DYNAMIC ANALYSIS OF VEHICLE-BRIDGE INTERACTION

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Abstract. *The main objective of this article is to study and develop a vehicle-bridge interaction model for dynamic analysis of bridges carrying moving loads. The vehicle is modeled as a group of mass, spring and damper connected elements. The bridge is a straight continuum beam with many spans and constant cross section area. The system vehicle-bridge is approximated by the finite element method. The main focus of this discussion is the interaction form between the vehicle and the bridge. Due to the continuously moving location of the variable loads on the bridge, the governing differential equations become rather complicated. Coupling equations between the characteristics of the vehicle and those of the bridge are developed. To solve the problem of coupling, Newmark's finite difference formulas are applied. Some examples are shown and the results are compared with those found in the literature.*

Keywords*: Dynamic Analysis; Vehicle-bridge interaction; Moving loads.*

1. INTRODUCTION

The structural design of ordinary bridges under moving loads is usually analyzed as a statically equivalent problem and the dynamic effects are introduced only by impact factors. To improve the structural analysis, however, it is important to consider the influence of the vehicle vibration and the bridge vibration on the moving loads. This is a typical couple problem, since the vehicle vibration affects the bridge vibration and vice versa. The coupled problem must be solved by an iterative process, which is not simple to solve. For further reference, detailed formulation may be found in the book of Frýba (1972, 1996), which presents the dynamic vibration problem of moving loads over bridges and different methods for solving this problem are carefully shown.

In the literature many works shows different models for vehicle-bridge interactions. Some of them are listed at the end of this paper. Cheng and Cheung (2001), for instance, have analyzed the vibration of railway bridges under a moving train and they used a bridge track-vehicle element. Another model was proposed by Esmailzadeh and Jalili (2003, 2004), for the case of uniform bridges traversed by moving vehicles. These authors presented a simple computer program to show the efficiency of the model proposed by them.

Nevertheless, by modeling both the bridge and the vehicle it is possible to consider some design aspects, like the passenger comfort, the vehicle stability and the effective impact of the vehicle over the bridge. Wu and Yang (2003) discussed the steady-state response and the riding comfort in the case of moving trains. The mass of the train body and the damping and stiffness of the train suspension are considered. Yau et al (1999) have examined the behavior of rail bridges in high speed cases and the riding comfort of rail cars.

This paper has worked with the Yang's model, which can be found in the articles of Yang and Yau (1997), Yang et al (1995, 1999). The coupled behavior of the bridge and the vehicle is analyzed through a special finite element whose formulation has incorporated the mutual relation between the vehicle and the bridge. In this way, the VBI element (vehicle bridge element interaction) may be introduced in a conventional finite element computational code without major difficulties.

The aim of this paper is to show the applicability of the Yang's model for simple cases of coupled vehicle- bridge vibrations, and to present some results to evaluate the influence of some parameters such as the suspension stiffness and damping, the ballast layer, and the velocity of the vehicle.

2. THE VBI ELEMENT

Consider the moving train crossing the span over a simple bridge as showed in Fig. 1. A schematic model is presented in Fig. 2, where it is possible to see that the car body is supported by a track with stiffness and damping. The train moves over an irregular surface. Due to the rail irregularity and the beam vertical displacement, the vehicle will have a transversal vibration which will excite the beam and vice- versa. Depending on the dynamic characteristics of the vehicle or the bridge, the dynamic effects may expand and a dynamic instability may occur. Usually, this instability affects only the passenger's comfort. In severe cases, however, it may cause a disaster, due to the absence of contact between the vehicle and the bridge, or even due to the resonant behavior of the bridge.

