STOCHASTIC FINITE ELEMENT ANALISYS OF ACOUSTIC PROBLEMS BY A NON INTRUSIVE COLLOCATION TECHNIQUE

Erb Ferreira Lins
COPPE - Programa de Engenharia Mecânica, Universidade Federal do Rio de Janeiro
Centro de Tecnologia, bloco I, sala I231, Cidade Universitária, Ilha do Fundão, 21945-970, Rio de Janeiro/RJ, Brasil
erb@mecsol.ufrj.br

Fernando Alves Rochinha
COPPE - Programa de Engenharia Mecânica, Universidade Federal do Rio de Janeiro
Centro de Tecnologia, bloco I, sala I231, Cidade Universitária, Ilha do Fundão, 21945-970, Rio de Janeiro/RJ, Brasil
rochinha@adc.coppe.ufrj.br

Sergio Bellizzi
Laboratoire de Mécanique et d'Acoustique – CNRS
31, chemin Joseph-Aiguier, 13402, Marseille cedex 20, França
bellizzi@lma.cnrs-mrs.fr

Abstract: *In this work, a stochastic collocation technique is applied to solve of interior acoustic problems in the frequency domain driven by stochastic input data. A finite element formulation is used to approximate the Helmholtz equation with sthocastic impedance boundary conditions in a tri-dimensional space. This FE model represents a room with the walls covered by a layer of sound absorbing material. The impedance of this layer is considered random and have a statistical description. The computation of statistical moments and the probability density function of the output pressure will be carried out by the Non-Intrusive Stochastic Galerkin (NISG) method, a collocation technique which uses a finite element mesh to represent the random support space and the joint probability distribution function of input random variables. The NISG statistical data is analyzed concerning error and convergence rate and compared with Monte Carlo technique.*

Keywords*: stochastic collocation, finite element, uncertainty quantification*

1. INTRODUCTION

The active and passive controls of noise are an active area of research. This research reflects an industrial need of reducing the sound intensity in some regions of an environment like vehicles cabins or increasing the intensity of the sound in acoustic rooms. The passive control of sound is a technique used to reduce the sound intensity in an enclosure by damping the sound energy using acoustic absorbers, placed in the walls of the enclosure. This technique has an excellent cost-benefit relationship (Cox and D'Antonio, 2004).

The analysis of time dependent sound propagation in incompressible fluids is done by the wave equation. In frequency domains, this equation can be transformed in the very well known Helmholtz equation, which can be solved by analytical means in few cases. Unfortunately, for a number of boundary conditions and domain geometries only numerical solutions are available for this problem. The finite element method is a very powerful technique which can be applied to obtain an approximate solution to Helmholtz equation. The FE solution to the Helmholtz equation has also been an object of study concerning the error estimation and propagation (see Ihlenburg and Babuška 1995a, 1995b and the references therein).

Since exact information about the nature of dissipation in most absorbers is very poor (the measurement of admittance of absorbing materials is a very complicated task as long as it depends of angle of the wave that hit the absorber, flow conditions in the environment, temperature, the interaction of the absorbing panel with the supporting structure etc.) we look for more robust models that take into account the random nature of this fluid structure interaction. The probabilistic modeling of mechanical properties can be used to this end. In this case, the probabilistic nature of physical parameters is considered in mathematical model, and the results obtained included some statistical information which can be used in design.

The Monte Carlo technique is a very powerful method to treat systems with random parameters. In this method, a large number of samplings of input variables are calculated and then the problem is solved for each sample of input variables. With the solution populace obtained, some statistical properties can be calculated. Unfortunately, this technique has poor convergence for mean and standard deviation of the solution, requiring a large number of samples to achieve good precision in results (Roberts and Spanos, 2003).

Another way to include the input statistical description in the mathematical formulation is using a stochastic collocation technique. In this case, an analytical treatment of stochastic input is done, and the input random field are represented by a Karhunen-Loève (KL) expansion (Sampaio and Volter, 2001, Ghanem and Spanos, 1991). The solution is obtained in the Non-Intrusive Stochastic Galerkin (NISG) analysis by finite element representation of stochasticquantities in a support space defined by the domain of input random variables obtained in KL expansion (Acharjee and Zabaras, 2007). For each point in support space a deterministic solution is calculated. As the number of these points is much smaller than the number of samples needed by Monte Carlo technique, the solution is much faster.

