DETERMINING ECONOMICAL SPEEDS FOR AIRCRAFT: A COMPARISON OF PERFORMANCE DATA PROVIDED BY THE SPECIFIC-ENERGY METHOD AND THE POINT-MASS MODEL

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Abstract. *The determination of economical flight trajectories is a very difficult task due to the non-linear nature of the aircraft's equations of motion. In order to simplify these equations, a method named "Specific Energy" can be used. In this method, the integration of the equations of motion is performed using the specific energy as an independent variable. An algorithm using this methodology and the calculus of variations was developed in the seventies for the calculation of optimized flight paths, in which the parameter to be minimized is the Direct Operational Cost. In this paper, the basic algorithm was implemented using performance data applicable for a modern medium-haul jet commercial aircraft. The optimized climb and descent profiles, using the fixed thrust and the free-thrust methods, were analyzed in order to determine a speed and thrust control law as a function of the flight altitude, the initial weight and the sigma (factor used to ponder the cost of time and the fuel cost). The performance figures obtained were compared to the ones provided by the point-mass model, using the same speed and thrust control laws produced by the optimization algorithm. Thus, the validity of using this simplified method can be assessed.*

Keywords*: flight mechanics, flight optimization, economical speeds, aircraft performance*

1. INTRODUCTION

The optimization of the climb and descent flight paths is a very complicated problem owing to the non-linear airplane's equations of motion. The simplification of this problem was proposed in some works, like the ones described in (Erzberger et al, 1975) and (Barman e Erzberger, 1975). These papers reorganize the equations of motion by replacing the time as the independent variable during the integration, substituting it by another parameter called "specific energy". The calculus of variations is also employed in order to determine the optimized flight path in climb and descent.

The basic algorithm, described in these papers, was used to generate the optimized flight trajectories based on performance data applicable to a modern medium-haul jet airliner. These optimized flight paths were synthesized to generate the speed profile in climb and descent as a function of the sigma (factor that measures the relative importance of the cost of time and the fuel cost) and the initial weight. In order to assess the differences between the performance given by these two methods ("point-mass" and "specific-energy"), the integration of the equations of motion were performed for the same weights and sigma, using the altitude as the independent variable. The differences of final time, total distance and total fuel consumption were thus analyzed.

2. PERFORMANCE INDEX DEFINITION

The operational costs in an airline are related to the aircraft's operations. These costs are divided in the "Direct Operating Costs" (fuel costs, flight and cabin crew wages, maintenance activities etc.) and in the "Indirect Operational Costs" (maintenance tooling and manuals, hangars and so on). Essentially, the DOC can be divided in two basic components: the cost of the fuel consumed to fly in given route and the cost of the time required. Thus, the DOC can be expressed mathematically through Eq. (1):

$$DOC = C_F.F + C_T.T \tag{1}$$

Where:
DOC: direct operating cost (in currency unit);
C_F: cost of one kg of fuel (currency unit/kg);
F: total fuel consumption in a given route (kg);
C_T: cost of time (currency unit/s);
T: total flight time (s).

The DOC can be parameterized by using a factor that measures the relative importance of the fuel cost and the cost of time. This factor, represented by the Greek letter σ, is expressed by Eq. (2):

$$\sigma = \frac{C_F / C_T}{1 + C_F / C_T} \quad (2)$$

σ varies from 0 (zero) up to 1 (one). When σ is equal to zero, it can be assumed that the fuel cost is negligible with regard the cost of time. If σ is equal to 1, it means that the flight trajectories must minimize the fuel consumption.

Thus, the DOC to fly a given route can be modeled in an alternative way and it can be expressed as a function of the σ as represented by Eq. (3):

$$J = \int_0^T [\sigma.FF + (1 - \sigma)] \mathrm{dt} \quad (3)$$

Where: J: equivalent cost (currency unit);
T: total flight time (s);
FF: fuel flow (kg/s).

3. OPTIMIZATION ALGORITHM

From the equations of motion, it is known that:

$$\dot{V} = \frac{F \bullet \cos\alpha - D}{m} - g \bullet sen\gamma \quad (4)$$

Where: $\dot{V}$: aircraft's acceleration (m/s^2);
F: total thrust (N);
α: angle of attack (radian);
D: drag (N);
m: mass (kg);
g: gravity's acceleration (m/s^2);
γ: climb angle (radian).

