A LOCKING-FREE LAMINATED COMPOSITE BEAM FINITE ELEMENT

Rafael Holdorf Lopez
INSA – Rouen - 76860 Saint-Etienne du Rouvray CEDEX - France
rafaelholdorf@gmail.com

João Elias Abdalla Filho
PUC/PR - Av. Imaculada Conceição, 1155 - Curitiba/PR, Brasil - CEP 80215-901
UTFPR - Av. Sete de Setembro, 3165 - Curitiba/PR, Brasil – CEP 80230-901
joao.abdalla@pucpr.br

Abstract. *This paper aimed to extend a finite element laminated composite beam model to analyze stresses at each laminae of a laminated composite. The expressions of normal and transverse shear stresses were developed using a first-order shear deformable theory. Such expressions were computationally implemented into the finite element software called LAMFEM, which was written in FORTRAN 77. A spurious term inherent to the formulation procedure was identified and removed from the model due to the use of a physically interpretable notation called Strain Gradient Notation. The results of normal and transverse shear stresses were obtained and compared to an analytical solution to validate the model. The model with spurious terms obtained a slower convergence for normal stress and it diverged from the analytical solution for the transverse shear stress. Moreover, the mode without spurious term converged for all cases.*
Keywords*: Laminated composites, strain gradient notation, Finite Element Method, locking.*

1. INTRODUCTION

The word composite in the term composite material signifies that two or more materials are combined on a macroscopic scale to form a useful third material (Jones, 1975). A very important class of composites is that of the laminated fiber-reinforced composites, commonly referred to as simply laminated composites. Such composites have been widely used in structural engineering (e.g. aerospace engineering, automotive engineering).

Laminated composites consist of at least two layers or laminae of fibrous materials that are bonded together. Strengths and stiffnesses of a laminated composite can be designed to achieve specific requirements of the structural element under consideration. The structural analysis of composite materials presents some peculiarities. For instance, if the arrangement of the constituent laminae of the laminate is not symmetric about its middle surface, coupling between deformation modes arises. The most important one is the coupling between bending and extension.

The finite element method has been widely employed to analyze structural systems composed of laminated composite components. However, spurious terms are introduced during the formulation procedure (Bird, 1988). Such spurious terms arise due to the use of incomplete and inconsistent polynomials and they may cause quantitative and/or qualitative errors (Abdalla, 1992).

Strain gradient notation (SGN) is a physically interpretable notation that can be used to develop finite elements and it allows for the identification and removal of such spurious terms (Dow and Huyer, 1986). It has been successfully used in the finite element analysis of laminated composite beams and plates as well as a curved beam.

This paper aims at extending a beam finite element model to calculate the stresses at each layer of a laminated composite beam. The first-order shear deformation theory is used to formulate the finite element. The equations used to obtain the stresses are computationally implemented in the finite element software called LAMFEM (which is written in FORTRAN 77). The spurious terms inherent to the formulation procedure are removed due to the use of SGN. The results of normal and transverse shear stresses are obtained for a laminated composite cantilever beam and compared to an analytical solution (Abdalla *et al.*, 2006).

2. BEAM FINITE ELEMENT

This section of the paper focuses on showing the identification and removal of the spurious terms. Moreover, it shows the expression used to obtain normal and transverse shear stresses. The entire formulation procedure can be seen in Abdalla *et al.* (2000) and Abdalla *et al.* (2006).

The assumptions for the mechanical behavior of the laminated beam are as follows: (a) plane sections remain plane, but not necessarily normal after bending (transverse shear deformation is taken into account); (b) there is perfect bond between laminae, and (c) strain and stresses normal to the middle surface are negligible.

