A SOLUTION THROUGH INTEGRAL TRANSFORMS FOR LAMINAR FLOW IN ANNULAR DUCTS WITH ROTATION

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Abstract. *A solution based on the Generalized Integral Transform Technique (GITT) is obtained for laminar flow of Newtonian fluids inside annular ducts with rotation. The mathematical formulation is constructed based on the cylindrical coordinates system within the hypothesis of validation of the boundary layer equations in the entrance region of the annular channel. Numerical results for the velocity field were produced for different values of the governing parameters, i.e., aspect ratio and Taylor numbers. The results were confronted with previously reported ones, providing critical comparisons while illustrating the employed integral transform approach.*

Keywords*: Annular ducts with rotation, Developing laminar flow, Integral transform.*

1. INTRODUCTION

Laminar flow in ducts of annular geometry involving Newtonian fluids is frequently found in several engineering applications, such as those in industrial processing plants, mainly in heat exchange devices, cooling systems and tubular heat exchangers. For these applications, it is important the knowledge of information with respect to the velocity field of the fluid flow, which may provide an adequately design or a better operation of such equipment. In dealing with the geometric configuration used in the construction of thermo-hydraulic equipments, the annular geometry is still extensively studied due to its great applicability in industries. Among these one can mention the polymer extrusion in the petrochemical industry, the deposition of paraffin in pipes during the pumping of petroleum in the petroleum industry, to name a few.

The literature survey for the flow in annular passages reveals an abundance of works, mainly due to its practical importance. Particularly, several authors have numerically studied the entrance region for the flow of a Newtonian fluid between two coaxial cylinders either with or without the rotation of the internal cylinder, and Coney and El-Shaarawi (1974, 1975) were one of the first researchers to conduct such analysis. In this same line of study, the fully developed flow and heat transfer of a non-Newtonian fluid that follows the rheological power-law model was analyzed by Batra and Sudarsan (1992) through the application of the finite element method for the solution of the governing equations. Later, these studies have again motivated interest in the works of Manglik and Fang (1995), Fang et al. (1999), Manglik and Fang (2002), Escudier et al. (2002) and Sayed-Ahmed and Sharaf-El-Din (2006) in which the effects of eccentricity and duct rotation were investigated for the flow and heat transfer of non-Newtonian fluids. In addition, the more recent work by Escudier et al. (2002) brings a detailed and up-to date literature review for flow and heat transfer in eccentric annular ducts involving Newtonian and non-Newtonian fluids.

In searching for solution of such problems, the methodology of the Generalized Integral Transform Technique (GITT), with its hybrid analytical-numerical nature, has been advanced as an alternative tool for benchmarking purposes and covalidation of purely numerical techniques (Cotta, 1993 and 1994). Recently, this hybrid approach has also been employed in the flow and heat transfer of Newtonian and non-Newtonian fluids in annular ducts (Pereira et al., 1998; Pereira et al., 2002; Monteiro et al., 2004).

In this context, the purpose of the present study is to solve the momentum equations for developing laminar flow in ducts with rotation of the inner wall by employing the GITT approach, and to establish reliable numerical results for the velocity field, for different values of the governing parameters. The present results are then confronted with previously reported ones, providing critical comparisons while illustrating the employed integral transform approach, and to assess the consistency of the final results, as well.

2. ANALYSIS

Laminar flow of a Newtonian fluid in an annular duct with rotation of the inner wall is considered as shown in Fig. 1. The continuity and incompressible steady Navier-Stokes equations in cylindrical coordinates are used to model the flow inside this annular geometry. The longitudinal velocity component v_z is assumed to be known at the channel entrance, and the inlet flow is assumed to be parallel ($v_r = v_\theta = 0$). Fully developed flow conditions are attained at a sufficiently large channel length, recovering the parabolic flow structure. Therefore, the governing equations within the hypothesis of the boundary layer, in the region $\gamma < r < 1$ and $z > 0$, are written in dimensionless form as:

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0; \quad \frac{v_\theta^2}{r} = \frac{1}{\xi^2}\frac{\partial p}{\partial r} \tag{1,2}$$

$$v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]; \quad v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) \tag{3,4}$$

Equations (1) to (4) are subjected to the following inlet and boundary conditions:

$$v_z(r,0) = 1; \quad v_\theta(r,0) = 0 \tag{5a,b}$$

$$v_r(\gamma,z) = v_z(\gamma,z) = 0; \quad v_\theta(\gamma,z) = 1 \tag{5c-e}$$

$$v_r(1,z) = v_z(1,z) = v_\theta(1,z) = 0 \tag{5f-h}$$

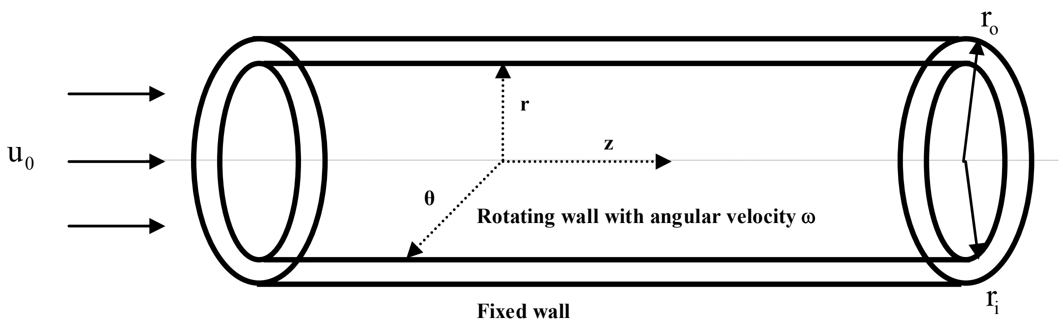

Figure 1. Geometry and coordinate system for developing annular flow with rotation of the inner wall.