Figure 1 – Moving vehicle over simple bridge

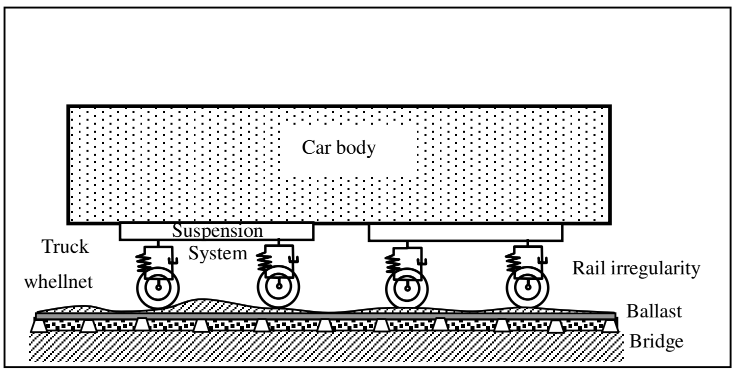


Figure 2 – Vehicle model over the beam

As it can be seen in the Fig. 3, the dynamic structural properties involved in this problem are: m_v, c_v, k_v which are respectively the vehicle mass, the damping and the stiffness; m_w is the whellnet mass; k_B is the ballast stiffness; $r_c(x)$ is the rail irregularity; $s(t)$, $v(t)$ are the position and the velocity of the vehicle; $f_c(t)$ is the contact force between the vehicle and the bridge; $z_1(t)$ and $z_2(t)$ are two degrees of relative freedom to the transversal position of the whellnet and the vehicle mass; x_j is the X-coordinate for the j^{th} node; u_1, u_2, u_3 and u_4 are the nodal displacements of a linear bridge finite element, whose values can be arranged in a vector form $\{u_b\}$.

This study is limited to the case of $f_c(t) \geq 0$.. This means that the vehicle will never escapes off the bridge surface, which is also a stability condition for traveling. Therefore, the contact force is applied simultaneously for the vehicle and the bridge. The difficult is that the contact force intensity is determined only iteratively because it depends on the displacements, velocities and accelerations of both, vehicle and the structure, which have different dynamic behaviors.

Consider { } as a column vector, ⟨ ⟩ as its transpose and [ ] as a square matrix. Suppose that $\{N_c\}$ is a vector with the hermitian polynomials for the local degrees of freedom. Since the contact force acts on the bridge, on the ballast and on the vehicle element simultaneously, it is possible to write the following equations:

$$f_c = k_B\left(\langle N_c\rangle\{u_b\} + r_c - z_1\right) \geq 0 \text{ - for the ballast} \tag{1}$$

$$[m_b]\{\ddot{u}_b\} + [c_b]\{\dot{u}_b\} + [k_b]\{u_b\} = \{N_c\}f_c \text{ - for the bridge element} \tag{2}$$

$$f_c = -p + m_w\ddot{z}_1 + M_v\ddot{z}_2 \text{ - for the vehicle element} \tag{3}$$