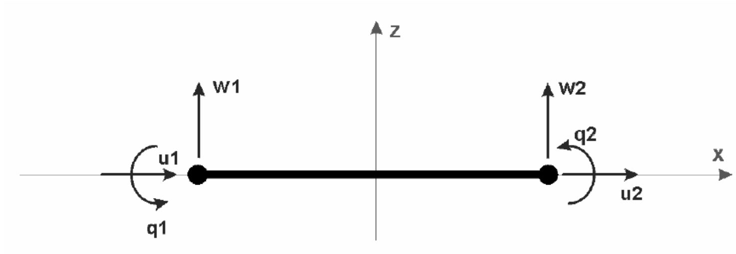

Figure 1. Two-dimensional beam finite element

The beam's displacement field is defined by:

$$u(x,z) = u_0(x) - zq(x) \tag{1}$$

$$w(x) = w_0(x) \tag{2}$$

where $u_0(x)$ is the axial displacement, $w_0(x)$ is the midsurface's vertical displacement, which defines the element's vertical displacement, and $q(x)$ is the in-plane rotation.

In SGN the beam's displacements become (the quantities in brackets are the strain gradients obtained using the SGN):

$$u(x,z) = \left([u]_0 + [\varepsilon_x]_0 x\right) - z\left(\left[-\frac{\gamma_{xz}}{2} - q_{xz}\right]_0 + [-\varepsilon_{x,z}]_0 x\right) \tag{3}$$

$$w\ (x) = [w]_0 + \left[\frac{\gamma_{xz}}{2} - q_{xz}\right]_0 x \tag{4}$$

where, $[u]_0$, $[w]_0$ and $[q]_0$ are rigid body displacements; $[\varepsilon_x]_0$ and $[\gamma_{xz}]_0$ are constant normal and transverse shear strains, and $[\varepsilon_{x,z}]_0$ is the flexural strain. The subscript o in the notation is employed to indicate that the strain gradients result from the evaluation of the corresponding Taylor series expansions at a given origin.

In Eq. (3) and (4), it can be noticed that instead of unknown coefficients used in regular notation, their coefficients are physically interpretable.

The strain field, which will be used to obtain the stresses, is given by the derivatives of the displacements. In SGN it becomes:

$$\varepsilon_x = [\varepsilon_x]_0 + [\varepsilon_{x,z}]_0 z \tag{5}$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\varepsilon_{x,z}]_0 x \tag{6}$$

By inspecting the Eq. (5), it can be seen that the coefficients are related to the normal strain. Thus, they are both legitimate. However, the expansion for the transverse shear strain (Eq. (6)), shows the presence of a shear strain term, but also of the flexural strain term $[\varepsilon_{x,z}]_0$. It indicates that when the beam undergoes a flexural deformation, there is an increase in shear strain. As this is physically impossible since there is no coupling between transverse shear and flexural strains, the latter term is certainly spurious. Such incorrect coupling is known as parasitic shear (Bird, 1988) and it causes quantitative (slow convergence) and/or qualitative error (Abdalla, 1992) (not convergence at all). It is important to note that the spurious term has only been identified *a-priori* due to the use of SGN. To free the element from the effects of parasitic shear, the term $[\varepsilon_{x,z}]_0$ must be removed from the transverse shear strain expression.

From this point on, the paper focuses in the stresses analysis through the thickness of the laminated composite beam.

Equations (7) and (8) are used to obtain the normal and transverse shear stresses in a typical lamina k. It is important to note that the normal stress varies along the z coordinate.

$$\sigma_x = \left(\overline{Q}_{11}\right)_k \left\{\frac{1}{L}(u_2 - u_1) - \frac{z}{L}(q_2 - q_1)\right\} \tag{7}$$

$$\tau_{xz} = \left(\overline{Q}_{55}\right)_k \left\{\frac{1}{L}(w_2 - w_1) - \frac{x}{L}(q_2 - q_1) - \frac{1}{2}(q_2 + q_1)\right\} \tag{8}$$

In these equations, Q_{11} and Q_{55} are stiffness properties of the lamina's material and u, v and w are the element's degrees of freedom as shown in Figure 1.