Assuming that the angle of attack is so small that it can be considered as negligible, Eq. (4) can be simplified as follows:

$$\dot{V} = \frac{F - D}{m} - g.sen\gamma \quad (5)$$

The aircraft's specific energy is defined by the following equation:

$$E = \frac{1}{2}.V^2 + g.h \quad (6)$$

Where: E: aircraft's specific energy (m^2/s^2);
V: aircraft's speed (m/s);
g: gravity's acceleration (m/s^2);
h: flight height (m).

The rate of variation of the aircraft's specific energy can be determined by deriving Eq. (6) with regard the time, resulting in:

$$\dot{E} = V.\dot{V} + g.\dot{h} \quad (7)$$

The rate of climb can be expressed by Eq. (8):

$$\dot{h} = V.sen\gamma \tag{8}$$

Inserting the Eq. (5) and (8) in Eq. (7), we obtain:

$$\dot{E} = V.\left(\frac{F-D}{m}\right) - V.g.sen\gamma + g.V.sen\gamma \Rightarrow \dot{E} = V.\left(\frac{F-D}{m}\right) \tag{9}$$

Another assumption is the negligibility of the climb angle. So, the lift is equal to the weight. Another consequence of this hypothesis is that the distance flown by the aircraft is expressed by the integral of the aircraft's speed, as demonstrated by Eq. (10):

$$R = \int_0^T (V + V_W)\,\mathrm{dt} \tag{10}$$

Where: R: distance to be flown (m);
V: aircraft's speed (m/s);
V_W: wind speed (m/s).

The determination of the optimum flight path during the climb and the descent is carried out by replacing the time as the independent variable during the integration of the aircraft's equation of motion. The integration is carried out by using the specific energy as the independent variable. This change is exposed in Eq. (11):

$$\dot{E} = \left(\frac{dE}{dt}\right) \Rightarrow dt = \left(\frac{dE}{\dot{E}}\right) \tag{11}$$

After this replacement, the total cost to fly a route composed of climb, cruise and descent, as expressed by Eq. (3), can be re-written as Eq. (12):

$$J = \int_{Ei}^{Ec}\left(\frac{P_{c\lim b}}{\dot{E}}\right)dE + (\mathrm{R} - \mathrm{d}_{\mathrm{climb}} - d_{descent}).\lambda + \int_{Ec}^{Ef}\left(\frac{P_{descent}}{\dot{E}}\right)\mathrm{dE} \tag{12}$$

Where: P: equivalent cost;
E_i: initial aircraft's specific energy (m^2/s^2);
E_c: aircraft's specific energy at the Top of Climb (TOC) (m^2/s^2);
E_f: final aircraft's specific energy (m^2/s^2);
R: total distance to be flown by the aircraft (m);
λ: cost of flight in cruise, per meter;
d_{climb}: climb distance (m);
$d_{descent}$: descent distance (m).

The equivalent cost P is expressed by Eq. (13):

$$P = \sigma.FF + (1-\sigma) \tag{13}$$

After the change of the variable of integration, the equations that describe the distances flown in climb and descent can be written as:

$$\mathrm{d}_{\mathrm{climb}} = \int_{Ei}^{Ec}\left(\frac{V_{\mathrm{climb}} + V_{w,\mathrm{climb}}}{\dot{E}}\right)dE \tag{14}$$

$$\mathrm{d}_{\mathrm{descent}} = \int_{Ec}^{Ef}\left(\frac{V_{\mathrm{descent}} + V_{w,\mathrm{descent}}}{\dot{E}}\right)dE \tag{15}$$

Thus, inserting Eq. (14) and (15) into Eq. (12), we obtain:

$$J = \int_{Ei}^{Ec} \left[\frac{P_{c\lim b}}{\dot{E}} - \lambda . \left(\frac{V_{c\lim b} + V_{w,c\lim b}}{\dot{E}} \right) \right] dE + \lambda.\mathrm{R} + \int_{Ec}^{Ef} \left[\frac{P_{descent}}{\dot{E}} - \lambda . \left(\frac{V_{descent} + V_{w,descent}}{\dot{E}} \right) \right] \mathrm{dE} \quad (16)$$

As we are considering only the climb and descent flight phases, so the costs to be minimized are:

$$J_{c\lim b} = \int_{Ei}^{Ec} \left[\frac{P_{c\lim b}}{\dot{E}} - \lambda . \left(\frac{V_{c\lim b} + V_{w,c\lim b}}{\dot{E}} \right) \right] dE \quad (17)$$

$$J_{descent} = \int_{Ec}^{Ef} \left[\frac{P_{descent}}{\dot{E}} - \lambda . \left(\frac{V_{descent} + V_{w,descent}}{\dot{E}} \right) \right] \mathrm{dE} \quad (18)$$