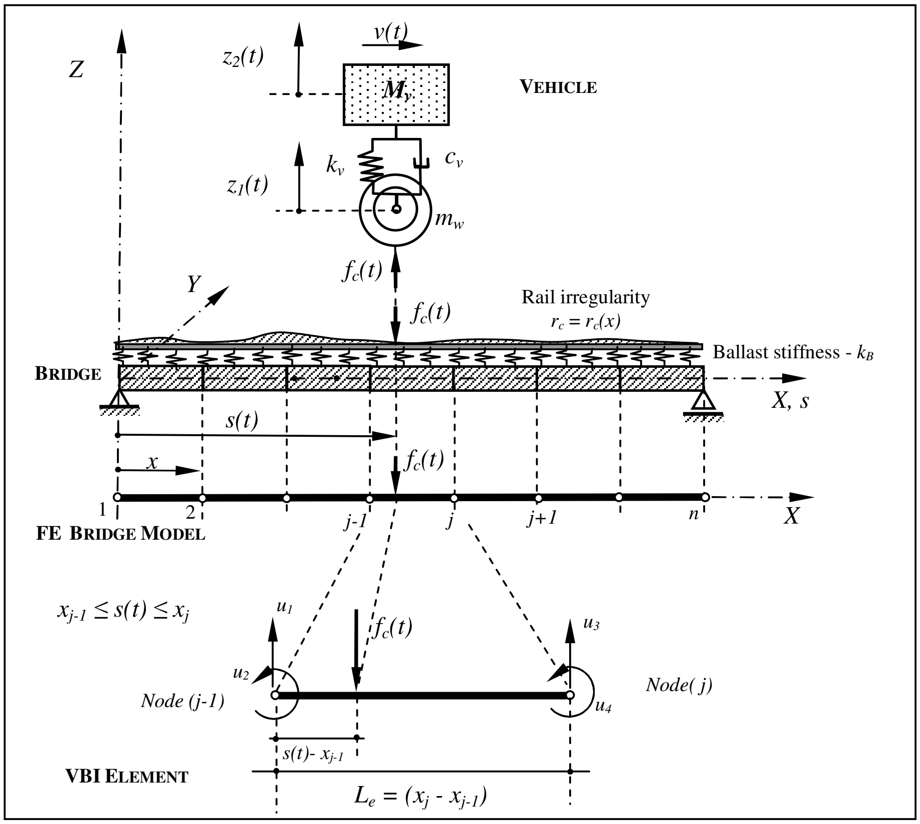


Figure 3 – Parameters for the VBI element

The equilibrium condition for each time step over imposes the equality of the three above equations. The solution, by its turn, must be obtained iteratively. Yang and Yao (1997) have proposed a method for solving the coupled condition, which is adopted in this paper. The main steps will be presented here.

2.1. Dynamic equations for the vehicle element

For the two degrees of freedom vehicle element, it is possible to write:

$$\begin{bmatrix} m_w & 0 \\ 0 & M_v \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} c_v & -c_v \\ -c_v & c_v \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} + \begin{bmatrix} k_v & -k_v \\ -k_v & k_v \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} p + f_c \\ 0 \end{Bmatrix} \tag{4}$$

Solving Eq. (4) in an incremental way, the discrete system equation will be equal to:

$$\begin{bmatrix} m_w & 0 \\ 0 & M_v \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix}_{t+\Delta t} + \begin{bmatrix} c_v & -c_v \\ -c_v & c_v \end{bmatrix} \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}_{t+\Delta t} + \begin{bmatrix} k_v + k_B & -k_v \\ -k_v & k_v \end{bmatrix} \begin{Bmatrix} \Delta z_1 \\ \Delta z_2 \end{Bmatrix} = \begin{Bmatrix} p + k_B(\langle N_c \rangle \{u_b\} + r_c \\ 0 \end{Bmatrix}_{t+\Delta t} - \begin{Bmatrix} q_{s1} \\ q_{s1} \end{Bmatrix}_t \tag{5}$$

where

$$\begin{Bmatrix} q_{s1} \\ q_{s2} \end{Bmatrix} = \begin{bmatrix} k_v + k_B & \\ -k_v & k_v \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_t \tag{6}$$

Using Newmark Method to approximate the temporal variables, one obtains:

$$\{\dot{z}\}_{t+\Delta t} = \{\dot{z}_t\} + [(1-\gamma)\{\ddot{z}\}_t + \gamma\{\ddot{z}\}_{t+\Delta t}]\Delta t \tag{7}$$

$$\{z\}_{t+\Delta t} = \{z\}_t + \{\dot{z}\}_t \Delta t + [(0.5-\beta)\{\ddot{z}_{t+\Delta t}\}]\Delta t \tag{8}$$

Taking the form

$$\{z\}_{t+\Delta t} = \{z\}_t + \{\Delta z\} \tag{9}$$

the acceleration and velocity vectors will be:

$$\{\ddot{z}\}_{t+\Delta t} = a_0\{\Delta z\} - a_2\{\dot{z}\}_t + a_3\{\ddot{z}\}_t \tag{10}$$

$$\{\dot{z}\}_{t+\Delta t} = \{\dot{z}\}_t + a_6\{\ddot{z}\}_t + a_7\{\ddot{z}\}_{t+\Delta t} \tag{11}$$

where

$$\begin{aligned} &a_0 = \frac{1}{\beta \Delta t^2}\,;\, a_1 = \frac{\gamma}{\beta \Delta t}\,;\, a_2 = \frac{1}{\beta \Delta t}\,;\, a_3 = \frac{1}{2\beta} - 1\,;\, a_4 = \frac{\gamma}{\beta} - 1 \\ &a_5 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right);\, a_6 = \Delta t(1-\gamma)\,;\, a_7 = \gamma \Delta t \end{aligned} \tag{11}$$