3. NUMERICAL ANALYSIS

This section presents the solution of a laminated composite cantilever beam comprised of two unidirectional lamina of graphite/epoxy (Abdalla *et al.*, 2006). The mechanical properties of each lamina are shown in Tab. 1:

Table 1: Mechanical properties of each lamina

	E_{11} (Pa)	E_{22}=E_{33} (Pa)	G_{12}=G_{23}=G_{31}(Pa)*	ν_{12}=ν_{23}=ν_{31}	h (m)	α
LÂMINA 1	1,38E+11	1,45E+10	5,00E+09	0,21	0,04	30º
LÂMINA 2	1,38E+11	1,45E+10	3,86E+09	0,21	0,06	0

* the shear strength values were set with different values only to provide different stresses at each lamina in the analysis, in a typical engineering problem they would be equal once the lamina's material is the same.

The beam is 2.0m long, 0.05m wide and it is loaded with 2000N/m along its length as shown in Fig. 2.

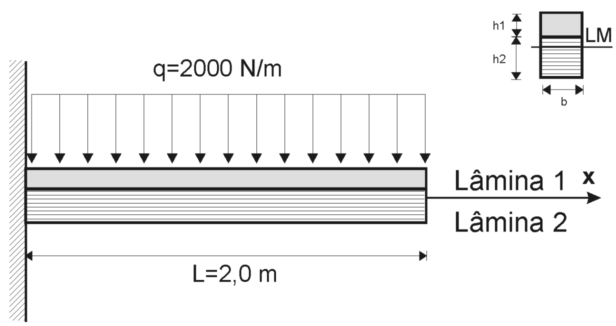

Figure 2. Laminated composite cantilever beam with distributed load

This problem has been analyzed using the finite element developed in the previous section. In order to validate the model, analytical solution for this beam problem should be available for comparison. Hence, an analytical solution for this beam derived through the minimization of the total potential energy is obtained from reference (Abdalla *et al.*, 2006). The resulting normal stress and transverse shear stress equations of the analytical solution can be seen in Eq. (9) and (10), respectively, where Q_o is the distributed load; $A_{11,}$ B_{11} and D_{11} are the extension, bending-extension and bending stiffnesses, respectively; $(\overline{Q}_{11})_k$ and $(\overline{Q}_{55})_k$ come from the constitutive equation of the typical lamina k; L and b are the beam's length and width, respectively.

$$\sigma_{xx}(x,z)=(\overline{Q}_{11})_k\left[\frac{Q_0 B_{11}}{b\left(B_{11}^2-A_{11}D_{11}\right)}\left(\frac{x^2}{2}-Lx+\frac{L^2}{2}\right)-\frac{zQ_0 A_{11}}{b\left(B_{11}^2-A_{11}D_{11}\right)}\left(\frac{x^2}{2}-Lx+\frac{L^2}{2}\right)\right] \tag{9}$$

$$\gamma_{xz}(x)=\left(\overline{Q}_{55}\right)_k\left[\frac{Q_0}{bA_{55}}(x-L)\right] \tag{10}$$

The normal and transverse shear stresses are analyzed at the fixed support (x=0) and at mid-span (x=1m). The results are generated by the model with and without the spurious term in order to show its deleterious effect in the finite element analysis. Two, four, eight, sixteen and thirty-two element meshes are employed. Fig. 3 and 4 show the normal stress variation through the thickness of the beam at x =0 (fixed support) employing the model without and with the spurious term, respectively. It can be observed that the model with the spurious term converges slightly slower than the model free of it. However, this difference is almost insignificant, and it may be said that both models converge well within the analytical solution with refinement.

Figures 5 and 6 show the normal stress through the thickness of the beam computed at mid-span employing the model without and with the spurious term, respectively. It can be seen that the convergence of the model without the spurious term is monotonic, different from the model containing the parasitic shear term. Figure 6 shows that numerical solutions oscillate around the analytical solution, not presenting a clear indication of convergence.