The minimization of Eq. (17) and (18) is carried out initially by minimizing λ. The optimum value of λ, represented by λ_{opt}, is given by Eq. (19):

$$\lambda_{opt}(E_{\max}) = \min_{H,V} \frac{[\sigma.FF + (1-\sigma)]}{V + V_W} \quad (19)$$

The program determines the optimum λ by iterating the value of altitude and speed. For each combination of altitude and speed (resulting in a given specific energy level), the required thrust means a certain fuel flow, resulting in different values of λ as demonstrated by Fig. 1.

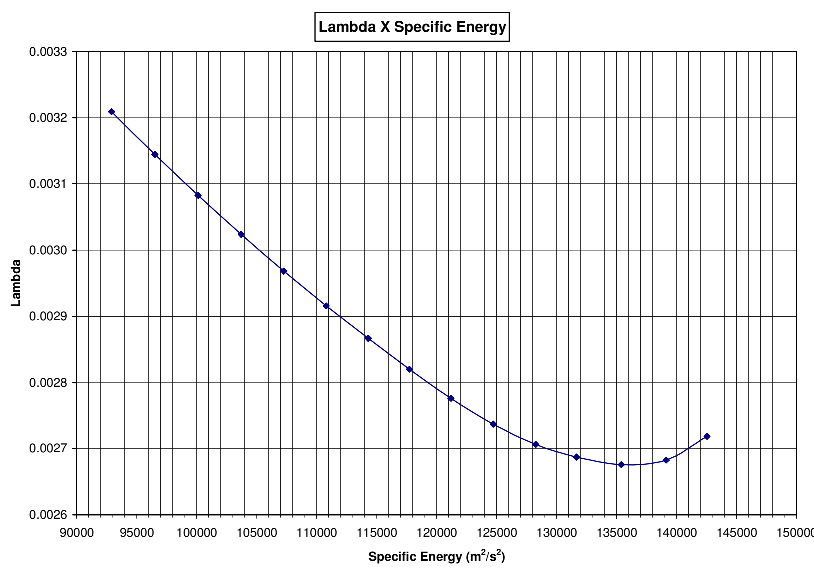

Figure 1. λ as a function of the specific energy (σ=1, no wind, standard conditions, mass = 60.000 kg).

The determination of the optimum λ depends on the weight at the end of the climb ("Top of Climb", TOC). This weight is not known beforehand, so the program assumes initially that the weight at the TOC is the same weight at the beginning of the flight. After reaching the TOC, the program determines a tentative weight equal to the final weight plus half of the fuel burned in climb. The optimum altitude and speed for this weight constitute the new final energy in climb, so the program starts the integration again using this new final energy as a limit for integration. The iteration ends when the difference between optimums altitudes is within 50 meters. The descent calculation doesn't require the same process.

During the integration, for each energy level, the optimum speed is determined by minimizing the integrand in Eq. (17) and (18). After determining the optimum speed, the altitude is obtained by solving Eq. (20).

$$h = \frac{(2.E - V^2)}{2.g} \tag{20}$$

Where: h: flight height (m);
E: specific energy (m^2/s^2);
V: aircraft's speed (m/s);
g: gravity's acceleration (m/s^2).

As the same specific energy can be obtained with a higher speed in a lower altitude than the optimum, the program checks if the current Mach number is higher than the optimum. If this limit is exceeded, so the speed is restricted. This characteristic assures that the final altitude will be exactly the optimum altitude. Moreover, the program doesn't allow a negative flight height or flight heights above the optimum altitude (during the descent). The search for the optimum speed is carried out in a range limited by the minimum flight speed (limited by stall or by engine thrust) and by the maximum speed (limited by engine thrust and the certified limits). In the case studied, it was assumed a maximum indicated airspeed (V_{MO}) equal to 340 KIAS and the maximum operating Mach number (M_{MO}) equal to 0.82.

For the point-mass method, in order to follow the same speed profile used by the specific-energy method in climb or descent, the equation that describes the acceleration is changed because the integration is no longer carried out using time as the independent variable. As the speed is a function of the altitude, the aircraft's acceleration is expressed as follows:

$$\dot{V} = \frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt} = \frac{dV}{dh}.V.\sin\gamma \tag{21}$$

4. AERODYNAMIC AND PROPULSIVE DATA

The performance and propulsion data applicable to a modern jet transport aircraft were ingested into this program. The drag polar (flaps up, landing gear up) is shown in Fig. 2.