The dimensionless groups employed in the equations above are defined as

$$r = r^*/r_0;\ v_z = v_z^*/u_0;\ v_r = v_r^* r_0/\nu;\ v_\theta = v_\theta^*/(\omega r_i);\ p = (p^* - p_0)/(\rho u_0^2);\ Re = D_h u_0/\nu;$$

$$z = 2z^*(1-\gamma)/(r_0\, Re);\ Ta = 2\omega^2 r_i^2 (r_0 - r_i)^3/[\nu^2 (r_0 + r_i)];\ \xi^2 = 2(1+\gamma)Ta/[(1-\gamma)Re^2] \tag{6}$$

where $D_h = 2r_0(1-\gamma)$ is the hydraulic diameter and γ is the aspect ratio.

Now, in order to express Eqs. (1) to (5) in the streamfunction formulation, which automatically satisfies the continuity equation and eliminates the pressure field. Therefore, the streamfunction is defined in terms of the dimensionless velocity components in the radial (r) and longitudinal (z) directions, respectively, as

$$v_r(r,z) = \frac{1}{r}\frac{\partial \psi(r,z)}{\partial z}; \quad v_z(r,z) = -\frac{1}{r}\frac{\partial \psi(r,z)}{\partial r} \tag{7,8}$$

Taking the derivative of Eq. (2) with relation to the z variable and the derivative of Eq. (4) with relation to the r variable, the results are subtracted, after that the definitions given by Eqs. (7,8) are introduced in such result as well as in Eq. (3), to yield

$$\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r^2}\frac{\partial \psi}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial v_\theta}{\partial z} = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \tag{9}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial z}\left(\frac{\partial^3 \psi}{\partial r^3} - \frac{3}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r^2}\frac{\partial \psi}{\partial r}\right) - \frac{1}{r}\frac{\partial \psi}{\partial r}\left(\frac{\partial^2 \psi}{\partial r^2 \partial z} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial z}\right) - 2\xi^2 v_\theta \frac{\partial v_\theta}{\partial z} = \frac{\partial^4 \psi}{\partial r^4} - \frac{2}{r}\frac{\partial^3 \psi}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2 \psi}{\partial r^2} - \frac{3}{r^3}\frac{\partial \psi}{\partial r} \tag{10}$$

Equations (9) and (10) require the specification of boundary conditions expressed in terms of the streamfunction, which are written as:

$$\psi(r,0) = C_1 - \frac{(r^2-\gamma^2)}{2}\ ; \quad v_\theta(r,0) = 0 \tag{11a,b}$$

$$\psi(r,\gamma) = C_1\ ; \quad \frac{\partial\psi(\gamma,z)}{\partial r} = 0\ ; \quad v_\theta(\gamma,z) = 1 \tag{11c-e}$$

$$\psi(1,z) = C_2\ ; \quad \frac{\partial\psi(1,z)}{\partial r} = 0\ ; \quad v_\theta(1,z) = 0 \tag{11f-h}$$

Constants C_1 and C_2 specify the streamfunction values at the duct walls, and are related by using the boundary conditions above as $C_2 = C_1 - (1-\gamma^2)/2$. One may arbitrarily specify $C_1 = 0$, so that $C_2 = -(1-\gamma^2)/2$.

Equations (9) and (10) and boundary conditions (11) complete the problem formulation in terms of the streamfunction and the tangential velocity component. Following the ideas in the generalized integral transform technique (Cotta, 1993; Cotta, 1994; Cotta, 1998; Santos et al., 2001; Cotta and Mikhailov, 2006), for improved computational performance, it is convenient to define filters that reproduce the fully developed flow solution in order to homogenize the boundary conditions in the r direction, which later will be the coordinate chosen for the specification of the eigenvalue problem. Therefore, the filters are written as

$$v_\theta(r,z) = v_{\theta,f}(r) + v_{\theta,F}(r,z)\ ; \quad v_{\theta,f}(r) = \frac{\gamma}{(1-\gamma^2)}\left(\frac{1}{r} - r\right) \tag{12a,b}$$

$$\psi(r,z) = \psi_\infty(r) + \phi(r,z)\ ; \quad \psi_\infty(r) = -\frac{2}{\beta}\left[r_m^2\left(r^2\ln r - \gamma^2\ln\gamma\right) - \frac{(r^2-\gamma^2)}{4}\left(2r_m^2 - 2 + r^2 + \gamma^2\right)\right] \tag{13a,b}$$