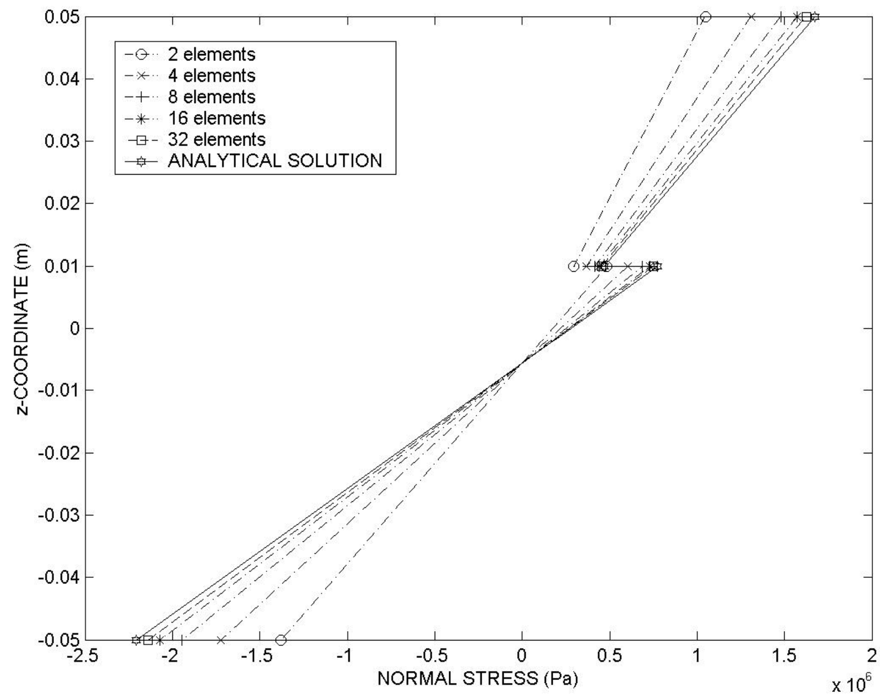

Figure 3. Normal stresses computed through the thickness of the laminate at x = 0, without the spurious term.

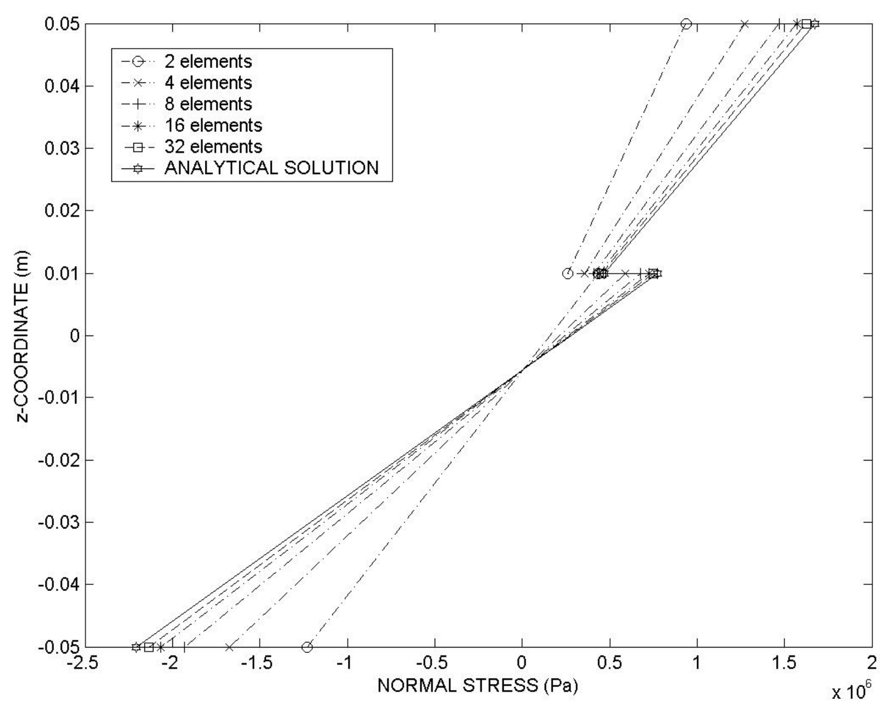

Figure 4. Normal stresses computed through the thickness of the laminate at x = 0, with the spurious term.