The incremental approximation results in:

$$\begin{bmatrix} k_v + k_B + a_0 m_\omega + a_1 c_v & -k_v - a_1 c_v \\ -k_v - a_1 c_v & k_v + a_0 M_v + a_1 c_v \end{bmatrix} \begin{Bmatrix} \Delta z_1 \\ \Delta z_2 \end{Bmatrix} = \begin{Bmatrix} p + k_B r_c + k_B \langle N_c \rangle \{u_b\} \\ 0 \end{Bmatrix}_{t+\Delta t} - \left(\begin{Bmatrix} q_{s1} \\ q_{s2} \end{Bmatrix} + \begin{Bmatrix} q_{e1} \\ q_{e2} \end{Bmatrix} \right) \tag{12}$$

where

$$q_{e1,t} = -m_\omega(a_2 \dot{z}_1 + a_3 \ddot{z}_1) - c_v[a_4(\dot{z}_1 - \dot{z}_2) + a_5(\ddot{z}_1 - \ddot{z}_2)] \tag{13}$$

$$q_{e2,t} = -M_v(a_2 \dot{z}_2 + a_3 \ddot{z}_2) - c_v[a_4(\dot{z}_2 - \dot{z}_1) + a_5(\ddot{z}_2 - \ddot{z}_1)] \tag{14}$$

Yang and Yao (1997) showed that the transverse increment displacements may be determined by:

$$\begin{aligned} \begin{Bmatrix} \Delta z_1 \\ \Delta z_2 \end{Bmatrix} = -\frac{1}{D} &\begin{Bmatrix} k_v + a_0 M_v + a_1 c_v \\ k_v + a_1 c_v \end{Bmatrix} \left(p + K_B r_{c,t+\Delta t} + k_B \langle N_c \rangle \{u_b\}_{t+\Delta t}\right) - \\ &\frac{1}{D} \begin{Bmatrix} (q_{s1,t} + q_{e1,t}) a_0 M_v + (\hat{q}_{s,t} + \hat{q}_{e,t})(k_v + a_1 c_v) \\ (q_{s1,t} + q_{e1,t})(a_0 m_\omega + k_B) + (\hat{q}_{s,t} + \hat{q}_{e,t}) \end{Bmatrix} \end{aligned} \tag{15}$$

where

$$\hat{q}_{e,t} = (q_{e1} + q_{e2})_t\ ,\ \hat{q}_{s,t} = (q_{s1} + q_{s2}) = k_B z_{1,t} \tag{16}$$

$$D = \begin{vmatrix} k_v + k_B + a_o m_\omega + a_1 c_v & -k_v - a_1 c_v \\ -k_v - a_1 c_v & k_v + a_0 M_v + a_1 c_v \end{vmatrix} \tag{17}$$

Taking the following matrix and vectors as:

$$[\hat{k}_b] = [k_b] + k_B \frac{a_0}{D}[(M_v + m_\omega) + (k_v + a_1 c_v) + a_0 M_v m_\omega]\{N_c\}\langle N_c \rangle \tag{18}$$

$$\{p_s\}_{t+\Delta t} = -k_B\left[r_{c,t+\Delta t} - \left(p + k_B r_{c,t+\Delta t}\right)\frac{1}{D}\left(k_v + a_0 M_v + a_1 c_v\right)\right]\{N_c\} \tag{20}$$

$$\{f_s\}_t = k_b\left[\frac{1}{D}\left\{\left(q_{s1,t} + q_{e1,t}\right)a_0 M_v + \left(\hat{q}_{s,t} + \hat{q}_{e,t}\right)\left(k_v + a_1 c_v\right)\right\} - z_{1,t}\right]\{N_c\} \tag{21}$$

Since the incremental transverse displacement vector is known, Eq. (15), the displacement, acceleration and velocity vectors for the vehicle element can be determined by:

$$\{z\}^i_{t+\Delta t} = \{z\}^i_{t+\Delta t} + \{\Delta z\} \tag{22}$$

$$\ddot{z}^i_{t+\Delta t} = a_0\{\Delta z\}^i - a_2\{\dot{z}\}^{i-1}_{t+\Delta t} - a_3\{\ddot{z}\}^{i-1}_{t+\Delta t} \tag{23}$$

$$\dot{z}^i_{t+\Delta t} = \{z\}^{i-1}_{t+\Delta t} - a_6\{\ddot{z}\}^{i-1}_{t+\Delta t} - a_7\{\ddot{z}\}^i_{t+\Delta t} \tag{24}$$

2.2. Dynamic equations for the local bridge element

The displacement bridge vector may be decomposed by:

$$\{u_b\}_{t+\Delta t} = \{u_b\}_t + \Delta u_b \tag{25}$$

In this way, the dynamic equation for the bridge element will be as:

$$[m_b]\{\ddot{u}_b\}_{t+\Delta t} + [c_b]\{\dot{u}_b\}_{t+\Delta t} + [k_b]\{u_b\}_{t+\Delta t} = p_{t+\Delta t} \tag{26}$$