$$\beta = 1 + \gamma^2 - 2r_m^2\ ; \quad r_m^2 = \frac{(1-\gamma^2)}{2\ln(1/\gamma)} \tag{13c,d}$$

where the filters $v_{\theta,f}(r)$ and $\psi_\infty(r) \equiv \psi(\infty,r)$ represent the tangential velocity for the situation of pure rotation flow and the streamfunction in the fully developed region, respectively, and $v_{\theta,F}(r,z)$ and $\phi(r,z)$ are now the filtered potentials to be solved for. The resulting problem formulation is then given by:

$$\frac{1}{r}\frac{\partial\phi}{\partial z}\left(\frac{\partial^3\phi}{\partial r^3} - \frac{3}{r}\frac{\partial^2\phi}{\partial r^2} + \frac{3}{r^2}\frac{\partial\phi}{\partial r} + \frac{d^3\psi_\infty}{dr^3} - \frac{3}{r}\frac{d^2\psi_\infty}{dr^2} + \frac{3}{r^2}\frac{d\psi_\infty}{dr}\right) - \frac{1}{r}\left(\frac{d\psi_\infty}{dr} + \frac{\partial\phi}{\partial r}\right)\left(\frac{\partial^3\phi}{\partial r^2\partial z} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial z}\right)$$

$$-2\xi^2\left(v_{\theta,f} + v_{\theta,F}\right)\frac{\partial v_{\theta,F}}{\partial z} = \frac{\partial^4\phi}{\partial r^4} - \frac{2}{r}\frac{\partial^3\phi}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2\phi}{\partial r^2} - \frac{3}{r^3}\frac{\partial\phi}{\partial r} \tag{14a}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial z}\left(\frac{\partial v_{\theta,f}}{\partial r} + \frac{\partial v_{\theta,F}}{\partial r}\right) + \left(\frac{v_{\theta,f}}{r^2} + \frac{v_{\theta,F}}{r^2}\right)\frac{\partial\phi}{\partial z} - \frac{1}{r}\left(\frac{d\psi_\infty}{dr} + \frac{\partial\phi}{\partial r}\right)\frac{\partial v_{\theta,F}}{\partial z} = \frac{\partial^2 v_{\theta,F}}{\partial r^2} + \frac{1}{r}\frac{\partial v_{\theta,F}}{\partial r} - \frac{v_{\theta,F}}{r^2} \tag{14b}$$

$$\phi(r,0) = \frac{2}{\beta}\left[r_m^2\left(r^2\ln r - \gamma^2\ln\gamma\right) - \frac{(r^2-\gamma^2)}{4}\left(2r_m^2 - 2 + r^2 + \gamma^2\right)\right] - \frac{(r^2-\gamma^2)}{2}\ ; \quad v_{\theta,F}(r,0) = -\frac{\gamma}{(1-\gamma^2)}\left(\frac{1}{r} - r\right) \tag{14c,d}$$

$$\phi(\gamma,z) = 0\ ; \quad \frac{\partial\phi(\gamma,z)}{\partial r} = 0\ ; \quad v_{\theta,F}(\gamma,z) = 0 \tag{14e-g}$$

$$\phi(1,z) = 0\ ; \quad \frac{\partial\phi(1,z)}{\partial r} = 0\ ; \quad v_{\theta,F}(1,z) = 0 \tag{14h-j}$$

2.1. Solution methodology

In applying the GITT approach in the solution of the PDE system given by Eqs. (14), due to the homogeneous characteristics of the boundary conditions in the r direction, it is more appropriate to choose this direction for the process of integral transformation. Therefore, the following auxiliary eigenvalue problems are taken as (Pereira et al., 1998):

- For the potential $\phi(r,z)$:

$$\left[\frac{d^2\Omega_i(r)}{dr^2}+\frac{1}{r}\frac{d\Omega_i(r)}{dr}\right]^2-\lambda_i^4\Omega_i(r)=0 \text{ in } \gamma<r<1 \tag{15a}$$

$$\Omega_i(\gamma)=0;\quad \frac{d\Omega_i(\gamma)}{dr}=0 \tag{15b,c}$$

$$\Omega_i(1)=0;\quad \frac{d\Omega_i(1)}{dr}=0 \tag{15d,e}$$

Problem (15) is analytically solved and written on a normalized form for improved computational performance, to furnish:

$$\Omega_i(r)=A_1J_0(\lambda_i r)+A_2Y_0(\lambda_i r)+A_3\frac{I_0(\lambda_i r)}{I_0(\lambda_i)}+A_4\frac{K_0(\lambda_i r)}{K_0(\lambda_i\gamma)} \tag{16a}$$