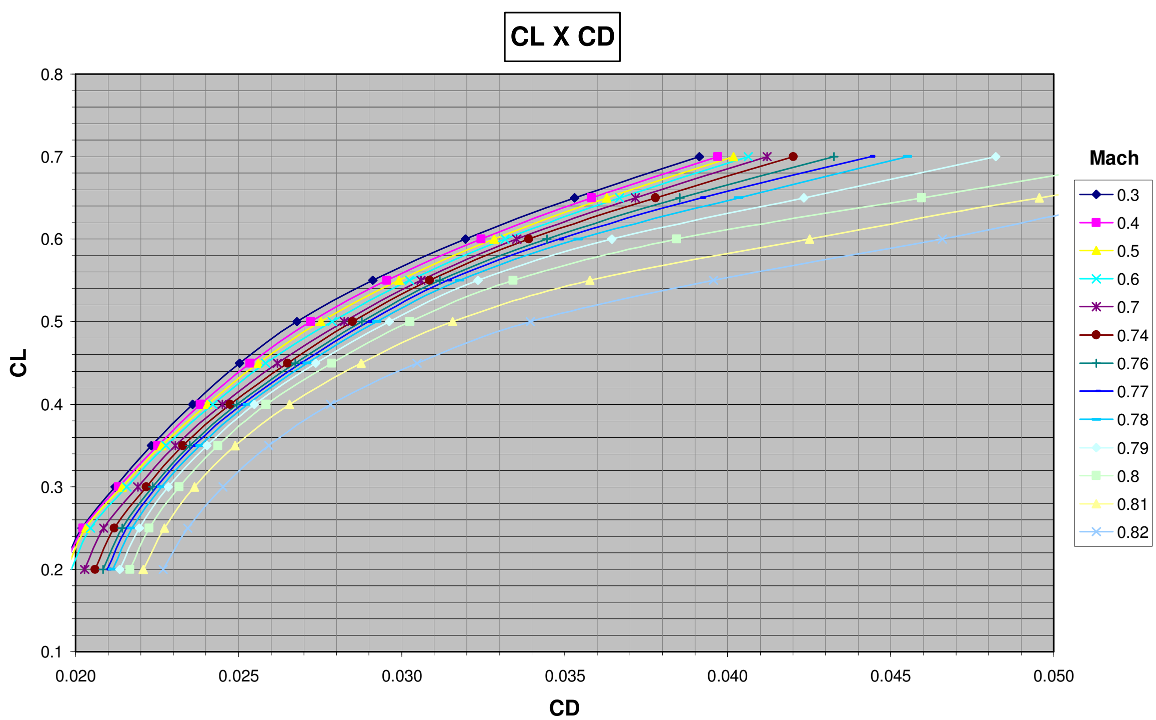

Figure 2. Drag polar for flaps up/landing gear up configuration.

The drag coefficient given by Fig. 2 is corrected by the Reynolds Number, provided by Fig. 3:

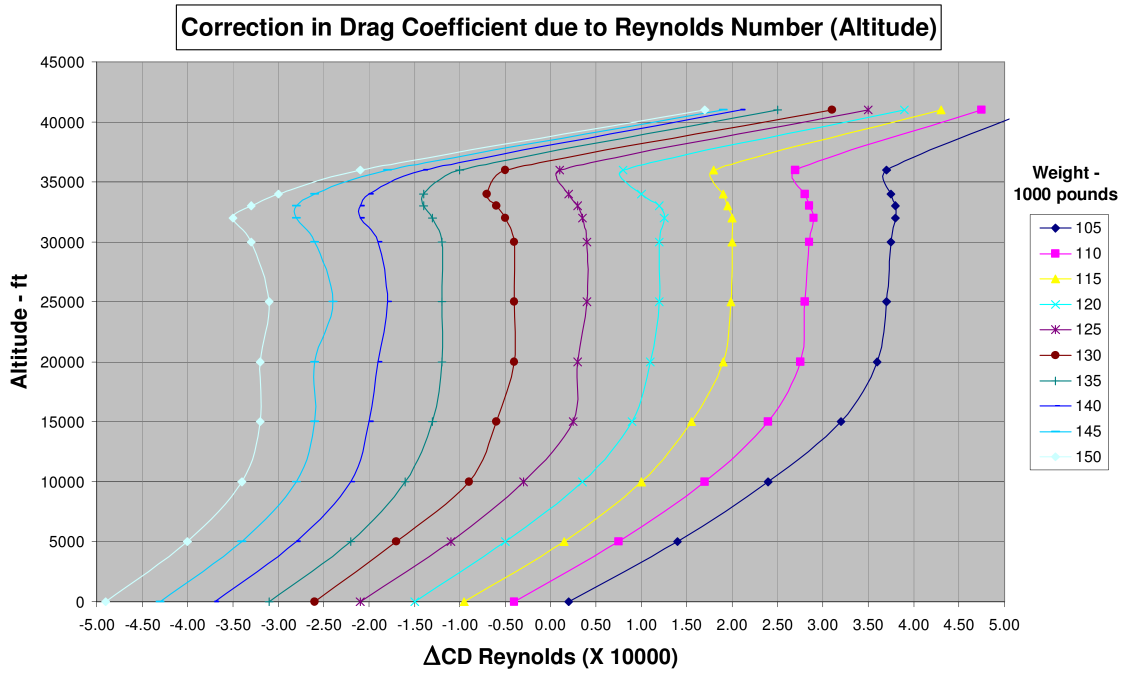

Fig. 3. Correction in drag coefficient due to Reynolds Number.

The determination of the thrust is done using a parameter called $N_1/\sqrt{\Theta}$, which reflects the actual thrust produced by the engine. This parameter can be obtained using the graph shown in Fig. 4.

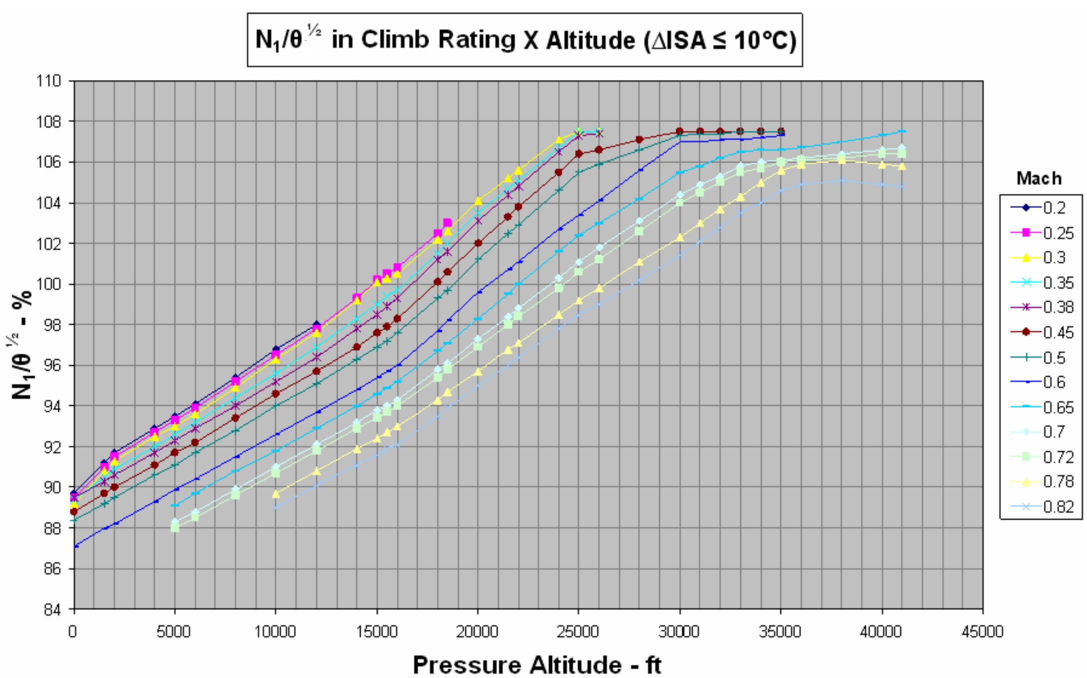

Figure 4. $N_1/\sqrt{\Theta}$ in climb rating.

In descent, the thrust level is the "flight idle". The $N_1/\sqrt{\Theta}$ for flight idle is given by Fig. 5.

