COMPUTATION OF THE EQUIVALENT SHEAR STRESS AMPLITUDE IN CRITICAL PLANE BASED MULTIAXIAL MODELS BY EVOLUTIONARY ALGORITHMS

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Abstract*: This work has as objective the development of an optimized numerical tool for evaluation of the critical plane in fatigue of components submitted to complex states of stress. This tool has the capacity to determine the plane containing the largest equivalent shear stress amplitude and its normal stress. The Modified Wöhler Curves Model (MWCM) is then considered to evaluate the resistance in fatigue of several experiments reported in the literature. In order to optimize the process of search of the critical plane through the several material planes in a state of stress the genetic algorithm technique was use. Such algorithm is based on evolutions concepts and is capable to foresee the critical plane without the need to determine the equivalent shear stress amplitude in "all" material planes. The results showed that the use of the genetic algorithm significantly reduced the computational cost associated with the determination of the plane which experienced the severest shear stress, i.e., the critical plane. Moreover, the results found for the estimation of the multiaxial fatigue resistance compared well with the experimental data collected in literature.*

Keywords: *Genetic algorithm, shear stress amplitude, critical plane, multiaxial fatigue.*

1. INTRODUCTION

Critical plane models were developed from a physical interpretation of the fatigue process where cracks were observed to initiate and grow on certain preferential material planes, Findley (1959). In such an approach, shear and normal stresses during the loading cycle are determined for several planes at the same point analyzed in the component. An empirical combination of these is used to predict the most severely loaded plane (critical plane), where cracks are expected to nucleate. In the setting of High Cycle Multiaxial Fatigue the critical plane is usually defined as the material plane experiencing the largest amplitude of the shear stress. Usually the radius of the minimum circle circumscribing the shear stress vector path in a material plane is used to characterize its amplitude, Susmel, L., Lazzarin, P. (2002), but more recent approaches suggest that measures associated with the minimum circumscribing ellipse, Araújo, J. A. & Mamiya E. N. (2002), or with the maximum prismatic hull, Araújo, J. A., Mamiya, E. N. & Dantas, A. P. (2007) may provide a better measure of the shear stress amplitude.

The main drawback associated with the use of critical plane criteria as a practical engineering design tool is the high computational cost involved in determining the critical plane. Such multiaxial models not only require that a large number of planes be investigated to get an accurate response, but also need to numerically compute the minimum sphere (or an equivalent measure) circumscribing the shear stress vector path for each material cut. In real components a Finite Element Analysis (FEA) usually will provide the cyclic stress history in a large number of nodes that will need to be checked under this time consuming critical plane searching process in order to determine the most threaten material point in fatigue terms. An optimization technique capable to reduce such numerical cost without significant loss of accuracy would constitute an important advance for the practical application of the multiaxial critical plane based criteria. In this setting, the aim of this work is to evaluate the performance of a numerical tool based on Genetic Algorithms to search the critical plane and calculate the shear stress amplitude by a minimum circumscribing circle concept.

2. CRITICAL PLANE

Consider a mechanical element submitted to oscillatory multiaxial loadings. In what follows, Sym^3 denotes the set of all 3X3 symmetric matrices. Let $\Sigma : [0,\mathrm{T}] \rightarrow Sym^3$ be the *stress path* associated with a given material point O along a time interval [0,T]. At each time instant $t \in [0,T]$, the point is subjected to internal forces represented by the *Cauchy stress tensor* $\mathbf{\Sigma}(t) \in Sym^3$.

$$\Sigma(t) = \begin{bmatrix} \sigma_{xx}(t) & \tau_{xy}(t) & \tau_{xz}(t) \\ \tau_{xy}(t) & \sigma_{yy}(t) & \tau_{yz}(t) \\ \tau_{xz}(t) & \tau_{yz}(t) & \sigma_{zz}(t) \end{bmatrix} \quad (1)$$