Substituting Eq. (16a) into the boundary conditions given by Eqs. (15b-e) one leads to the following system of four algebraic equations:

$$\sum_{j=1}^{4}P_{ij}A_j=0;\quad i=1,2,3,4;\quad \mathbf{P}=\begin{bmatrix} J_0(\lambda_i\gamma) & Y_0(\lambda_i\gamma) & \dfrac{I_0(\lambda_i\gamma)}{I_0(\lambda_i)} & 1 \\ J_0(\lambda_i) & Y_0(\lambda_i) & 1 & \dfrac{K_0(\lambda_i)}{K_0(\lambda_i\gamma)} \\ -J_1(\lambda_i\gamma) & -Y_1(\lambda_i\gamma) & \dfrac{I_1(\lambda_i\gamma)}{I_0(\lambda_i)} & -\dfrac{K_1(\lambda_i\gamma)}{K_0(\lambda_i\gamma)} \\ -J_1(\lambda_i) & -Y_1(\lambda_i) & \dfrac{I_1(\lambda_i)}{I_0(\lambda_i)} & -\dfrac{K_1(\lambda_i)}{K_0(\lambda_i\gamma)} \end{bmatrix} \tag{16b,c}$$

The non-trivial solution of system (16b) requires:

$$\text{Det}(\mathbf{P})=0 \tag{16d}$$

which provides a transcendental equation for the computation of eigenvalues λ_i's. The eigenfunctions given by Eqs. (16a) are evaluated by making $A_1 = 1$ for convenience, and the remaining A_i's are calculated from system (16b), for each eigenvalue λ_i.

The eigenfunctions satisfy the following orthogonality property:

$$\int_\gamma^1 r\Omega_i(r)\Omega_j(r)dr=\begin{cases}0, & i\neq j\\ M_i, & i=j\end{cases} \tag{16e}$$

The normalization integral M_i is then computed from:

$$M_i=\int_\gamma^1 r\Omega_i^2(r)dr=[J_0(\lambda_i)+A_2Y_0(\lambda_i)]^2-\gamma^2[J_0(\lambda_i\gamma)+A_2Y_0(\lambda_i\gamma)]^2 \tag{16f}$$

- For the potential $v_{\theta,F}(r,z)$:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dX_i(r)}{dr}\right]-\left(\frac{1}{r^2}-\mu_i^2\right)X_i(r)=0 \text{ in } \gamma<r<1 \tag{17a}$$

$$X_i(\gamma)=0;\quad X_i(1)=0 \tag{17b,c}$$

Similarly, problem (17) is analytically solved, to furnish the eigenfunctions, transcendental equation to compute the eigenvalues, orthogonality property and normalization integral, respectively, as

$$X_i(r)=\frac{J_1(\mu_i r)}{J_1(\mu_i)}-\frac{Y_1(\mu_i r)}{Y_1(\mu_i)};\quad J_1(\mu_i)Y_1(\mu_i\gamma)-J_1(\mu_i\gamma)Y_1(\mu_i)=0,\quad i=1,2,3... \tag{18a,b}$$

$$\int_\gamma^1 rX_i(r)X_j(r)dr=\begin{cases}0, & i\neq j\\ N_i, & i=j\end{cases};\quad N_i=\int_0^1 rX_i^2(r)dr=\frac{2}{\pi^2}\frac{[J_1^2(\mu_i\gamma)-J_1^2(\mu_i)]}{\mu_i^2Y_1^2(\mu_i)J_1^2(\mu_i\gamma)J_1^2(\mu_i)} \tag{18c,d}$$

The eigenvalue problems defined by Eqs. (15) to (18) allow for the definition of the following integral transform pairs:

- For the streamfunction field:

$$\bar{\phi}_i(z)=\int_\gamma^1 r\tilde{\Omega}_i(r)\phi(r,z)dr\,,\qquad \text{transform} \tag{19a}$$

$$\phi(r,z)=\sum_{i=1}^{\infty}\tilde{\Omega}_i(r)\bar{\phi}_i(z)\,,\qquad \text{inverse} \tag{19b}$$

- For the tangential velocity component:

$$\bar{v}_{\theta,i}(z)=\int_\gamma^1 r\tilde{X}_i(r)v_{\theta,F}(r,z)dr\,,\qquad \text{transform} \tag{20a}$$

$$v_{\theta,F}(r,z)=\sum_{i=1}^{\infty}\tilde{X}_i(r)\bar{v}_{\theta,i}(z)\,,\qquad \text{inverse} \tag{20b}$$

where $\tilde{\Omega}_i(r)=\Omega_i(r)/M_i^{1/2}$ and $\tilde{X}_i(r)=X_i(r)/N_i^{1/2}$ are the normalized eigenfunctions.

The next step is thus to accomplish the integral transformation of the original partial differential system given by Eqs. (14). For this purpose, Eq. (14a) and the inlet condition (14c) are multiplied by $[r\Omega_i(r)]$, integrated over the domain $[\gamma,1]$ in r, and the inverse formula given by Eq. (19b) is employed. Similarly, Eq. (14b) and the inlet condition (14d) are multiplied by $[rX_i(r)]$, also integrated over the domain $[\gamma,1]$ in the r-direction, and the inverse formulae given by Eqs. (19b) and (20b) are employed. After the appropriate manipulations, the following coupled ordinary differential system results, for the calculation of the transformed potentials $\bar{\phi}_i(z)$ and $\bar{v}_{\theta,i}(z)$:

$$\sum_{j=1}^{\infty}A_{ij}\frac{d\bar{\phi}_j(z)}{dz}+\sum_{j=1}^{\infty}B_{ij}\frac{d\bar{v}_{\theta,j}(z)}{dz}=\sum_{j=1}^{\infty}C_{ij}\bar{\phi}_j(z),\quad i=1,2,3... \tag{21a}$$

