THE COLLOCATION POINT POSITION IN THE DUAL BOUNDARY ELEMENT EQUATION WITH THE TANGENTIAL DIFFERENTIAL OPERATOR

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Abstract. *The dual boundary element formulation using the tangential differential operator in the boundary integral equation (BIE) for tractions was study in [1] and a good accuracy of the obtained results with expected values of the literature was shown even considering the reduction of the order of the singularity due to the use of the operator. The collocation point position is analyzed in the present study for dual formulations with the strong singularity in the traction BIE as well as that using the tangential differential operator. Isoparametric linear elements are employed with same expressions in shape functions for conformal and non-conformal interpolations. Conformal interpolations are applied on the crack surface and the obtained results are compared with solutions available from the literature.*

***Keywords**: Tangential Differential Operator, Dual Boundary Element Method, Stress Intensity Factor.*

1. INTRODUCTION

The presented study began when the implementation presented in [Portela, Aliabadi and Rooke (1992)] was revisited in [Almeida and Palermo (2004)] with the purpose to apply conformal interpolations along the crack surface. Portella et alli employed non-conformal interpolations along the crack surface using isoparametric quadratic elements and collocation points positioned at the element nodes. The strategy in [Almeida and Palermo (2004)] used same expressions for shape functions in conformal and non-conformal interpolations, collocations points shifted to the interior of the elements and nodal parameters fixed at the ends of boundary elements. The obtained results with quadratic and linear elements had a good agreement with those presented in the literature when the internal position (ξ') of the collocation point was $\xi'=\pm0.67$ in the range (-1, 1). The next step was the introduction of the tangential differential operator (TDO) in the traction BIE to reduce the order of the strong singularity. The kernel of the integral containing the TDO employs displacements derivatives and several studies in the literature introduce another shape function to map the displacements derivatives. The shape function employed to approximate displacements on the elements was differentiated in [Palermo et all (2006)]. The use of linear elements carried to a constant function for the displacement derivative and the efficiency of TDO with low order elements was the background purpose of the last paper.

Different positions for collocation points on opposite elements at the crack surfaces are studied in the present paper. The nodes of elements placed along one face of the crack are the positions for a set of collocation points whereas internal points of elements along the opposite face were the positions for another set of collocation points. A short explanation on the basic steps to obtain the dual boundary element method and the TDO will be presented next.

The differentiation of the kernels of integrals in the displacement BIE to obtain one for stresses increase the order of kernel singularities and require special treatment of improper integrals when the integrations are performed. The use of the tangential differential operator (TDO) in Kelvin type fundamental solutions reduces the strong singularity in the BIE for stresses. The BIE for the gradient at an internal point x is next written using the differentiation in terms of field variables:

$$u_{i,m}(x)=\int_{\Gamma}T_{ij,m}(x,y)u_j(y)\,d\Gamma(y)-\int_{\Gamma}U_{ij,m}(x,y)t_j(y)\,d\Gamma(y) \tag{1}$$

U_{ij} (x, y) and T_{ij} (x, y) are the displacement and the traction, respectively, in the direction j at the boundary point y due to a singular load in the direction i at the collocation point x, according to the Kelvin solution for two-dimensional problems. $u_j(y)$ and $t_j(y)$ are the displacement and the traction at the field point, respectively.

The first and the second integrals of eq. (1) are regular for internal points and exhibit singularities of order $1/r^2$ and $1/r$, respectively, when the field point approaches the collocation point. The introduction of the TDO on the first integral of the right member of eq. (1) reduces the order of the strong singularity ($1/r^2$). After some algebraic manipulation and employing properties of Kelvin type fundamental solutions, the TDO is obtained from integration by parts [Bonnet (1999), Palermo et all (2006)] and the BIE for the gradient at an internal point x with TDO is given by:

$$u_{i,m}(x)=\int_{\Gamma}\sigma_{ibj}(x,y)\,D_{mb}\left[u_j(y)\right]d\Gamma(y)-\int_{\Gamma}U_{ij,m}(x,y)\,t_j(y)\,d\Gamma(y) \tag{2}$$

$D_{bm}(\,)$ is the tangential differential operator, which has the following definition:

$$D_{mb}\left[u_j(y)\right]=n_m(y)\,\mathrm{u}_{j,b}(y)-n_b(y)\,u_{j,m}(y) \tag{3}$$