Consider now, a material plane characterized by its unit normal vector $\mathbf{n}$, passing through point O. This in turn described by its spherical angles ϕ and θ, as shown in Fig. 1. ϕ is the angle between $\mathbf{n}$ and the z-axis, whereas θ is the angle between the projection of $\mathbf{n}$ in the xy plane and the x-axis. Let $\mathbf{t} : [0,\mathrm{T}]$ denotes the *stress vector* acting on any material plane $\mathbf{n}$ along a time interval $[0,\mathrm{T}]$. It is possible to describe the stress vector path in terms of a curve Ψ in R^3 where each point $\mathbf{t}(t) \in R^3$ of such path can be decomposed in two other vectors: a stress vector normal to the plane, denoted as $\boldsymbol{\sigma}(t)$ and a shear stress vector $\boldsymbol{\tau}(t)$, acting on the plane (Fig. 1). It can be observed that the direction of the normal stress vector $\boldsymbol{\sigma}(t)$ does not vary during its loading, however its magnitude is modified in each instant t. The shear stress vector $\boldsymbol{\tau}(t)$ changes magnitude and direction on the plane $\mathbf{n}$ during the cyclic loading. The projection of the curve Ψ on the plane $\mathbf{n}$ is the curve $\Psi_{\mathbf{n}}$ that describes the history of the shear stress vector, which will be different for other planes cutting point O. This means that the amplitude of shear stress $\tau_{\max}$ depends on the orientation of the plane where it acts, *i.e.*, is function of angles ϕ and θ. An appropriate algorithm to compute the shear stress amplitude, which is given by the radius of the minimum circle circumscribing the stress path $\Psi_{\mathbf{n}}$ is given in Dang Van (1989). Briefly, a small radius for the circle is initially set and starting from the geometrical center of $\Psi_{\mathbf{n}}$ the shear stress path is traversed from point to point. If the new point lies outside the circle, its radius is slightly increased (by an expansion factor χ) and its center translated. Convergence is achieved when all points of the stress path are completely engulfed and the center of the circle remains stable.

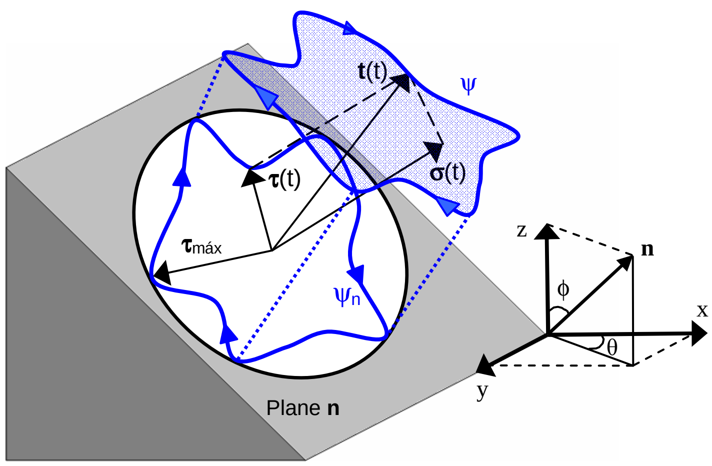

Figure 1: Plane **n** passing through the point O and the minimum circumscribing circle that contains the loading history Ψ described by the shear stress vector $\boldsymbol{\tau}(t)$.

Different researches have considered criteria of multiaxial fatigue of high cycle based on the critical plane concept. They consider that cracks of fatigue have origin in determined material planes, where combinations of shear and normal stress or deformation are particularly severe. Therefore, these criteria are capable to not only supply the resistance in fatigue of the material and the place of initiation of cracks, but also its orientation.

3.0 GENETIC ALGORITHM

The Genetic Algorithms (GA) are based on the Charles Darwin's theory of evolution of species and they have been first considered by John Holland. These algorithms, based on concepts of the biological evolution, are used in the optimization of mathematical problems. The Genetic Algorithms have several applications mainly due to its comparative advantages to other methods of optimization. It is well suited for cases where gradient information is not available or is computationally expensive. Moreover, the GA's work with a set of individuals and not with only one point, thus being able to make parallel searches in different areas of the solution space.