$$\sum_{j=1}^{\infty}D_{ij}\frac{d\bar{\phi}_j(z)}{dz}+\sum_{j=1}^{\infty}E_{ij}\frac{d\bar{v}_{\theta,j}(z)}{dz}=\sum_{j=1}^{\infty}F_{ij}\bar{v}_{\theta,j}(z),\quad i=1,2,3... \tag{21b}$$

$$\bar{\phi}_i(0)=\bar{f}_i;\quad \bar{v}_{\theta,i}(0)=\bar{g}_i \tag{21c,d}$$

where,

$$A_{ij}=G_{ij}+\sum_{k=1}^{\infty}H_{ijk}\bar{\phi}_k(z);\ B_{ij}=I_{ij}+\sum_{k=1}^{\infty}J_{ijk}\bar{v}_{\theta,k}(z);\ C_{ij}=\int_{\gamma}^{1}\tilde{\Omega}_i(r)\left[r\tilde{\Omega}_j^{(iv)}(r)-2\tilde{\Omega}_j'''(r)+\frac{3}{r}\tilde{\Omega}_j''(r)-\frac{3}{r^2}\tilde{\Omega}_j'(r)\right]dr \quad (22a\text{-}c)$$

$$D_{ij}=K_{ij}+\sum_{k=1}^{\infty}L_{ijk}\bar{v}_{\theta,k}(z);\quad E_{ij}=M_{ij}+\sum_{k=1}^{\infty}N_{ijk}\bar{\phi}_k(z);\ F_{ij}=-\delta_{ij}\mu_i^2 \quad (22d\text{-}f)$$

$$G_{ij}=\int_{\gamma}^{1}\tilde{\Omega}_i(r)\left\{\tilde{\Omega}_j(r)\left[\psi_\infty'''(r)-\frac{3}{r}\psi_\infty''(r)+\frac{3}{r^2}\psi_\infty'(r)\right]-\tilde{\Omega}_j''(r)\psi_\infty'(r)+\frac{1}{r}\tilde{\Omega}_j'(r)\psi_\infty'(r)\right\}dr \quad (22g)$$

$$H_{ijk}=\int_{\gamma}^{1}\tilde{\Omega}_i(r)\left\{\tilde{\Omega}_j(r)\left[\tilde{\Omega}_k'''(r)-\frac{3}{r}\tilde{\Omega}_k''(r)+\frac{3}{r^2}\tilde{\Omega}_k'(r)\right]-\tilde{\Omega}_j''(r)\tilde{\Omega}_k'(r)+\frac{1}{r}\tilde{\Omega}_j'(r)\tilde{\Omega}_k'(r)\right\}dr \quad (22h)$$

$$I_{ij}=-2\xi^2\int_{\gamma}^{1}r\tilde{\Omega}_i(r)\tilde{X}_j(r)v_{\theta,f}(r)dr;\ J_{ijk}=-2\xi^2\int_{\gamma}^{1}r\tilde{\Omega}_i(r)\tilde{X}_j(r)\tilde{X}_k(r)dr \quad (22i,j)$$

$$K_{ij}=\int_{\gamma}^{1}\tilde{X}_i(r)\tilde{\Omega}_j(r)\left[\frac{1}{r}v_{\theta,f}(r)+v_{\theta,f}'(r)\right]dr;\ L_{ijk}=\int_{\gamma}^{1}\tilde{X}_i(r)\tilde{\Omega}_j(r)\left[\frac{1}{r}\tilde{X}_k(r)+\tilde{X}_k'(r)\right]dr \quad (22k,l)$$

$$M_{ij}=-\int_{\gamma}^{1}\tilde{X}_i(r)\tilde{X}_j(r)\psi_\infty'(r)dr;\ N_{ijk}=-\int_{\gamma}^{1}\tilde{X}_i(r)\tilde{X}_j(r)\tilde{\Omega}_k'(r)dr \quad (22m,n)$$

$$\bar{f}_i=-\int_{\gamma}^{1}r\tilde{\Omega}_i(r)\left[\psi_\infty(r)+\frac{(r^2-\gamma^2)}{2}\right]dr;\quad \bar{g}_i=-\int_{\gamma}^{1}r\tilde{X}_i(r)v_{\theta,f}(r)dr \quad (22o,p)$$

In order to numerically handle the ODE system given by Eqs. (21) through the subroutine DIVPAG of the IMSL Library (1991), it is necessary to truncate the infinite series in a sufficiently high number of terms (NSF and NTV for the streamfunction and tangential velocity, respectively) so as to guarantee the requested relative error in obtaining the original potentials. This subroutine solves initial value problems with stiff behavior, and provides the important feature of automatically controlling the relative error in the solution of the ordinary differential equations system, allowing the user to establish error targets for the transformed potentials.