Substituting Eq. (25) into (26) results:

$$[m_b]\{\ddot{u}_b\}_{t+\Delta t} + [c_b]\{\dot{u}_b\}_{t+\Delta t} + [k_b]\{\Delta u_b\} = \{p_b\}_{t+\Delta t} + p\{N_c\} - [k_b]\{u_b\}_t \tag{27}$$

Using equations (18) to (20) one obtains:

$$[m_b]\{\ddot{u}_b\}_{t+\Delta t} + [c_b]\{\dot{u}_b\}_{t+\Delta t} + \left[\hat{k}_b\right]\{\Delta u_b\} = \left(\{p_{t+\Delta t}\} + \{p_s\}_{t+\Delta t}\right) - \left(\{f_s\}_t + \left[\hat{k}_b\right]\{u_b\}_t\right) \tag{28}$$

2.3. Dynamic equations for the global bridge structure

Equations (25) and (26) are related to the VBI element, which is determined only for that finite element over which the vehicle is passing on. For free vehicle elements, the dynamic vectors and matrices are developed in a conventional FE way. It must be remembered that, since the vehicle is moving, the conditions whether or not the element is a free vehicle must be checked in each time step. In the sense to build the global equation system for the bridge, and taking $[M_b]$, $[C_b]$, $[K_b]$ the global bridge matrices; $\{U_b\}$ and $\{\Delta U_b\}$ the global displacements and incremental displacements vectors respectively; $\{P_b\}$ and $\{F_b\}$ the external load vector and equivalent load vector applied, one obtains:

$$[M_b]\{\ddot{U}_b\}_{t+\Delta t} + [C\ ]\{\dot{U}_b\}_{t+\Delta t} + [K_b]\{\Delta U_b\} = \{P_b\}_{t+\Delta t} - \{F_b\}_t \tag{29}$$

where

$$\{U_b\}_{t+\Delta t} = \{U_b\}_t + \{\Delta U_b\} \tag{30}$$

$$\{P_b\}_{t+\Delta t} = \sum_{eim=1}^{n}\left[\{p_b\}_{t+\Delta tt} + \{p_s\}_{t+\Delta t}\right] \tag{31}$$

$$\{F_b\}_t = \sum_{eim=1}^{n}\left[\{f_s\}_t + \left[\hat{k}_b\right]\{u_b\}_t\right] \tag{32}$$

$$\left\{\ddot{U}_b\right\}_{t+\Delta t} = a_0\left\{\Delta U_b\right\} - a_2\left\{\dot{U}_b\right\}_t - a_3\left\{\ddot{U}\right\}_t \tag{33}$$

$$\left\{\dot{U}_b\right\}_{t+\Delta t} = \left\{\dot{U}_b\right\}_t + a_6\left\{\ddot{U}_b\right\}_t + a_7\left\{\ddot{U}\right\}_{t+\Delta t} \tag{34}$$

Equation (29) may be simplified to:

$$\left[\overline{K}_b\right]_{t+\Delta t}\left\{\Delta U_b\right\} = \left\{P_b\right\}_{t+\Delta t} - \left\{\overline{F}_b\right\}_t \tag{35}$$

where, using the Newmark's coefficients, oneobtains:

$$\left[\bar{K}_b\right]_{t+\Delta t} = a_0\left[M_b\right] + a_1\left[C_b\right] + \left[K_b\right] \tag{36}$$

and

$$\left\{F_b\right\}_t = \left\{F_b\right\}_t - \left[M_b\right]\left(a_2\left\{\dot{U}_b\right\}_t + a_3\left\{\ddot{U}_b\right\}_t\right) - \left[C_b\right]\left(a_4\left\{\dot{U}\right\}_t + a_5\left\{\ddot{U}_b\right\}_t\right) \tag{37}$$

So, the incremental solution is taking in the form:

$$\left[K\right]_{t+\Delta t}\left\{\Delta U_b\right\}^i = \left\{P_b\right\}_{t+\Delta t} - \left\{F_b\right\}_{t+\Delta t}^{i-1} \tag{38}$$

$$\left\{F\right\}_{t+\Delta t}^i = \left\{F_b\right\}_{t+\Delta t}^{i-1} - \left[M_b\right]\left(a_2\left\{\dot{U}_b\right\}_{t+\Delta t}^{i-1} + a_3\left\{\ddot{U}_b\right\}_{t+\Delta t}^{i-1}\right) - \left[C_b\right]\left(a_4\left\{\dot{U}_b\right\}_{t+\Delta t}^{i-1} + a_5\left\{\dot{U}_b\right\}_{t+\Delta t}^{i-1}\right) \tag{39}$$