A typical run of a Genetic Algorithm (GA) starts with a random population of individuals and then evolve these individuals by using some genetic operators: fitness calculation, selection, crossover and mutation. In the end of each iteration, a new generation of individuals is created and this new population represents a better approximation to the unknown solution of the optimization problem than the previous one (Fig. 2).

In this work, the individuals are binary string representing the angles ϕ and θ. The fitness value, for each individual, is the maximum radius of the minimum circle circumscribing the shear stress path, plus the corresponding normal stress value weighted by 0.01 (Fig. 1). The better fitted individuals are selected to randomly mate, in this step, crossover and mutation are used, respectively, to exchange genes between the mates and to add new genetic information in the new population. The fitness evaluation and the creation of new generations are repeated until an adequate solution is found or until the maximum number of generations is reached, Goldberg D.E. (1989).

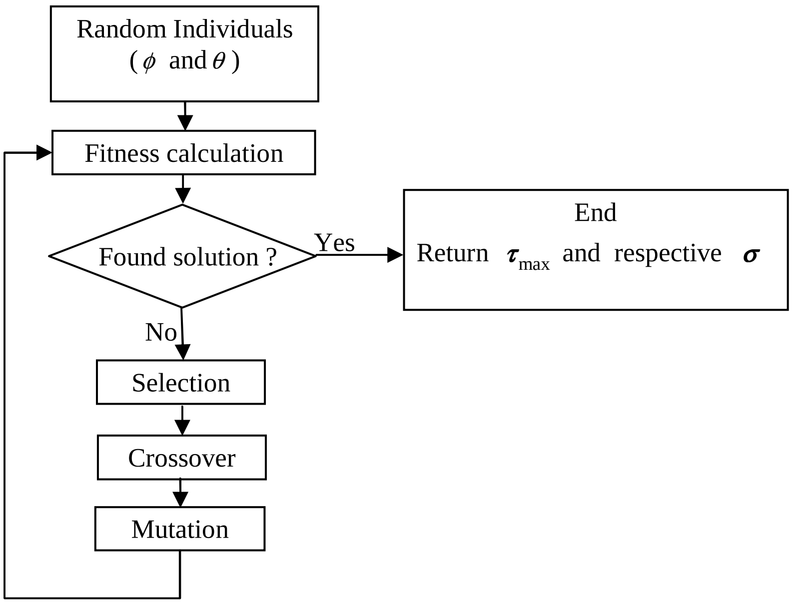

Figure 2: Diagram representing the steps of a typical GA

4.0 RESULTS AND DISCUSSIONS

In order to evaluate the results obtained by applying Genetic Algorithms to compute the critical plane and its associated equivalent shear stress amplitude we considered the stress history for twenty two multiaxial fatigue tests generated by Heindereich *et al.*(1985) on 34Cr4 and by Nishihara & Kawamoto (1945) on 0,51%C hard steel. Both materials were fatigue tested under combined bending and torsion loading. Synchronous, in phase and out-of-phase sinusoidal waves were applied with and without mean normal and shear stresses superimposed. Such stress history can be described by Eq. (2):

$$\begin{aligned}\sigma(t) &= \sigma_m + \sigma_a \, sin(\omega t),\\ \tau(t) &= \tau_m + \tau_a \, sin(\lambda\omega t - \beta),\end{aligned} \qquad (2)$$

where the subscript *a* stands for the amplitude of stresses while *m* represents the mean value, σ and τ are normal and shear stresses, while β is the phase difference and λ is the load frequency, which is always equal to one for iso-frequency tests. Tables 1 and 2 report these data. f_{-1} and t_{-1} are the fatigue limits under fully reversed bending and torsion, respectively.

Table 1 – Synchronous biaxial fatigue data generated by Heidenreich et al. (1985). for 34Cr4 $f_{-1} = 410\,\mathrm{MPa}$ $t_{-1} = 256\,\mathrm{MPa}$ $f_{-1}/t_{-1} = 1.6 \in]1.3, \sqrt{3}[$.