In this work, the NISG will be applied to the stochastic modeling of impedance of an absorbing material. A two-dimensional cavity propagation problem will be solved by the finite element method in the spatial dimension. The Helmholtz equation with Dirichlet and Robin boundary conditions will be used to represent this physical domain.

2. MATHEMATICAL MODEL

In a compressible ideal fluid, with small perturbations of acoustic pressure p, the sound propagation is described, in the time domain, by the wave equation (Ihlenburg, 1998)

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = F\left(x\right) \qquad em \qquad \Omega_f \tag{1}$$

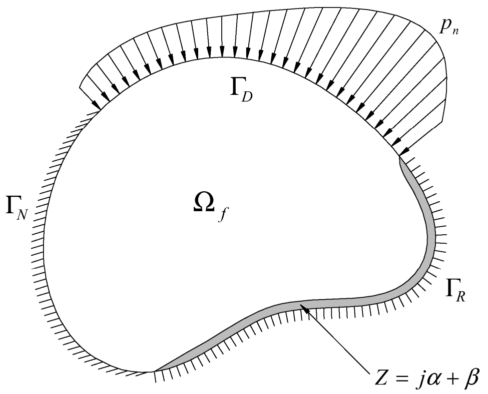

Figure 1 – Model description

A closed domain Ω_f is shown in figure 1. c represents the velocity of sound propagation in the medium. This domain is limited by a boundary Γ divided in three part where the following relations hold $\Gamma = \Gamma_N \cup \Gamma_D \cup \Gamma_R$ and $\Gamma_D \cap \Gamma_N = \Gamma_D \cap \Gamma_R = \Gamma_N \cap \Gamma_R = \varnothing$. In the frequency domain, we obtain the Helmholtz equation:

$$\nabla^2 p + k^2 p = f\left(x\right) \tag{2}$$

This represents the wave propagation of harmonic waves. $f\left(x\right)$ could represents loads likes sources or body forces. Where $k = \omega/c = 2\pi\lambda^{-1}$ and ω and λ are the angular frequency and wavelength of sinusoidal wave, respectively. At the boundary surfaces, three kinds of boundary conditions can be set (Irimie and Bouillard, 2001)

$$\begin{array}{llll} Dirichlet: & p = p_n & at & \Gamma_D \\ Neumann: & \nabla p \cdot \mathbf{n} = -j\rho ck\ \mathbf{v}_n \cdot \mathbf{n} & at & \Gamma_N \\ Robin: & \nabla p \cdot \mathbf{n} = -j\dfrac{\rho ck}{Z_n} p & at & \Gamma_R \end{array} \tag{3}$$

Where ρ is the fluid density, $\mathbf{n}$ is the vector normal to the boundary surface, p_n is the prescribed pressure at boundary Γ_D, Z_n is the impedance at boundary Γ_R, $\mathbf{v}_n$ is the fluid velocity in the boundary Γ_N.

We consider in this analysis a two dimensional domain shown in fig. 1. This domain represents a cavity with a pressure source in one of the corner.

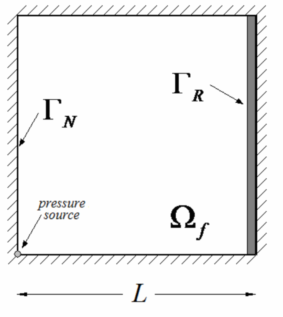

Figura 2 – Two-dimensional cavity with an absorption boundary

We consider that in this closed domain, one of the boundaries is covered with a layer of absorption material and all others boundary are rigid walls. In one of the corners a pressure source generates harmonic waves in this domain. This is a square cavity with size of $L = 1$. To this problem, the mathematical formulation with Helmholtz eq.(2) can be set as

$$\nabla^2 p + k^2 p = f(x) \tag{4}$$

$$\begin{aligned} &\nabla p \cdot \mathbf{n} = 0 && at\ \Gamma_N \\ &\nabla p \cdot \mathbf{n} - \frac{j\rho\omega}{Z} p = 0 && at\ \Gamma_R \end{aligned} \tag{5}$$

