n_1}(z)\alpha_{r,n-n_1}(z))\sin(\theta(z)) - \frac{4}{\Pi_d}\sum_{n_1=0}^{n}\Bigg[\Bigg(\sum_{l=1}^{n-n_1}\frac{1}{l!}D^l g(j_{g,0}) \\ &\sum_{j_1+\ldots+j_l=n-n_1} j_{g,j_1}\ldots j_{g,j_l}\Bigg)\Bigg[\sum_{n_2=0}^{n_1}\Bigg(\sum_{l=1}^{n_1-n_2}\frac{1}{l!}D^l f_m(R_{e,m,0},\frac{\epsilon_r}{D})\sum_{j_1+\ldots+j_l=n_1-n_2} R_{e,m,j_1}\ldots R_{e,m,j_l}\Bigg)\alpha_{n_2}\Bigg]\Bigg] \\ &+\frac{4}{\Pi_d}\sum_{n_1=0}^{n}\Bigg[\Bigg(\sum_{l=1}^{n-n_1}\frac{1}{l!}D^l g(j_{g,0})\sum_{k_1+\ldots+k_l=n-n_1} j_{g,k_1}\cdots j_{g,k_l}\Bigg)\Bigg[\sum_{n_2=0}^{n_1}\Bigg(\sum_{l=1}^{n_1-n_2}\frac{1}{l!}D^l f_m(R_{e,m,0},\frac{\epsilon_r}{D}) \\ &\sum_{k_1+\ldots+k_l=n_1-n_2} R_{e,m,k_1}\ldots R_{e,m,k_l}\Bigg)\sum_{n_3=0}^{n_2}\alpha_{r,n_2-n_3}(z)P_{n_3}(z)\Bigg]\Bigg]\Bigg\} dz \end{aligned} \tag{45}$$

$$j_{g,n+1}(s) = \frac{\dot{m}_{g0}}{P_{n+1}(s)} \tag{46}$$

$$\alpha_{r,n+1}(s) = \frac{j_{g,n+1}(s)}{C_d(s)\,[1+j_{g,n+1}(s)] + U_d(s)}, \tag{47}$$

for $n \geq 1$.

5. Results.

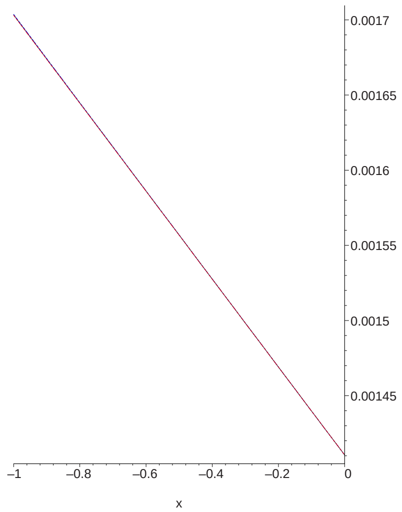
We compare the asymptotic solution obtained in the previous section with the closed form solution for a vertical straight riser under the assumption of no friction. We use pipeline-riser system parameters used in the experiment described in Schmidt et al. (1980). The parameters and their values are

$$\begin{array}{llll} P_t = 10^5\ Pa, & D = 0.0508\ m, & T = 293.15\ K, & Q_{l0}/A = 0.127\ m/sec \\ L = 30,48\ m, & \theta = 90^o, & R = 287.336\ m^2 sec^{-2} K^{-1}, & \\ L_r = 15,24\ m, & \beta = 5^o, & \rho_l = 824.952\ kg.m^{-3}. & \end{array}$$

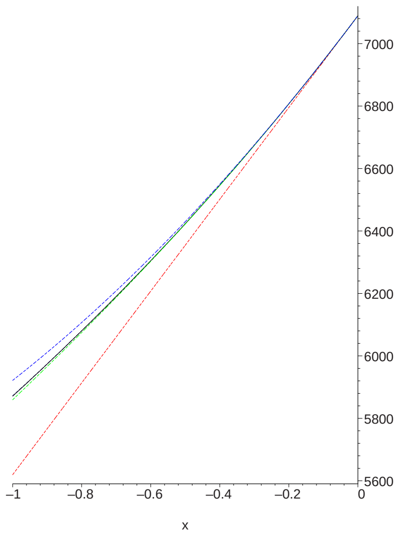
For these values of parameters, the non-dimensional number $\Pi_L = 0.00174$. It satisfies the necessary assumption to apply the perturbation approach described in the previous section. We give in Fig. 2 graphs of the asymptotic solutions for $P(s), j_g(s)$ and $\alpha_r(s)$ up to the order $O(\epsilon^4)$ for the vertical straight riser used in the experiments reported in Schmidt et al. (1980) under the assumption of no friction. This special case has closed form solution given in section 3.1.1 and is used here to assess the performance of the asymptotic approach. We consider two values of the non-dimensional parameter $\dot{m}_{g0}$, which represents the ratio between the gas and liquid mass flow rates. According to Fig. 2, the agreement between the asymptotic solution and the closed form solution is very good and increases for larger values of $\dot{m}_{g0}$.

6. Discussion and Conclusions.

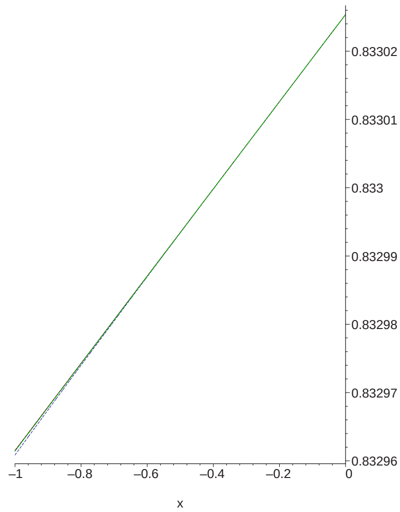
The nice agreement between the closed form solution and the asymptotic solution for the case of a straight vertical riser suggest that the asymptotic approach provide a good approximation for the stationary state. Under the perspective of the linear stability analysis, the asymptotic explicit solution is more useful than the implicit closed form solution since the coefficients of the equations for the perturbations of the stationary state are polynomials in s instead of implicit functions of s, what may allow us to obtain an expression in terms of the system parameters as a stability criteria for a vertical straight riser.