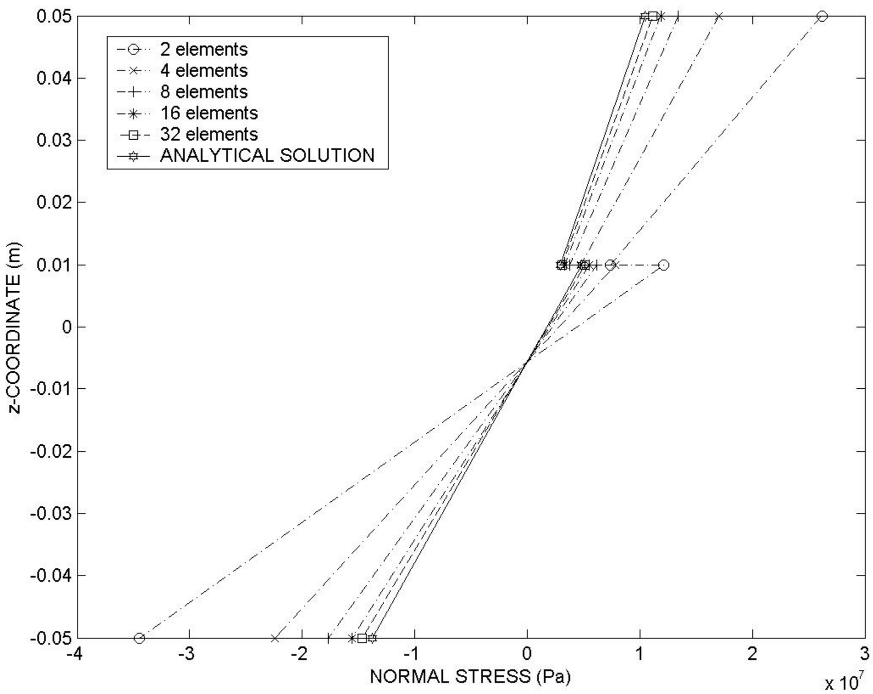

Figure 5. Normal stresses computed through the thickness of the laminate at x = 1m, without the spurious term.

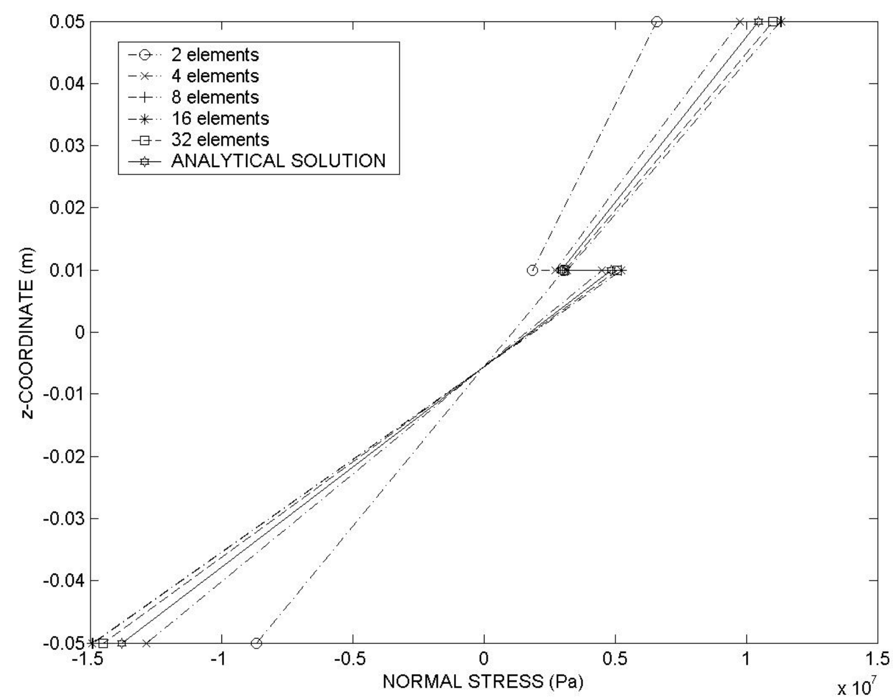

Figure 6. Normal stresses computed through the thickness of the laminate at x = 1m, with the spurious term.