Test N°	σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	λ
1	314	0	157	0	0	1
2	315	0	158	0	60	1
3	316	0	158	0	90	1
4	315	0	158	0	120	1
5	224	0	224	0	90	1
6	380	0	95	0	90	1
7	316	0	158	158	0	1
8	314	0	157	157	60	1
9	315	0	158	158	90	1
10	279	279	140	0	0	1
11	284	284	142	0	90	1
12	212	212	212	0	90	1

Table 2 – Synchronous biaxial fatigue data generated by Nishihara & Kawamoto (1945) for hard steel, $f_{-1} = 313.9\,\mathrm{MPa}$, $t_{-1} = 196.2\,\mathrm{MPa}$, $f_{-1}/t_{-1} = 1.6 \in]1.3, \sqrt{3}[$.

Test N°	σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (°)	λ
1	138,1	0	167,1	0	0	1
2	140,1	0	169,9	0	30	1
3	145,7	0	176,3	0	60	1
4	150,2	0	181,7	0	90	1
5	245,3	0	122,6	0	0	1
6	249,7	0	124,8	0	30	1
7	252,4	0	126,2	0	60	1
8	258	0	129	0	90	1
9	299,1	0	62,8	0	0	1
10	304,5	0	63,9	0	90	1

To compute the equivalent shear stress amplitude only the traditional minimum circumscribing sphere methodology described in section “critical plane” was used, although new strategies based on the minimum circumscribing ellipsoid Bin Li, Santos, J. L., Freitas, M. A (2000) and on the prismatic hull Gonçalves, C. A., Araújo, J. A. & Mamiya, E. N.(2005) could also be considered.

The sphere expansion coefficient χ was defined as 0.05 in all cases studied. For each test reported in Tab. 1 and 2 the location of the critical plane and the computational cost associated to its search were obtained by two methods. The first searched the critical plane looking at a number of material planes defined by angle increments $\Delta\phi = \Delta\theta = 1°$. Such method requires that a combination of 181 by 181 different planes be analyzed in order to find the critical one for a single material point, and is denominated here on as the Plane Increment Method (PIM). It is worth of notice that such searching process is extremely time consuming as it not only requires that a large number planes be investigated to get an accurate response, but also needs to numerically compute the minimum sphere circumscribing the shear stress vector path $\boldsymbol{\tau}(t)$ for each material cut. The plane containing the largest sphere among all the searched planes is the critical one.

The second method used Genetic Algorithm (GA) as an optimization technique to find the critical plane. To carry out the analysis, we initially took a population of ten individuals randomly generated within a pre-defined range. In the context of this work, each individual is a material plane defined by the angles ϕ and θ in the range $0° \le \phi \le 180°$ and $0° \le \theta \le 180°$. All individuals (planes) are evaluated by the fitness function. The fitness value, for each individual, is the radius of the minimum circle circumscribing the shear stress path, plus the corresponding normal stress value weighted by 0.01. New genetic information, which corresponds to the binary coded strings associated to the plane orientation, was introduced in some individuals with a mutation probability of 1%. The fitness evaluation and the creation of new generations were repeated until convergence was achieved or twenty generations were reached. In this work, twenty generations of ten individuals would be produced, in case of no prior convergence, resulting in the creation of 200 individuals, therefore, with the 1% mutation probability, two individuals would be mutated in the procedure. In this study, all cases achieved convergence before the limit of twenty generations.

Figures 3 and 4 depict the values of the shear stress amplitude and the computational cost provided the PIM and the GA methods for the tests reported in Tab. 1 and 2. It seems clear that both methods provide essentially the same value of the equivalent shear stress amplitude. On the other hand, as the computational cost is an important factor, the time required by the GA method was significantly smaller compared with the PIM method. For instance, in test two for the 34Cr4 (Fig. 3), the PIM method found the critical plane in 748 seconds while the GA method took 33 seconds only. In test 7 for the hard steel (Fig. 4) such cost was 765 and 35 seconds respectively. The analysis was always carried out in a same Pentium IV computer with a 2.2 GHz processor.