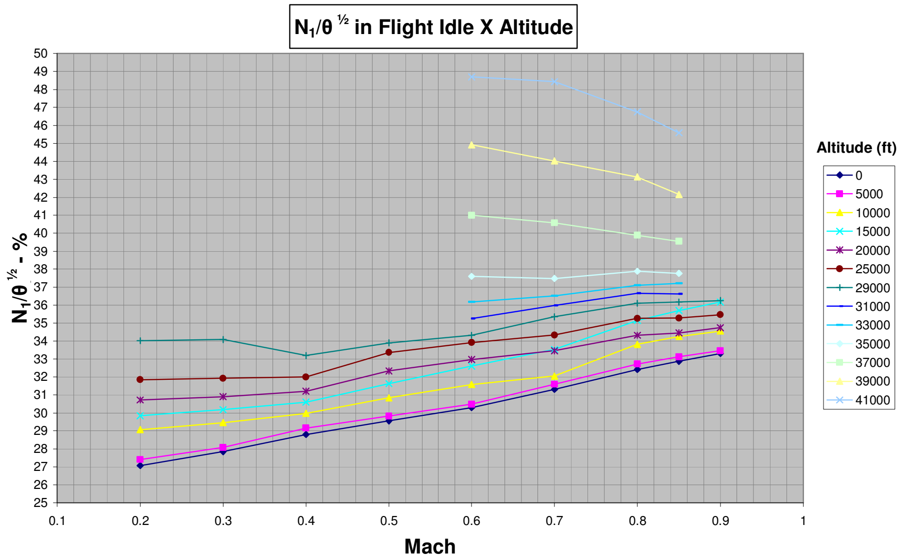

Figure 5. $N_1/\sqrt{\Theta}$ in flight idle.

Once the $N_1/\sqrt{\Theta}$ is determined, the actual thrust produced per engine can be calculated using the graph exposed in Fig. 6.

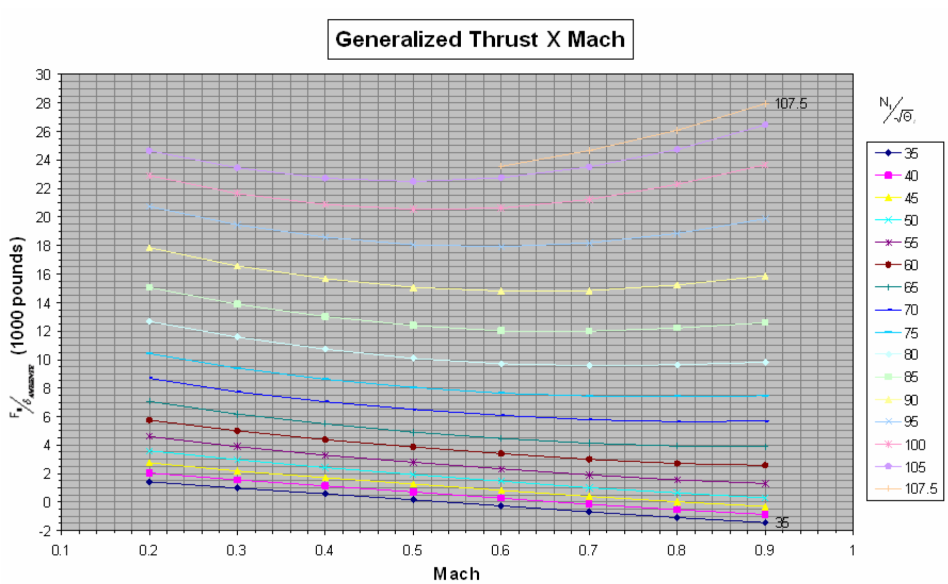

Figure 6. Generalized thrust per engine.

The fuel flow per engine was parameterized according to a method described in Boeing (1989). It is given by Eq. (22):

$$FF = ((-2.3142.M^3+1.4963.M^2+0.5927.M+0.3117).\left(F_N/\delta\right)^3 +$$
$$(78.502.M^3-68,604.M^2-0.2899.M-9.8521).\left(F_N/\delta\right)^2 +$$
$$(-509.68.M^3+382.57.M^2+397.1.M+398.94).\left(F_N/\delta\right) +$$
$$(-1437.7.M^3+3419.2.M^2+521.81.M+248.89)).\ \delta\ .\sqrt{\theta} \quad (22)$$

Where: FF: fuel flow per engine (pounds per hour);
M: Mach number;
F_N: net thrust per engine (pounds);
δ: pressure ratio (dimensionless);
θ: temperature ratio (dimensionless).

δ is given by Eq. (23) and (24):

(pressure altitudes less than or equal 11,000 m, or 36,089.24 feet)

$$\delta = \left(\frac{288.15 - 0.0019812.H_P}{288.15} \right)^{5.25588} \tag{23}$$

(pressure altitudes greater than 11,000 m, or 36,089.24 feet)

$$\delta = 0.22336.e^{\left(\frac{36089.24 - H_P}{20805,.} \right)} \tag{24}$$

Where: H_P: pressure altitude (feet).