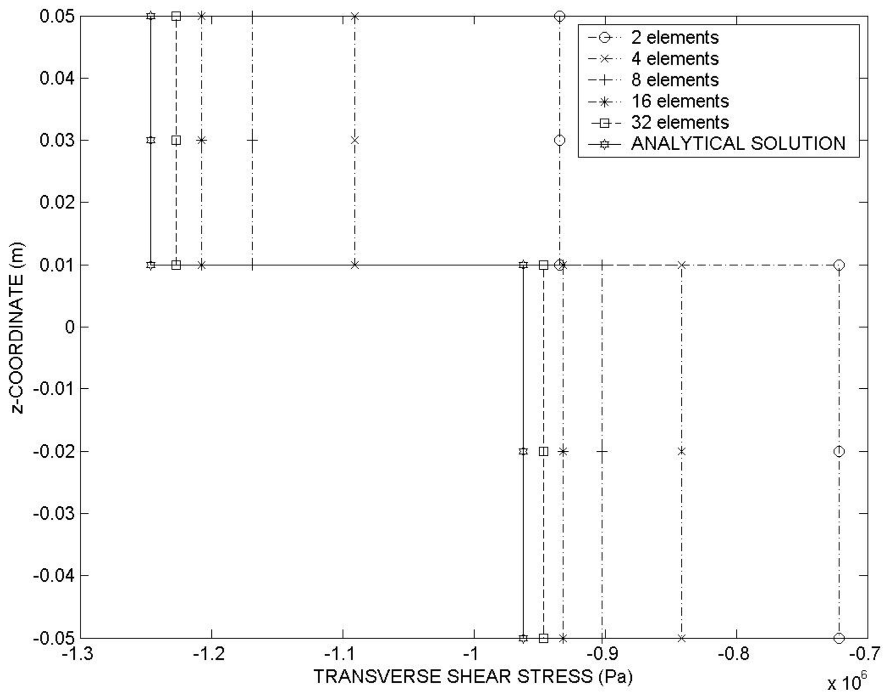

Figure 7. Transverse shear stress computed through the thickness of the laminate at x = 0m, without the spurious term.