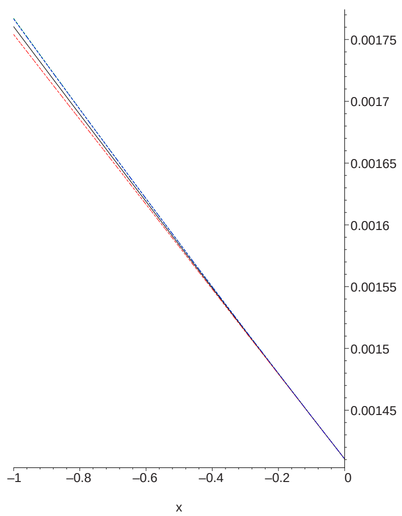
(a) Pressure - $\dot{m}_{g0} = 10$



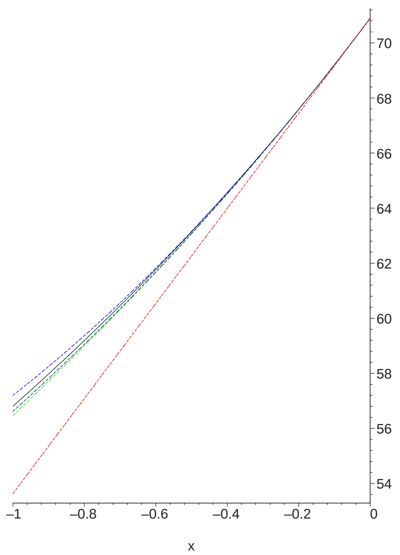
(b) Gas superficial velocity - $\dot{m}_{g0} = 10$



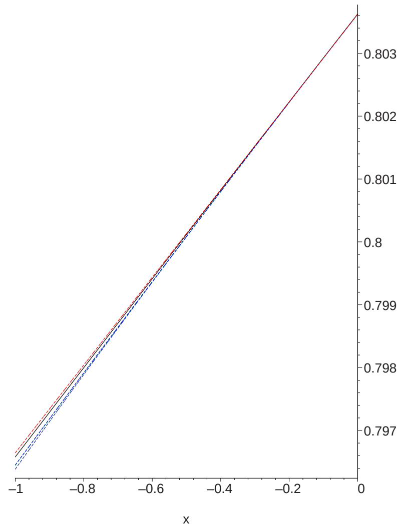
(c) Void fraction - $\dot{m}_{g0} = 10$



(d) Pressure - $\dot{m}_{g0} = 0.1$



(e) Gas superficial velocity - $\dot{m}_{g0} = 0.1$



(f) Void fraction - $\dot{m}_{g0} = 0.1$

Figure 2. Horizontal axis: $x = s - 1$, and vertical axis: dependent variable. Solid line - closed form solution given in section 3.1.1 Dashed lines - asymptotic solution. Red line - asymptotic solution up to $O(\epsilon)$, Blue line - asymptotic solution up to $O(\epsilon^2)$, Green line - asymptotic solution up to $O(\epsilon^3)$, Navy Blue line - asymptotic solution up to $O(\epsilon^4)$.

Since our long term goal is to obtain a stability criteria for a pipeline-riser system, we need to construct assymptotic approximations for the stationary state for the system parameters values near the region where the stationary state loses stability. This occurs for small values of $\dot{m}_{g0}$, and consequently, small superficial velocities. The Fanning friction coefficient f_m usualy assumes values $f_m < 0.1$, and for the experiment described in Schmidt et. al (1980), $\Pi_D = 61.7745$, what imply that $4f_m/\Pi_d \ll 1$. These facts suggest that the frictional effect are small for system parameters values near the region where the stationary state loses stability, and they may be not important and could be neglected. Without friction forces, the asymptotic solution simplifies and we may be able to integrate in closed form the integrals in Eqs. (42) and (45) for risers with more general geometry if the equation for $\sin(\theta(s))$ is simple enough.

This paper is a report on work in progress.

7. Acknowledgments

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8. References

Bendiksen, K. H., 1984 , "An Experimental Investigation of the Motion of the Long Bubbles in Inclined Tubes", International Journal of Multiphase Flow, Vol 10, pp. 467-483.

Chen, N. H., 1979 , "An explicit equation for friction factor in pipe", Ind. Engng. Fundam., vol. 18, pp. 296-297.

Kokal, S. L. and Stanislav, J. F. 1989 , "An experimental study of two-phase flow in slightly inclined pipes-II.Liquid holdup and pressure drop", Chem. Engng. Sci., vol. 44, pp 665-679.

Schmidt, Z.; Brill, J. P. & Beggs, H. D. (1980) "Experimental study of severe slugging in a two-phase-flow pipeline-riser pipe system", SPE J., OCT 1980, pp. 407-414.

Zuber, N. and Findley, J. A., 1965 "Average Volumetric Concentration in Two-phase Flow systems", Journal of Heat Transfer, Vol. 87, pp. 453-468.

Yemada, T. and Dukler, A. E., 1976 , "A model for Predicting Flow Regimes Transition in Horizontal and Near Horizontal Gas-liquid Flow", AIChE Journal, Vol. 12, pp. 47-55.

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