Where Z is the impedance of absorbing material. A simple model to represent this impedance is a viscoelastic model could be (Gamallo, 2002)

$$Z = \alpha \mathbf{u}_f \cdot \mathbf{n} + \beta \mathbf{v}_f \cdot \mathbf{n} \tag{6}$$

In this case, the relation between pressure and velocity in the boundary is giving by two terms: one proportional to displacement of the fluid, representing an elastic response of this material and a second component, proportional to the velocity of the fluid, representing the viscous damping. In frequency domain, this condition simplify to

$$Z = \beta - j\frac{\alpha}{\omega} \tag{7}$$

The two material constants β e α are strictly positive and can be experimentally measured. In this work, we use the experimental data given by Bermúdez and Rodríguez (1999) for glass wool Manville.

3. FINITE ELEMENT FORMULATION

The finite element formulation is based in a weak form of eq. (4). To obtain this weak form we multiply eq. (4) by a test function $q \in V_q$ and integrate in the domain Ω_f. After some mathematical operation, the problem can be stated as: find $p \in V_p$ such

$$\mathscr{B}(p,q) = \mathscr{L}(q) \qquad \forall q \in V_q \tag{8}$$

Where

$$\begin{aligned} \mathscr{B}(p,q) &= \int_{\Omega} \nabla p \nabla q \, d\Omega - k^2 \int_{\Omega} pq \, d\Omega + \frac{j\rho ck}{Z_n} \int_{\Gamma_R} pq \, d\Gamma \\ \mathscr{L}(q) &= \int_{\Gamma_N} \mathbf{v}_N \cdot \mathbf{n} \, q \, d\Gamma - \int_{\Omega} f \, q \, d\Omega \end{aligned} \tag{9}$$

The functional space V will be such that

$$V_p = H_D^1\left(\Omega_f\right) = \left\{p \in H^1\left(\Omega_f\right)\right\} \tag{10}$$

$$V_q = H_0^1\left(\Omega_f\right) = \left\{q \in H^1\left(\Omega_f\right) \middle| q\left(\mathbf{x}_D\right) = 0, \mathbf{x}_D \in \Gamma_D\right\} \tag{11}$$

And $H^1\left(\Omega_f\right)$ is the Sobolev space given by the set of function with complex values with derivatives square integrable in Ω_f:

$$H^1\left(\Omega_f\right) = \left\{y : \Omega_f \to \mathbb{C}, \qquad y \in L^2\left(\Omega_f\right), \frac{\partial y}{\partial x_i} \in L^2\left(\Omega_f\right); \qquad \forall i = 1,2\right\} \tag{12}$$

We look for a FE approximation of eq.(8). Suppose that the domain could be split in a set of *Nel* elements, and each element will have his piece of domain represented by Ω_e as depicted in fig.(3). The parameter N controls the number of finite elements used in each direction.

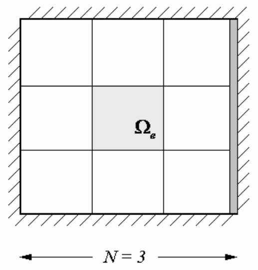

Figure 3 – Quadrilateral mesh used in discretization of domain

Let the approximate spaces $S_h \subset V$. The domain is the sum of all quadrilateral elements and in every element the interpolate function are lagrangian polynomial function of degree 1, corresponding to the functional space $S_h^{1,0}$. So the approximate problem could be stated as: find $p_h \in V_h$ such as

$$\mathscr{B}\left(p_h, q_h\right) = \mathscr{L}\left(q_h\right) \qquad p_h \in S_P^h, \; \forall q_h \in S_q^h \tag{13}$$

As usual in the FE method, we set an interpolation matrix as

$$p\left(\mathbf{x}\right) = \mathbf{H}\left(\mathbf{x}\right)\mathbf{P} \tag{14}$$

Which will result in the following linear system,

$$\left(\mathbf{K} + jk\mathbf{C} - k^2\mathbf{M}\right)\mathbf{P} = \mathbf{F} \tag{15}$$

And the matrices are given by

$$\begin{aligned} \mathbf{K} &= \int_\Omega \left(\nabla\mathbf{H}\right)^T \nabla\mathbf{H} d\Omega & \mathbf{M} &= \int_\Omega \mathbf{H}^T\mathbf{H} d\Omega \\ \mathbf{C} &= \frac{\rho c}{Z_n}\int_{\Gamma_R} \mathbf{H}^T\mathbf{H}\, d\Gamma & \mathbf{F} &= \int_\Omega \mathbf{H}\, f\left(x\right) d\Omega \end{aligned} \tag{16}$$