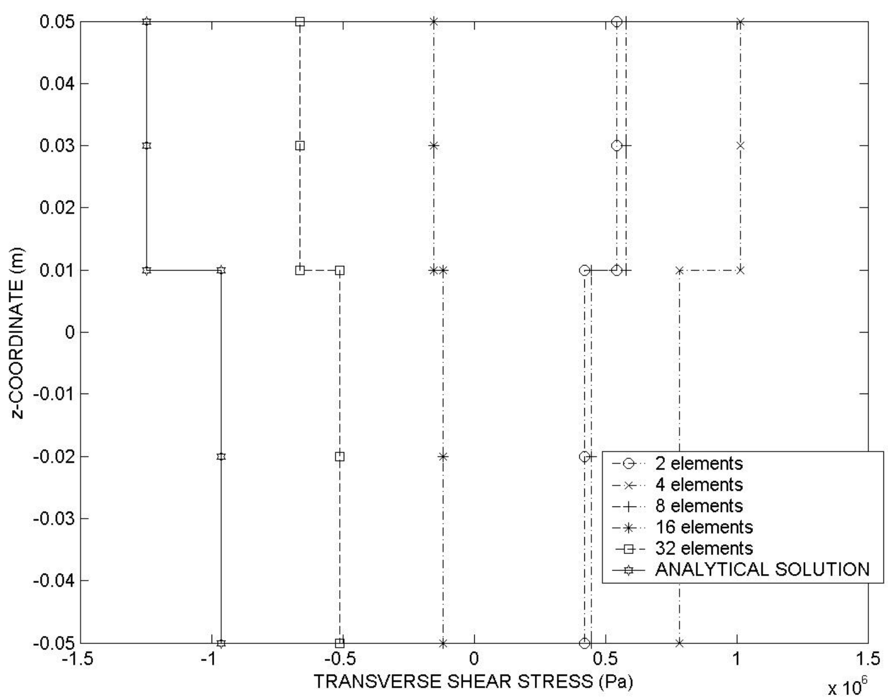

Figure 8. Transverse shear stress computed through the thickness of the laminate at x = 0m, with spurious term.

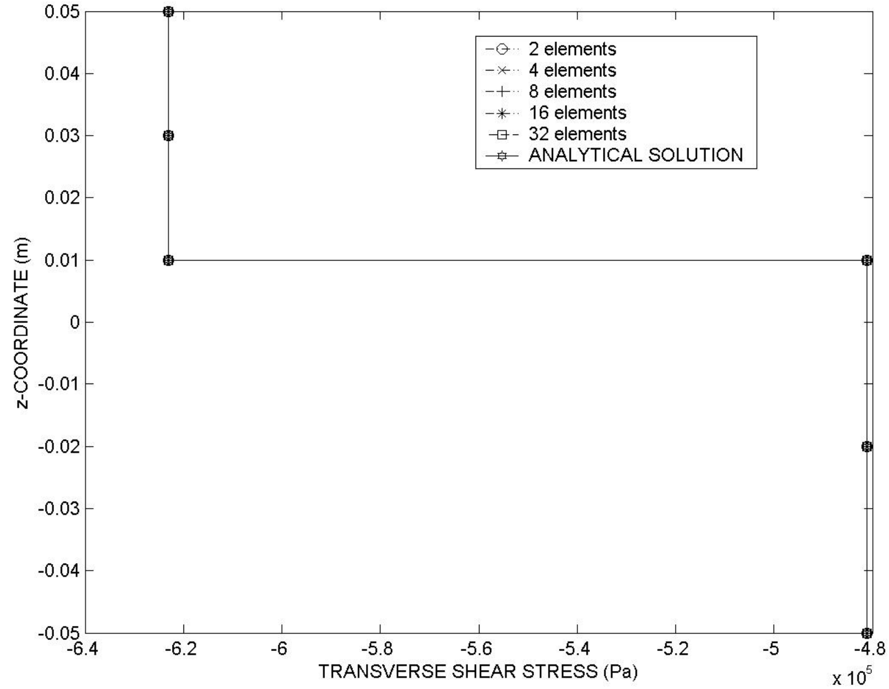

Figure 9 –Transverse shear stress computed through the thickness of the laminate at x = 1m, without the spurious term.