Once the transformed potentials $\bar{\phi}_i(z)$ and $\bar{v}_{\theta,i}(z)$ are available, the radial and longitudinal velocity components are obtained from the definition of the streamfunction given by Eqs. (7) and (8), after introducing the inverse formula (19b), as well as the tangential velocity component is computed from Eq. (12a), after introducing the inverse formula (20b), to yield:

$$v_r(r,z)=\sum_{i=1}^{\infty}\frac{\tilde{\Omega}_i(r)}{r}\frac{d\bar{\phi}_i(z)}{dz} \quad (23a)$$

$$v_z(r,z)=\frac{2}{\beta}\left(1-r^2+2r_m^2\ln r\right)-\sum_{i=1}^{\infty}\frac{\tilde{\Omega}_i'(r)}{r}\bar{\phi}_i(z) \quad (23b)$$

$$v_\theta(r,z)=\frac{\gamma}{(1-\gamma^2)}\left(\frac{1}{r}-r\right)+\sum_{i=1}^{\infty}\tilde{X}_i(r)\bar{v}_{\theta,i}(z) \quad (23c)$$

3. RESULTS AND DISCUSSION

Numerical results for the streamfunction filed and velocity components were produced along the entrance region of annular channel, within the governing parameters, i.e., aspect ratios γ = 0.2 and 0.5, and Rt = Re^2/Ta = 1. The computational code was developed in FORTRAN 90/95 programming language and implemented on a PENTIUM-IV 1.3 GHz computer. The routine DIVPAG from the IMSL Library (1991) was used to numerically handle the truncated version of the system of ordinary differential equations (Eqs. (21)), with a relative error target of 10^{-8} prescribed by the user, for the transformed potentials. Also, the results were produced with different truncation orders NT = NST = NTV $\leq$ 60 for γ = 0.2 and N = NST = NTV $\leq$ 100 for γ = 0.5.

First, Tabs. 1 and 2 brings the convergence behavior of the streamfunction at different radial positions of the annular duct and for Z = 0.003 and Z = 0.030. It can be observed the excellent convergence rates for the two aspect ratios analyzed. In an overall analysis at least four significant digits are fully converged in both cases studied.

Table 1. Convergence behavior of the streamfunction at different axial positions for Rt = Re^2/Ta = 1 and γ = 0.2.

r	$\psi(r,z)$ for Rt = Re^2/Ta = 1, γ = 0.2							
	NT = 10		NT = 20		NT = 40		NT = 60	
	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030
0.20	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.25	-0.005213	-0.003913	-0.005176	-0.003914	-0.005154	-0.003915	-0.005146	-0.003915
0.30	-0.017104	-0.013795	-0.017053	-0.013806	-0.016997	-0.013808	-0.016977	-0.013809
0.35	-0.034244	-0.029815	-0.034232	-0.029838	-0.034170	-0.029845	-0.034145	-0.029846
0.40	-0.058930	-0.055993	-0.059094	-0.056040	-0.059091	-0.056054	-0.059079	-0.056057
0.45	-0.083268	-0.084387	-0.083655	-0.084462	-0.083750	-0.084486	-0.083762	-0.084491
0.50	-0.110280	-0.117690	-0.110970	-0.117800	-0.111190	-0.117830	-0.111230	-0.117840
0.55	-0.145160	-0.161810	-0.146260	-0.161950	-0.146660	-0.162000	-0.146750	-0.162010
0.60	-0.177950	-0.203170	-0.179450	-0.203340	-0.180020	-0.203400	-0.180150	-0.203420
0.65	-0.213430	-0.246740	-0.215360	-0.246920	-0.216110	-0.246990	-0.222580	-0.247000
0.70	-0.258260	-0.298660	-0.260700	-0.298840	-0.261670	-0.298910	-0.261910	-0.298920
0.75	-0.299690	-0.342400	-0.302490	-0.342560	-0.303620	-0.342620	-0.303890	-0.342630
0.80	-0.343770	-0.383560	-0.346630	-0.383690	-0.347820	-0.383740	-0.348110	-0.383750
0.85	-0.396690	-0.425660	-0.398930	-0.425740	-0.399900	-0.425770	-0.400150	-0.425780
0.90	-0.438230	-0.454140	-0.439460	-0.454180	-0.440030	-0.454200	-0.440170	-0.454200
0.95	-0.468480	-0.473100	-0.468830	-0.473110	-0.469000	-0.473110	-0.469040	-0.473110
1.00	-0.480000	-0.480000	-0.480000	-0.480000	-0.480000	-0.480000	-0.480000	-0.480000

Table 2. Convergence behavior of the streamfunction at different axial positions for Rt = Re^2/Ta = 1 and γ = 0.5.