The Helmholtz equation has been studied by several researchers (Strouboulis *et al.*, 2006, Ihlenburg and Babuška, 1995a-b, Ihlenburg, 1998, Paulino and Rochinha, 2005, Irimie and Bouillard, 2001). The error dispersion in FE mesh, pollution due to operator loss of ellipticity, post processing are very well analyzed. In this work, we just use the information obtained by these researches to keep the error in FE solution low. This will allow us to focus our attention in the error in stochastic formulation. We consider that the relative error in H^1 seminormis bound by the following equation (Ihlenburg and Babuška, 1995a)

$$e_h < C_1 kh + C_2 k^3 h^2 \tag{17}$$

Where C_1 e C_2 are two constants independent of wave number and element size h.

The constant physical values and fluid properties that will be used in the numerical solution are shown in tab.(1).

Table 1 – Physical parameters used

$\rho = 1 kg/m^3$ $c = 343 m/s$	$P_0 = 1Pa$ $L = 1m$

4. NON INTRUSIVE STOCHASTIC FORMULATION

Some natural phenomena are intrinsically random. Some examples are the formation and position of voids in forged metal components, turbulence if flows with high Reynolds numbers, the distribution of fibers in glass wool etc. Other source of randomness is the value of some physical property of material. In vast majority of cases, it is impossible to know the exact value of these properties. An accurate experimental value can be obtained for some properties, like the Young modulus of elasticity in solids. Unfortunately, there is no way to accurate measure the sound absorption coefficient for a large range of sound insulating materials. This property depends of factor like humidity of air (in porous materials), flow velocity, sound wave incident angle, temperature etc. In order to achieve a better design this random nature should be take into account in modeling phase.

If complete statistical information about the random processes or random field is available, we can use a spectral expansion like the Karhunen-Loève expansion to include this information of random properties in the mathematical formulation. The KLE is an expansion of the stochastic process by a denumerable set of random variables (see Ghanem and Spanos (1991) for further details) and has some desirable properties: is unique, optimal in least squares sense, which means that the error in representation is minimized, and is almost surely convergent for Gaussian processes.

If statistical information is not known *a priori*, one can use the Polynomial Chaos (Ghanem and Spanos, 1991) or the Stochastic Collocation Method (SCM) (Keese, 2003, Babuska *et al.*, 2004). In this work, we will use the Non intrusive stochastic Galerkin approach developed by Acharjee and Zabaras (2007), which was derived directly from Stochastic Collocation Method.

Consider that the two constants α e β representing the impedance given by (7) are random variables so

$$Z = \beta(\theta) - j\frac{\alpha(\theta)}{\omega} \tag{18}$$

Consider also that they are uniformly distributed. For any frequency ω given it is available the mean and standard deviation of α e β so

$$\begin{aligned} \alpha(\theta,\omega) &= \mu_\alpha(\omega)\left(1+\sigma_\alpha(\omega)\xi_\alpha\right) \\ \beta(\theta,\omega) &= \mu_\beta(\omega)\left(1+\sigma_\beta(\omega)\xi_\beta\right) \end{aligned} \tag{19}$$

μ_α, μ_β are the mean, $\sigma_\alpha, \sigma_\beta$ are the standard deviation values of α e β. ξ_α e ξ_β represents two normalized uniform variables with range $[-1,1]$. This range represents the support space for each normalized random variable.

Now, consider eq.(15), if the impedance is as a stochastic variable this naturally leads to

$$\left(\mathbf{K} + jk\mathbf{C}(\theta) - k^2\mathbf{M}\right)\mathbf{P}(\theta) = \mathbf{F} \tag{20}$$

The NISG is a technique can be described as follow:

1) With the pdf of input random variables, calculate the joint pdf $f_Z(\boldsymbol{\xi})$. Here, as the two variables are independent and uniform in the interval $[-1,1]$, we have $f_Z(\boldsymbol{\xi}) = f_{\xi_\alpha} f_{\xi_\beta} = \frac{1}{4}$
2) Determine the dimensionality of support space. Apply the FEM to discretize this space (in this work this space is a square region $\Theta:[-1,1]\times[-1,1]$);