For each time step, the initial condition is adopted by:

$$\left\{F\right\}_{t+\Delta t}^0 = \left\{F\right\}_t^i \qquad \left\{U_b\right\}_{t+\Delta t}^0 = \left\{U\right\}_t^i \tag{40}$$

Solving the equation system for each time step, the global bridge displacements, accelerations and velocities vectors will be obtained through:

$$\left\{U_b\right\}_{t+\Delta t}^i = \left\{U_b\right\}_{t+\Delta t}^{i-1} + \left\{\Delta U_b\right\}^i \tag{41}$$

$$\left\{\ddot{U}_b\right\}_{t+\Delta t}^i = a_0\left\{\Delta U_b\right\}^i - a_2\left\{\dot{U}_b\right\}_{t+\Delta t}^{i-1} - a_3\left\{\ddot{U}\right\}_{t+\Delta t}^{i-1} \tag{42}$$

$$\left\{\dot{U}_b\right\}_{t+\Delta t}^i = \left\{\dot{U}_b\right\}_{t+\Delta t}^{i-1} - a_6\left\{\ddot{U}_b\right\}_{t+\Delta t}^{i-1} - a_7\left\{\ddot{U}_b\right\}_{t+\Delta t}^i \tag{43}$$

It must be noted that Eq. (38) depends on the values of $\{P_b\}$ and $\{F_b\}$ vectors. But these values are determined by Eq. (31) and (32), which depends on the vectors and matrices of Eq. (18) to (20). So, the solution must be obtained for each time step in an iterative form.

3. APPLICATIONS

To show the applicability of this method, let us consider a bridge with a straight uniform beam, simple supported, whose geometric and material properties are: L = 25 m; I = 2.90 m^4; m = 2.303 kg/m and E = 2.87 GPa. Assumes, in addition, the following dynamic properties: M_v = 5750 kg; m_w = 0,10 Mv; k_v = 1595 kN/m; c_v = 0,06 k_v. All examples were developed with ten linear bridge elements. The vehicle velocity is equal to 27.8 m/s. The results of the present paper are in continuum lines. Dashed lines are refereed to the Yang and Yao (1997) results.

Figure 4 shows the mid-span beam displacement for the case of moving spring-mass vehicle, where the damping effect was neglected. The incremental steps were equal to 10^{-3} s. The results show accordance between the present and the refereed article. A little delay has been observed, but extremes values are coincident.

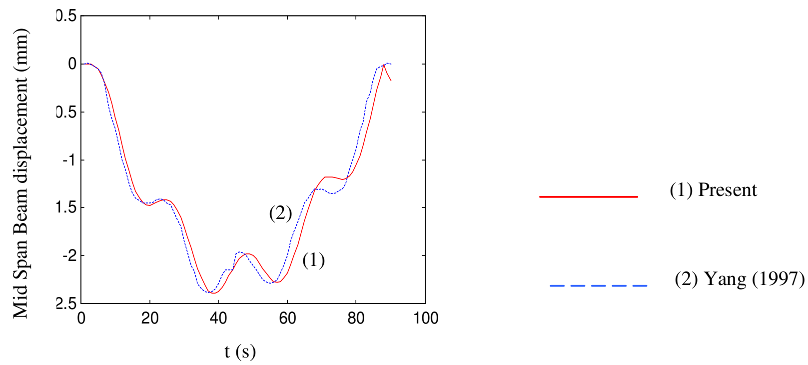
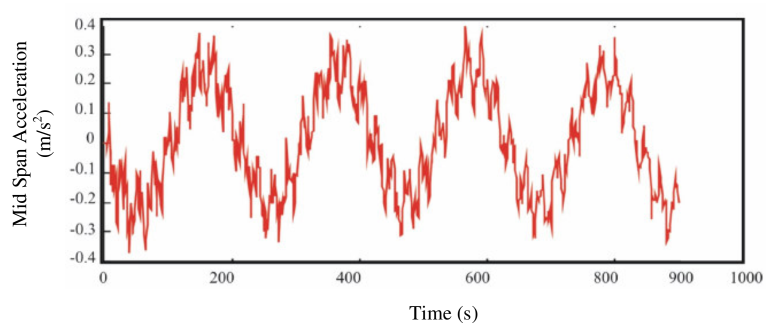
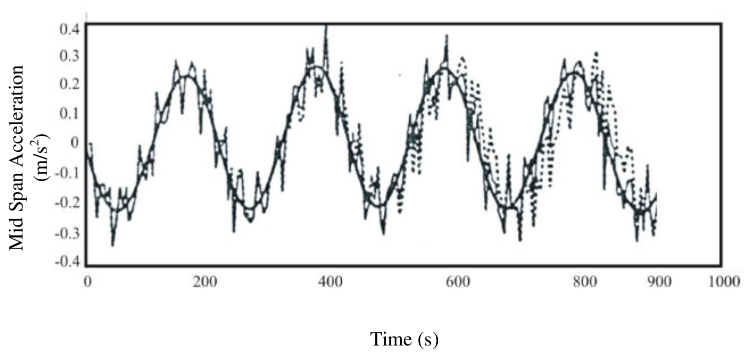