r	$\psi(r,z)$ for Rt = Re^2/Ta = 1, γ = 0.5									
	NT = 10		NT=20		NT=40		NT=60		NT=100	
	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030	Z = 0.003	Z = 0.030
0.5	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.55	-0.012424	-0.010378	-0.012379	-0.010378	-0.012352	-0.010379	-0.012344	-0.010379	-0.012339	-0.010379
0.60	-0.040649	-0.036204	-0.040564	-0.036206	-0.040508	-0.036206	-0.040490	-0.036206	-0.040479	-0.036207
0.65	-0.078677	-0.074862	-0.078684	-0.074865	-0.078653	-0.074866	-0.078640	-0.074866	-0.078631	-0.074866
0.70	-0.122260	-0.123120	-0.122510	-0.123120	-0.122570	-0.123130	-0.122580	-0.123130	-0.122580	-0.123130
0.75	-0.169560	-0.177290	-0.170120	-0.177290	-0.170310	-0.177290	-0.170350	-0.177290	-0.170370	-0.177290
0.80	-0.220000	-0.233210	-0.220810	-0.233220	-0.221110	-0.233220	-0.221170	-0.233220	-0.221210	-0.233220
0.85	-0.272290	-0.286300	-0.273140	-0.286300	-0.273450	-0.286300	-0.273530	-0.286300	-0.273570	-0.286300
0.90	-0.321940	-0.331430	-0.322510	-0.331430	-0.322730	-0.331430	-0.322790	-0.331430	-0.322820	-0.331430
0.95	-0.359910	-0.363030	-0.360100	-0.365220	-0.360170	-0.363030	-0.360190	-0.363030	-0.360200	-0.363030
1.00	-0.375000	-0.375000	-0.375000	-0.375000	-0.375000	-0.375000	-0.375000	-0.375000	-0.375000	-0.375000

Figures 2 and 3 show the convergence behavior of the v_z velocity component at Z = 0.002, for $\gamma = 0.2$ and 0.5, and Rt = Re^2/Ta = 1, respectively with different truncations orders NT. It is observed an excellent graphical convergence and a good agreement with the results of Sayed-Ahmed and Sharaf-El-Din (2006). Also, one can see that as the aspect ratio increases the velocity profile tends to be symmetrical.

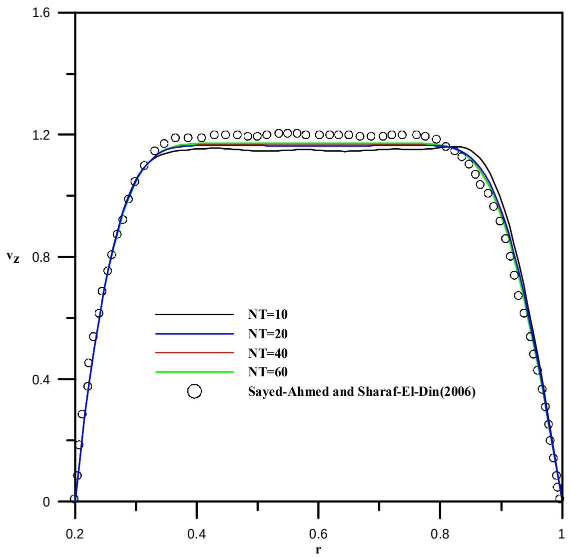

Figure 2. Velocity profile v_z for $\gamma = 0.2$ at Z = 0.002.

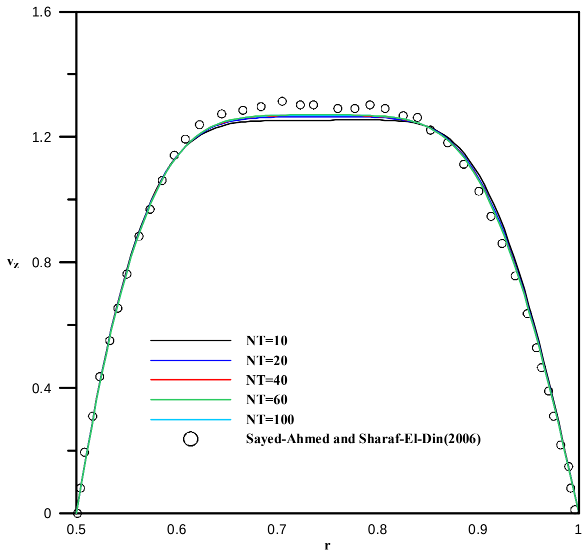

Figure 3. Velocity profile v_z for $\gamma = 0.5$ at Z = 0.002.

Similar analysis is shown in Figs. 4 and 5 for the convergence behavior of the v_r velocity component. Also, it is observed an excellent graphical convergence, but only a poor agreement with those results of Sayed-Ahmed and Sharaf-El-Din (2006). This, difference among the results can be attributed to the numerical scheme employed for these authors. In their works, it is not given any mention about the mesh analysis in order to find fully converged results.

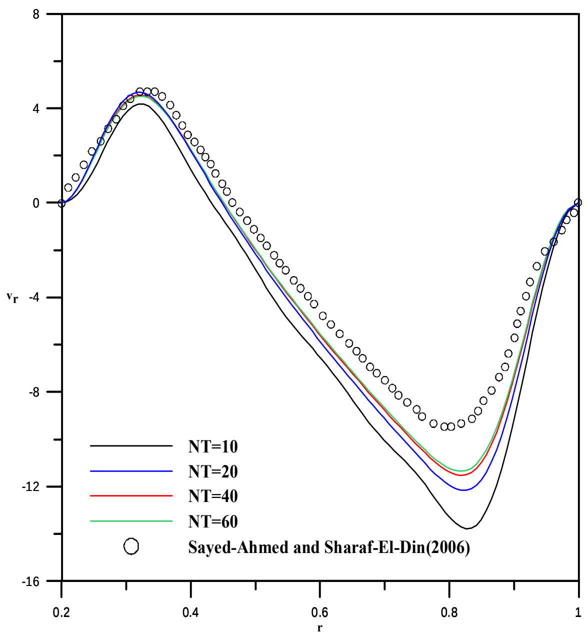

Figure 4. Velocity profile v_r for $\gamma = 0.2$ at Z = 0.002.