3) For each element in support space FE mesh, calculate the gauss integration points for this element and the correspondent value of ξ_α e ξ_β in this point. Calculate $\alpha(\theta,\omega)$ e $\beta(\theta,\omega)$ in this point and solve (20) with these values. This is a deterministic solution. Store this solution in the vector $p_j^h(\mathbf{x},\omega)$. j should index all integration points in the mesh.
4) Using least squares, calculate the value of $p_i^h(\mathbf{x},\omega)$ at nodes of support FE mesh. Using this nodal information, the solution in any point of mesh could be calculated with the use of interpolation functions Φ_i

$$p(\mathbf{x},\omega,\theta) = \sum_{i=1}^{Nodes} p_i^h(\mathbf{x},\omega)\Phi_i \tag{21}$$

5) The sth statistical moment could be calculated as

$$M_s = \int_\Theta \left(p(\mathbf{x},\omega,\theta) \right)^s f_Z(\theta) d\theta \tag{22}$$

Using the FEM representation considered in eq.(21) results

$$M_s = \int_\Theta \left(\sum_{i=1}^{Nodes} p_i^h(\mathbf{x},\omega)\Phi_i \right)^s f(\theta)d\theta = \sum_{e=1}^{Nel} \int_{\Theta^e} \left(p_i^h(\mathbf{x},\omega)\Phi_i \right)^s f(\theta_e)d\theta_e \tag{23}$$

The transformation above is possible since the functions Φ_i are locally supported. We calculate the integral in eq. (23) using the quadrature rule with the points j defined in step 3

$$M_s = \sum_{e=1}^{Nel} \sum_{j=1}^{Ng} w_i \left(p_i^h(\mathbf{x},\omega) \right)^s f(\theta_i^e) \tag{24}$$

In the above equation, θ_i^e and w_i represent the abcissae and weight of integration point. $p_i^h(\mathbf{x},\omega)$ is the deterministic evaluation obtained using θ_i^e. This solution involves no change of the preexistent computational codes: all the statistical data are obtained using the output of FEM software, which can be considered a non-intrusive approach. This representation leads to several advantages: the solution is obtained using uncoupled determinist evaluations, can easily deals with non-linearities in the input data and the stochastic representation closely follow the FEM theory, concerning the convergence and error analysis.

5. RESULTS AND DISCUSSIONS

The mean value used in the impedance of absorbing layer will be taken from Bermúdez and Rodríguez (1999). As long as we have no information regarding others statistical properties we will consider that the standard deviation is ±20% of this mean value and the stochastic distribution is uniform. So eq.(19) can be read

$$\begin{aligned} \alpha(\theta,\omega) &= \mu_\alpha(\omega)\left(1+0.2\,\xi_\alpha\right) \\ \beta(\theta,\omega) &= \mu_\beta(\omega)\left(1+0.2\,\xi_\beta\right) \end{aligned} \tag{25}$$

And ξ_α e ξ_β are uniformly distributed in the range $[-1,1]$. Figure 3 shows this variation in the frequency range used:

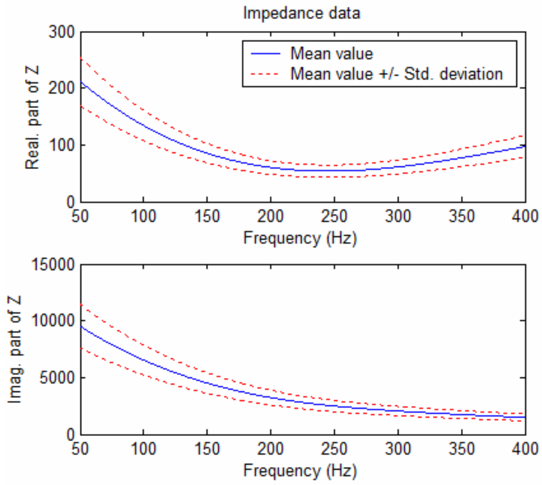

Figure 4 – Data used to stochastic model of Z

All calculations for NISG were done with MATLAB 6. The Deterministic finite element code for Helmholtz equation was written in FORTRAN 90. A mesh of 20x20 Finite elements has been used in the spatial domain. Table 1 show the numerical values obtained for different numbers of elements in support space for a point at the center of cavity. It should be observed the excellent convergence obtained for frequency $f = 400Hz$, the highest used in this work. For frequency $f = 180Hz$ the convergence is not so good, but it should be noted that this frequency is near a resonant frequency of this model (see figure 8). Compared with the Monte Carlo technique, this results shows good agreement. However, the number of deterministic solutions is just a small fraction of the number of solutions needed by Monte Carlo method.