θ is given by Eq. (25) and (26):

(pressure altitudes less than or equal 11,000 m, or 36,089.24 feet)

$$\theta = \frac{288.15 - (0.0019812.H_P) + \Delta ISA}{288.15} \tag{25}$$

(pressure altitudes greater than 11,000 m, or 36,089.24 feet)

$$\theta = 0.7519 + \Delta \text{ISA} \tag{26}$$

Where: ΔISA : deviation from the temperature for the same pressure altitude in the ISA atmosphere.

5. RESULTS

The results for climb and descent flight trajectories according to the two methods were compared by calculating their differences in percentage. The performance difference for fuel, for instance, is calculated as follows:

$$\Delta Fuel\,(\%) = 100.\frac{\left(Fuel_{Specific-energy} - Fuel_{Point-mass} \right)}{Fuel_{Point-mass}} \tag{27}$$

The same approach was used to determine the time and distance differences. All these differences are exposed in Tab. 1 to 4.

Table 1. Differences (%) according to the specific-energy and point-mass methods for the time in climb.

	Sigma										
Weight (kg)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
50000	1.01	1.01	0.30	0.30	0.10	-0.30	-0.37	0.56	1.57	1.89	2.03
52000	-0.54	-0.54	-0.26	-0.45	-0.38	-0.27	-0.37	0.62	1.43	1.84	2.03
54000	-0.39	-0.39	-0.14	-0.50	-0.43	-0.30	-0.33	0.61	1.25	1.85	2.03
56000	-0.15	-0.15	-0.33	-0.58	-0.38	-0.17	-0.34	0.51	1.09	1.46	2.06
58000	-0.57	-0.57	-0.38	-0.58	-0.35	-0.27	-0.33	0.33	1.02	2.57	2.11
60000	-0.33	-0.33	-0.42	-0.52	-0.46	-0.25	-0.31	0.77	1.15	1.38	1.87
62000	-0.76	-0.76	-0.38	-0.53	-0.41	-0.22	-0.25	0.64	1.09	1.75	1.80
64000	-0.03	-0.03	-0.40	-0.49	-0.40	-0.32	-0.27	0.71	1.27	1.70	1.99
66000	-0.11	-0.22	-0.40	-0.42	-0.44	-0.29	-0.29	0.19	0.87	1.58	1.91
68000	-0.19	-0.19	-0.26	-0.38	-0.39	-0.28	-0.26	0.12	0.97	1.37	1.96
70000	0.54	0.54	-0.32	-0.48	-0.30	-0.22	-0.24	0.06	1.59	1.18	1.86

Table 2. Differences (%) according to the specific-energy and point-mass methods for the fuel burned in climb.

	Sigma										
Weight (kg)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
50000	0.86	0.86	0.15	0.18	0.03	-0.32	-0.53	1.03	2.57	3.20	3.44
52000	-0.54	-0.54	-0.25	-0.45	-0.38	-0.27	-0.57	1.01	2.37	3.10	3.38
54000	-0.37	-0.37	-0.19	-0.48	-0.45	-0.33	-0.46	1.03	2.15	2.98	3.32
56000	-0.20	-0.20	-0.32	-0.55	-0.41	-0.21	-0.49	0.83	1.91	2.76	3.33
58000	-0.60	-0.60	-0.38	-0.58	-0.33	-0.29	-0.46	0.56	1.81	3.42	3.28
60000	-0.37	-0.37	-0.43	-0.51	-0.44	-0.28	-0.42	0.90	1.82	2.57	3.02
62000	-0.72	-0.72	-0.37	-0.51	-0.38	-0.21	-0.38	0.82	1.73	2.76	2.92
64000	-0.09	-0.09	-0.40	-0.45	-0.38	-0.33	-0.34	0.78	1.94	2.67	3.00
66000	-0.13	-0.25	-0.39	-0.37	-0.43	-0.28	-0.33	0.37	1.64	2.56	2.86
68000	-0.19	-0.19	-0.24	-0.31	-0.38	-0.27	-0.29	0.25	1.61	2.42	2.92
70000	0.41	0.41	-0.31	-0.42	-0.30	-0.20	-0.27	0.17	2.10	2.22	2.83

Table 3. Differences (%) according to the specific-energy and point-mass methods for the time in descent.