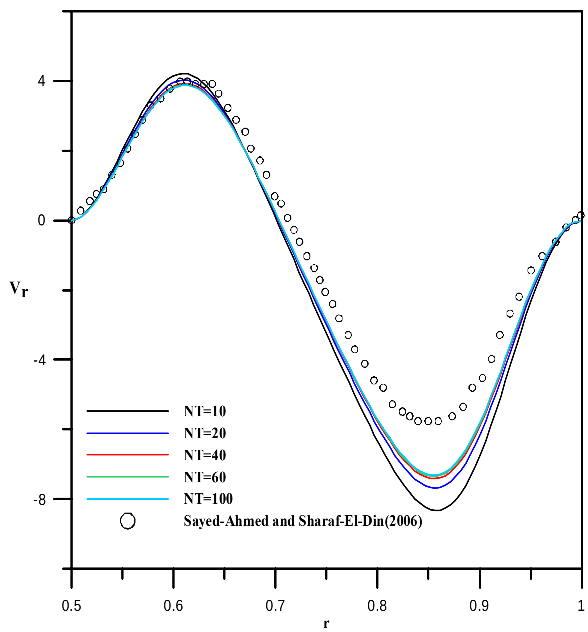

Figure 5. Velocity profile v_r for $\gamma = 0.5$ at Z = 0.002.

Finally, in Figs. 6 and 7 is shown the convergence behavior of the v_θ velocity component with the same governing parameters adopted in the above analysis. For this velocity component, the convergence rate is faster, as can be seen that lower truncations orders are needed to achieve fully converged graphical results. Also, an excellent agreement is found with the results of Sayed-Ahmed and Sharaf-El-Din (2006). In addition, the tangential velocity decreases from one to zero, at the inner to the outer wall of the channel, respectively, and for lower aspect ratio the tangential velocity gradient is stepper than for higher ones.

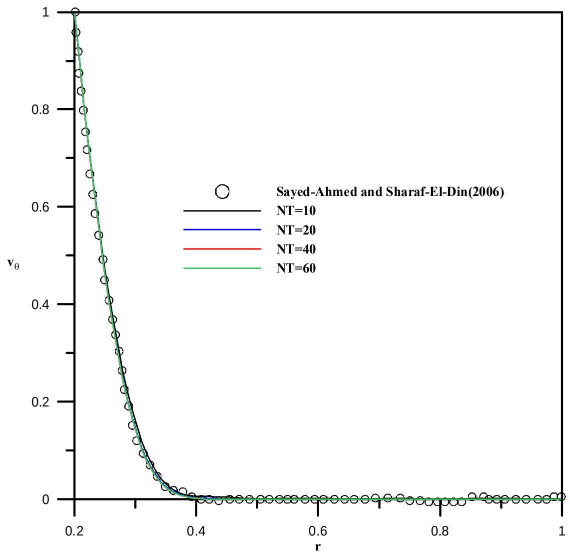

Figure 6. Velocity profile v_θ for $\gamma = 0.2$ at $Z = 0.002$.

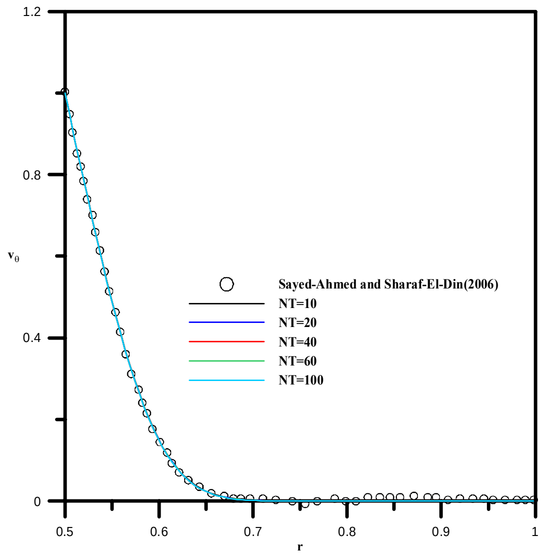

Figure 7. Velocity profile v_θ for $\gamma = 0.5$ at $Z = 0.002$.

4. CONCLUSIONS

The Generalized Integral Transform Technique (GITT) was further extended towards the hybrid numerical-analytical solution of developing laminar flow in annular ducts with rotation of the inner wall, here modeled by the boundary layer equations in cylindrical coordinates, with the employment of the streamfunction formulation. Numerical results for the streamfunction and the velocity components thus produced. The excellent agreement of the present results with previously reported ones demonstrates the consistency of this approach and adequacy for benchmarking such class of problems. An only reasonable agreement of the present converged results with those of Sayed-Ahmed and Sharaf-El-Din (2006) for the radial velocity component is attributed to the numerical scheme adopted by the above referred authors, which was not capable to generate fully converged results for this problem.

5. ACKNOWLEDGEMENTS

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