In this work we used 4x4 elements in the two dimensional stochastic dimension, which implies the solution of 64 deterministic problems (4 gauss points in each element).

Tabela 2 – Mean Pressure and standard deviation at $x = 0.5, y = 0.5$ for different numbers of elements in support space

		Frequency = $180Hz$		Frequency = $400Hz$	
	N_s	Mean Pressure	Std. Deviation	Mean Pressure	Std. Deviation
NISG	1×1	0.926318374 - 0.080427994i	0.344119974	0.785426092+ 0.035077524i	0.061918480
	2×2	0.956325938 - 0.095279371i	0.452627359	0.785425326+ 0.035077784i	0.061932046
	4×4	0.963091996 - 0.100435849i	0.490351715	0.785425273+ 0.035077802i	0.061932929
Monte Carlo (2×10^5 samples)		0.964387815 - 0.101365122i	0.498141994	0.785555120 + 0.035132683i	0.061827349

Figure 5-7 shows the mean values and calculated to frequency of 50, 200 and 400 Hz. The pressure distribution for mean is very similar to the ones obtained when we use a deterministic model. The standard deviation values are low compared with mean values obtained, but it strongly depends of the frequency used.

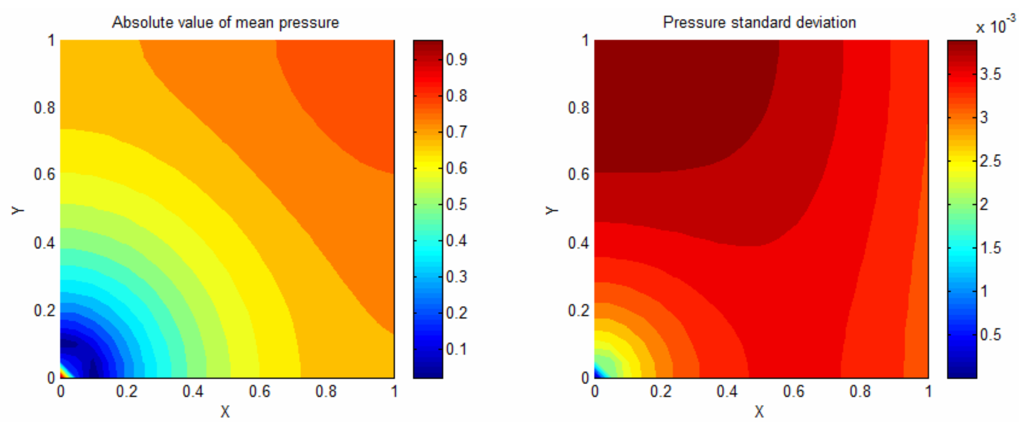

Figure 5 - Pressure distribution (Absolute mean value and std. deviation) for frequency $f = 50Hz$