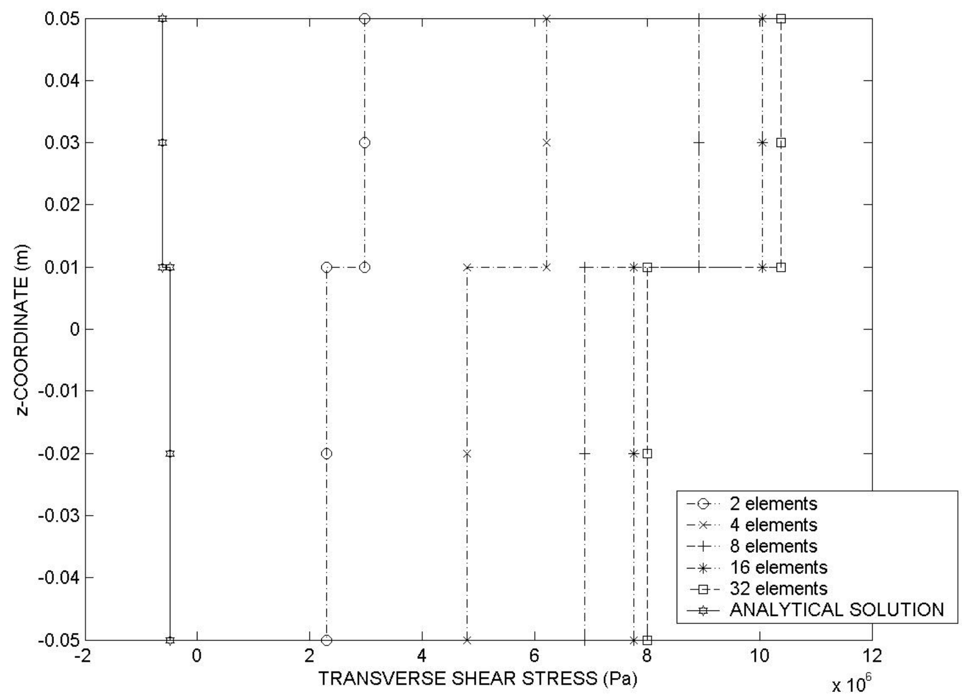

Figure 10 –Transverse shear stress computed through the thickness of the laminate at x = 1m, with spurious term.

Figures 7 and 8 show the transverse shear stress through the thickness of the beam computed at x = 0 (fixed support). The model corrected for the spurious term provides solution which converge monotonically to the analytical one (see Fig. 7). On the other hand, Fig. 8 shows that the model with the spurious term does not converge well. Actually, the coarser meshes provide divergent solutions as they get farther away from the analytical one. It means that the parasitic shear term causes a quantitative error, that is, locking, in the finite element analysis.

Figure 9 and 10 show the variation of the transverse shear stress through the thickness of the beam at x = 1 m (mid-span). For the model without spurious term (Fig. 9), all meshes provide very good agreement with the analytical solution. However, in the model with parasitic shear (Fig. 10), mesh refinement causes the solution to diverge from the analytical solution. That is, the spurious term causes not only locking of the model, but also a misleading solution. Therefore, it may be considered a qualitative error.

4. SUMMARY AND CONCLUSIONS

A beam finite element for the analysis of laminated composites has been formulated using strain gradient notation. Through the notation, it was possible to identify and eliminate an erroneous term present in the polynomial expansion of the transverse shear strain. Such term is a flexural term which increases the stiffness of the element unduly when it is activated. It is therefore a parasitic shear term.

Numerical analyses reveal that the spurious term usually causes locking, that is, convergence delay. It has also been shown that transverse shear strain solutions affected by the spurious term may diverge from the correct solution. This qualitative error is misleading to the analyst. Further, solutions corrected for the spurious term converge monotonically well to analytical solutions.

Therefore, following conclusions can be drawn:

1) The finite element formulation is efficient, because the results obtained using the model free from error converged to the analytical solutions.

2) Strain gradient notation provides means to identify and eliminate spurious terms inherent to the formulation procedure.

3) Spurious terms must be eliminated because they are responsible for serious deleterious effects in the results, more importantly, in the transverse shear ones.

5. REFERENCES

Jones, R.M., 1975, “Mechanics of Composite Materials”, Hemisphere Publishing Corporation, New York, 1975.

Byrd, D. E., 1988, “Identification and elimination of errors in finite element analysis”, PhD Dissertation, University of Colorado, Boulder, CO.

Abdalla, J.E., 1992, “Qualitative and discretization error analysis of laminated composite plate models”, PhD Dissertation, University of Colorado, Boulder, CO.

Dow, J. O. and Huyer, S.A., 1986, “Continuum models of space station Structures”, ASCE Journal for Aerospace Engineering, Vol. 2, pp. 212-230.

Abdalla J.E., Fagundes F.A. and Machado R.D., 2006, “Identification and elimination of parasitic shear in a laminated composite beam finite element”, Advances in Engineering Software, Vol. 37, pp. 522-532.

5. RESPONSIBILITY NOTICE