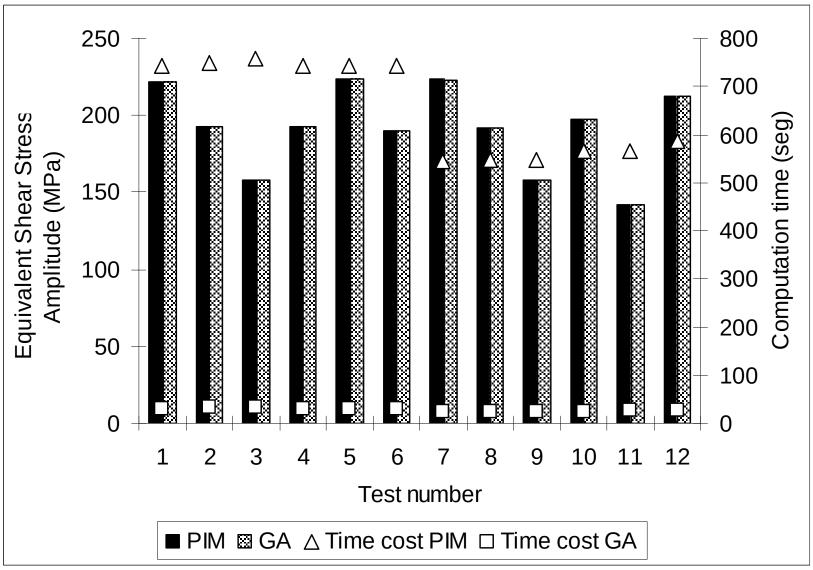

Figure 3: Maximum Shear Stress Amplitude and computational cost provided by PIM and GA method for 34Cr4.

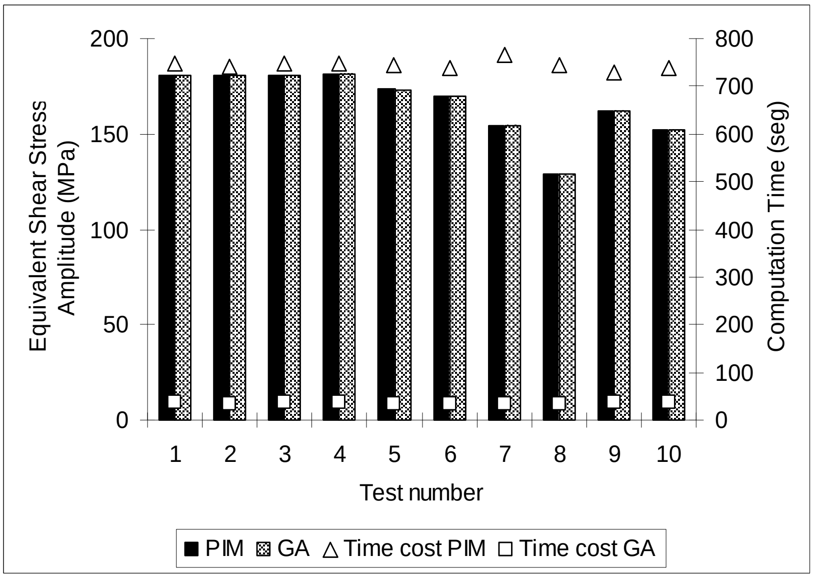

Figure 4: Maximum Shear Stress Amplitude and computational cost provided by PIM and GA method for hard steel

5. CONCLUSIONS

An optimization technique based on Genetic Algorithm (GA) was proposed to compute the equivalent shear stress amplitude in multiaxial fatigue. Comparable to the Plane Increment Method (PIM), the Genetic Algorithm shows a significant reduction in the run-time associated with the search of the critical plane and also provides accurate values of the maximum shear stress amplitude. Therefore the GA proved to be a powerful tool for optimization in fatigue analysis based on critical plane concepts. Further as such algorithm is not based on derivative concepts it do not face problems of determining local maximum or local minimum. As a consequence it always will converge to the global maximum/minimum but the results may be less precise than what could be obtained by a derivative based method. Considering the analysis carried in this work such drawback was not relevant as the computed values of the shear stress amplitude were essentially the same comparing the GA method with the PIM method.

6. ACKNOWLEDGEMENTS

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