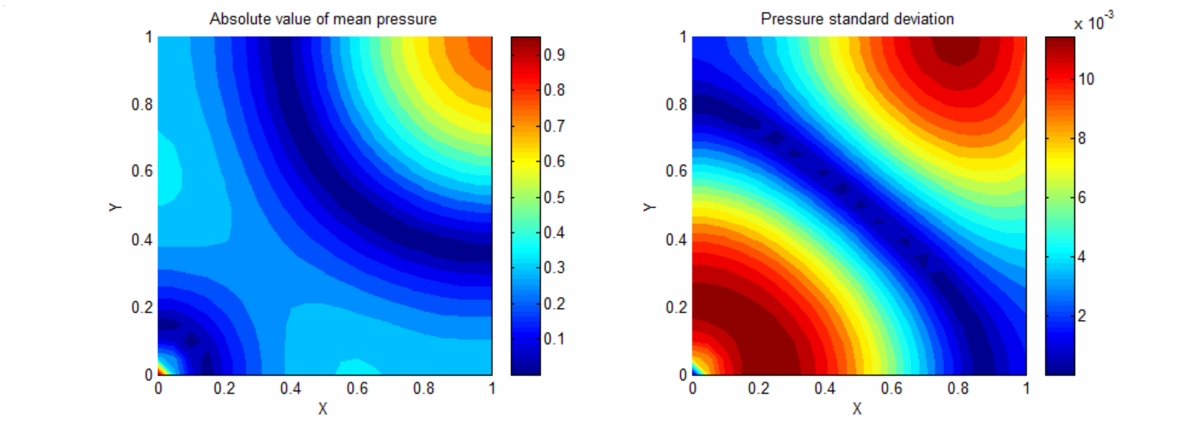

Figure 6 - Pressure distribution (Absolute mean value and std. deviation) for frequency $f = 200Hz$

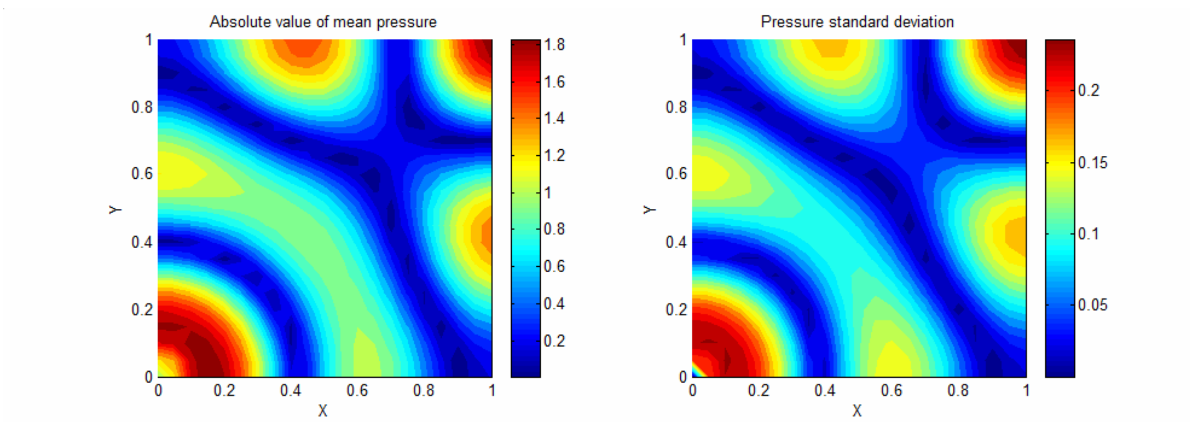

Figure 7 - Pressure distribution (Absolute mean value and std. deviation) for frequency $f = 400Hz$