	Sigma										
Weight (kg)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
50000	-1.24	-1.24	-1.34	-1.27	-1.35	-1.23	-1.17	0.29	0.22	0.23	0.65
52000	-1.03	-1.03	-1.22	-1.35	-1.16	-1.03	-0.74	0.69	0.60	1.13	0.11
54000	-0.93	-0.93	-1.35	-1.24	-1.08	-1.00	-0.65	0.70	0.57	1.22	0.10
56000	-0.87	-0.87	-1.30	-1.16	-1.05	-0.92	-0.76	0.46	0.47	0.60	0.13
58000	-0.90	-0.90	-1.05	-1.07	-1.00	-0.94	-0.87	-0.23	0.30	0.97	0.23
60000	-0.95	-0.95	-0.95	-1.01	-0.78	-0.91	-0.44	0.23	0.28	1.16	0.25
62000	-0.87	-0.87	-1.06	-0.96	-0.70	-0.71	-0.32	0.41	1.39	0.41	0.37
64000	-0.80	-0.80	-1.01	-0.88	-0.79	-0.57	-0.06	0.17	0.37	1.32	0.31
66000	-0.75	-0.75	-0.89	-0.99	-0.85	-0.43	0.21	1.01	1.32	1.21	0.22
68000	-0.73	-0.73	-0.93	-0.81	-0.71	-0.86	0.08	0.81	1.12	0.59	0.43
70000	-0.74	-0.74	-0.81	-0.81	-0.69	-0.44	0.09	0.19	1.06	0.49	0.51

Table 4. Differences (%) according to the specific-energy and point-mass methods for the fuel burned in descent.

	Sigma										
Weight (kg)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
50000	-1.14	-1.14	-1.22	-1.17	-1.22	-1.14	-1.10	1.26	1.10	1.04	0.87
52000	-0.98	-0.98	-1.12	-1.21	-1.08	-1.00	-0.81	1.44	1.19	1.74	0.03
54000	-0.91	-0.91	-1.19	-1.11	-1.03	-0.97	-0.73	1.44	1.02	1.88	-0.05
56000	-0.85	-0.85	-1.14	-1.06	-0.97	-0.88	-0.79	1.02	0.77	0.70	0.01
58000	-0.86	-0.86	-0.97	-1.00	-0.94	-0.90	-0.85	0.23	0.50	1.18	0.06
60000	-0.89	-0.89	-0.89	-0.93	-0.80	-0.87	-0.57	0.61	0.34	1.44	0.08
62000	-0.82	-0.82	-0.97	-0.89	-0.75	-0.74	-0.50	0.76	1.97	0.17	0.14
64000	-0.78	-0.78	-0.92	-0.84	-0.79	-0.64	-0.35	0.28	0.44	1.51	0.09
66000	-0.74	-0.74	-0.84	-0.90	-0.79	-0.56	-0.19	1.71	1.87	1.27	0.04
68000	-0.71	-0.71	-0.85	-0.78	-0.70	-0.78	-0.26	1.36	1.46	0.26	0.15
70000	-0.71	-0.71	-0.77	-0.75	-0.68	-0.54	-0.24	0.74	1.44	0.10	0.21

6. CONCLUSIONS

The results for climb show that the differences in performance figures are the highest for high sigma values. This can be explained by the fact that high sigma generally means low climb speeds, which favors the climb angle due to the higher difference between thrust and drag. As the specific-energy method is based on the assumption that lift is equal to drag, which is true only for leveled flight, these high differences are expected as the sigma increases. On the other hand, low sigma means climb with very high speeds. As the climb angles are small in high speeds, the differences tend to be smaller for low sigma. The same effect can be observed as the initial weight increase. As the climb angle for a high weight is small, so the performance difference between the two methods is also reduced. These differences are very noticeable for low weights, especially for sigma above 0.7.

For descent, it was observed that, for $\sigma \leq 0.4$, the differences tend to be more negative. These differences are also very noticeable when the weight is low. This could be explained by the fact that, when the aircraft is light, the descent angle must be increased in order to maintain the high speeds required by the low sigma.

These results demonstrate that the cost reduction obtained through the application of optimization techniques exposed by Pradines *et al.* (2006) can be affected if the flight path is calculated using the point-mass model, instead of the specific-energy method. This effect could be assessed in future papers. Another aspect that was explored by Erzberger *et al*. (1975) is the effect of the free-thrust climb and descent, which was not studied in this paper. It is expected that the performance differences would be even greater, due to the reduced climb angle obtained with the free-thrust optimized flight paths.

7. REFERENCES

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