Figure 4 – Mid-span beam displacement for the mass-spring vehicle



(a) Present Work



(b) Yang and Yau (1997)

Figure 5 – Mid Span Beam acceleration: (a) Present Work; (b) Yang and Yau (1997)

To compare the different behaviors of the moving mass and force, it is shown in Fig. 5 the mid-span beam vertical displacement. It is possible to see that the maximum deflection of the moving mass is similar to the moving force, but in the along the beam, moving mass deflections are as bigger than moving force.

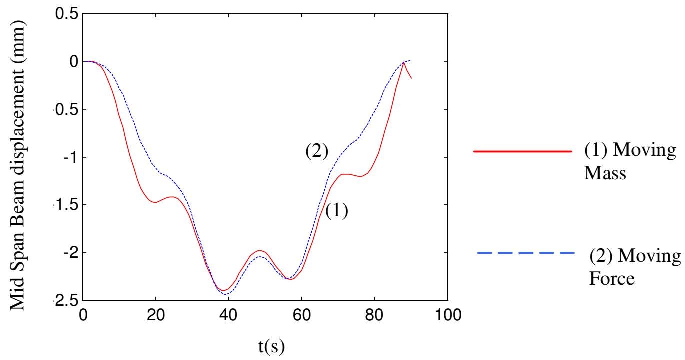


Figure 5 – Mid-span beam displacement for the moving mass and moving force.

The vertical displacement of the vehicle is affected by its traveling velocity. Figure 6 shows two cases, on the first one the velocity is equal to 20 m/s and, on the other, it is equal to 40 m/s. The higher reaches the maximum displacement vehicle values and its oscillation amplitude is more accentuated than the other.

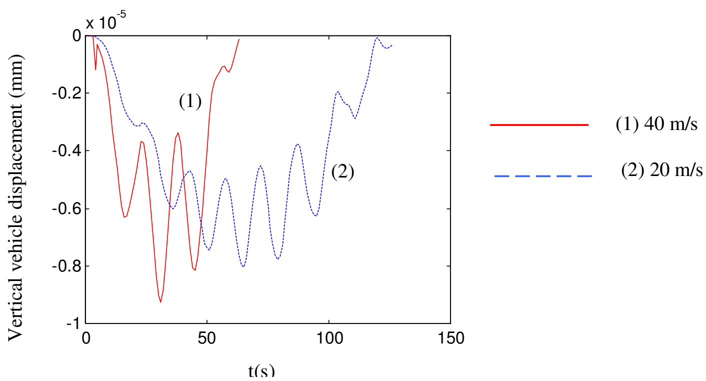


Figure 7 – VBI vertical displacement for different velocities

The influence of ballast stiffness for two different values is shown in Figure 8 for the beam and for the vehicle. Figures 9 and 10 show the dynamic influence of the vehicle stiffness and damping.