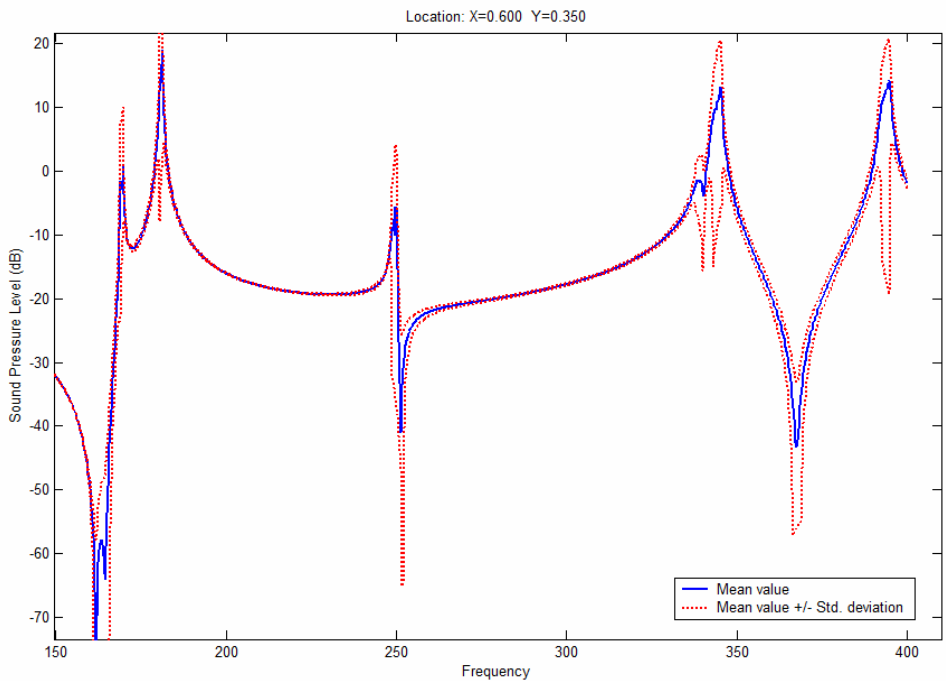

Figure 8 – Sound Pressure level $SPL(dB) = 20\log_{10}\left(mean(p)/p_0\right)$ for point $x = 0.6$ and $y = 0.35$: Mean value and deviation from mean.

Figure 8 show the sound pressure level at the point located at $x = 0.6$ and $y = 0.35$. As can be seen, the standard deviation values can be very high (compared to the mean values) in the regions near the resonance frequencies. In frequencies higher than $350Hz$, this uncertainty level also has a significant values.

Another way to quantify the level of uncertainty is calculate the ratio between standard deviation and mean values in a given point. This is shown in figure 9 for the point located at $x = 0.6, y = 0.35$ and $x = 0.25, y = 0.2$. The dotted line represents the level of uncertainty of input values (20%). As said before, the uncertainty level grows with frequency and it can show high values in the vicinity of resonance frequencies.

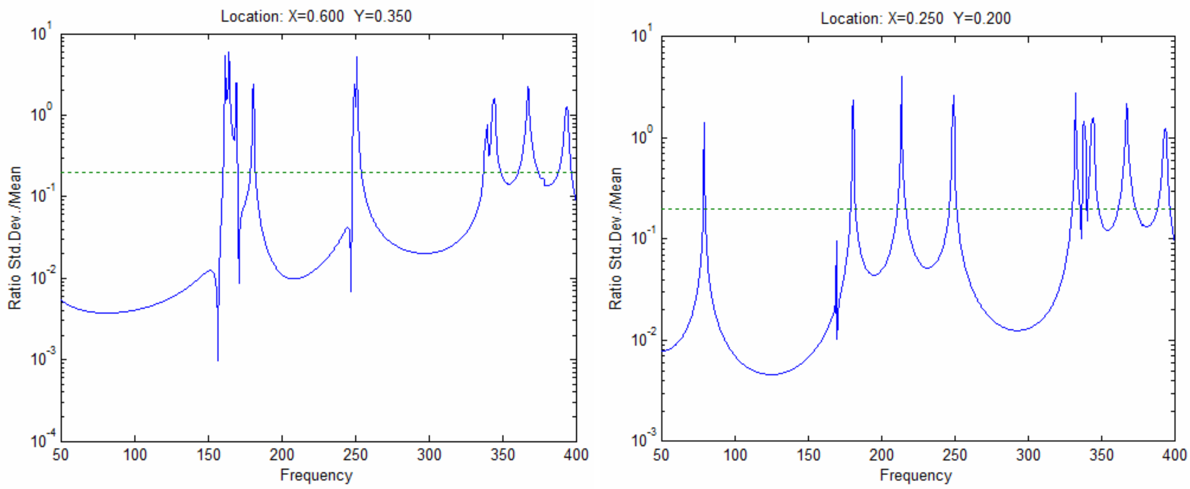

Figure 9 – Ratio σ_x / μ_x for points $x = 0.6, y = 0.35$ (a) and $x = 0.25, y = 0.2$ (b).

6. CONCLUSIONS

In this work, the Non-Intrusive Stochastic Galerkin (NISG) approach is applied to the solution of acoustic problem in a two dimensional rigid cavity, with an absorbing layer at one boundary. The impedance at this boundary is considered a stochastic variable with known probabilistic distribution. The spatial two-dimensional domain was discretized using the finite element technique, followed by FE discretization of stochastic support space. The statistical data obtained shown that the uncertainty level of output can be higher than the uncertainty level of input for frequencies near the resonant ones. Besides this, higher values of frequencies had shown an increase of this uncertainty output level. The technique developed here can be used to obtain the statistics of output of interest using only a small number of deterministic solutions.

7. ACKNOWLEDGEMENTS

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