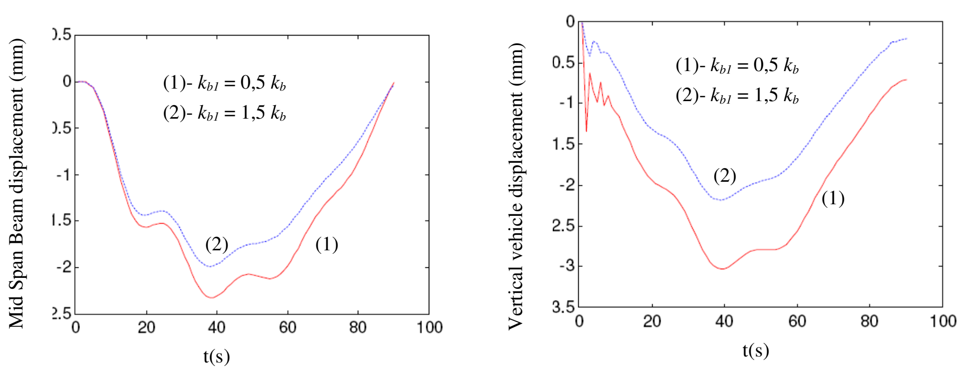


Figure 8 - Influence of ballast stiffness: (a) Beam displacement; (b) Vehicle displacement

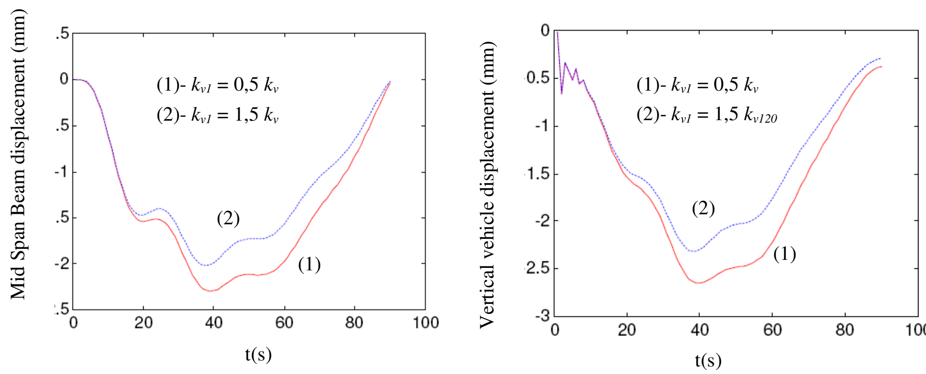


Figure 9 – Influence of vehicle stiffness: (a) Beam displacement; (b) Vehicle displacement

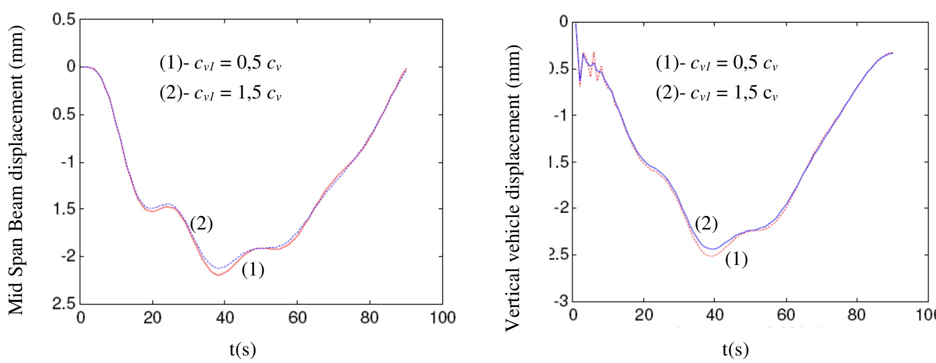


Figure 10 – Influence of Vehicle damping: (a) Beam displacement; (b) Vehicle displacement

4. CONCLUSIONS

This article showed a coupled model of vehicle bridge interactions. The model was proposed by Yang and Yao (1997) and it considers the influence of the vehicle dynamic responses to the bridge responses and vice-e-versa. The coupling implies iterative solution. The main hypothesis is that there are no gaps between the vehicle and the bridge, which means that the vehicle is in constant contact on the structure. The contact force is achieved iteratively since it affects the vehicle, the ballast and the bridge structure. The VBI element matrices are developed only for the FE elements which were supporting the moving car. For vehicle free elements, the matrices are the same as the FE Method. Newmark's procedure was used to approximate the transient solution. In each time step, the dynamic equation system was solved iteratively. Some examples were presented to show the applicability of the method. For other applications, it is possible to implement multiples VBIs to simulate a train composition or a real dynamic automotive analysis.

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