The integrals of eq. (2) are regular for internal points and exhibit singularities of order 1/r when the field point approaches the collocation point. The BIE for stress is obtained from eq. (1) or eq. (2) using the Hooke tensor and the symmetry property of $U_{ij,\,m}$ (x, y). The limiting form of the stress BIE at an internal point when it is led to a point on the boundary defines the BIE for stresses at a boundary point, which has the following expression at the point x' on a smooth boundary:

$$\frac{1}{2}\sigma_{ak}(x')=C_{\mathrm{aki\,m}}\int_{\Gamma}T_{ij,m}(x',y)\,u_j(y)\,d\Gamma(y)-\int_{\Gamma}\sigma_{jak}(x',y)\,t_j(y)\,d\Gamma(y) \tag{4}$$

$$\frac{1}{2}\sigma_{ak}(x')=C_{\mathrm{aki\,m}}\int_{\Gamma}\sigma_{ibj}(x',y)\,D_{mb}\left[u_j(y)\right]d\Gamma(y)-\int_{\Gamma}\sigma_{jak}(x',y)\,t_j(y)\,d\Gamma(y) \tag{5}$$

$$C_{akim}=\mu\left(\frac{2\nu}{1-2\nu}\delta_{ak}\delta_{im}+\delta_{ai}\delta_{km}+\delta_{am}\delta_{ki}\right) \tag{6}$$

C_{akim} is the Hooke tensor for isotropic media, ν is the Poisson ratio and μ is equal to the shear modulus. δ_{ij} is the Kronecker delta

It is important to note on the continuity requirement for the derivative of the displacement function at the collocation point x´. The traction BIE is obtained from eq. (4) or (5) when the stress tensor at the boundary point x' is multiplied by direction cosines of the outward normal at this point (n'_a). The corresponding traction BIEs from eq. (4) and (5) are given by:

$$\frac{1}{2}t_k(x')=n'_a(x')C_{\mathrm{aki\,m}}\int_{\Gamma}T_{ij,m}(x',y)\,u_j(y)\,d\Gamma(y)-n'_a(x')\int_{\Gamma}\sigma_{jak}(x',y)\,t_j(y)\,d\Gamma(y) \tag{7}$$

$$\frac{1}{2}t_k(x')=n'_a(x')C_{\mathrm{aki\,m}}\int_{\Gamma}\sigma_{ibj}(x',y)\,D_{mb}\left[u_j(y)\right]d\Gamma(y)-n'_a(x')\int_{\Gamma}\sigma_{jak}(x',y)\,t_j(y)\,d\Gamma(y) \tag{8}$$

2. THE DUAL BOUNDARY INTEGRAL EQUATIONS

The boundary integral formulation is degenerated in a structure containing a mathematically sharp crack due to the coincidence of the two crack surfaces. The same geometric position of collocation points positioned on each face of the crack surface is the main challenge to present the boundary element method directly. Several strategies are shown in the literature to treat this problem like partitioning the domain into multi-domains, the use of crack Green's functions, the displacement discontinuity technique and the dual boundary element method (DBEM). The DBEM employs a single domain formulation to treat general mixed-mode crack problems by using the displacement BIE applied to one of the crack surfaces and the traction equation to the other. Although the integration path is the same for coincident points on the crack surfaces, the respective boundary integral equations are now distinct. The collocation point needed to perform the traction boundary integral equation and the strategy used to treat improper integrals are the essential features of the formulation. The displacement BIE used in the DBEM has the following expression for the collocation point on a smooth boundary:

$$\frac{1}{2}u_i(x')+\int_{\Gamma}T_{ij}(x',y)\,u_j(y)\,d\Gamma(y)=\int_{\Gamma}U_{ij}(x',y)\,t_j(y)\,d\Gamma(y) \tag{9}$$

The continuity requirements for each BIE have to be satisfied at the collocation point position. The continuity of the displacement function at x', required for the displacement BIE, is satisfied for collocation points shifted to the interior of the boundary element or placed at the ends of the element in case of conformal interpolations. The continuity of the displacement derivative at x', required for the traction BIE, is satisfied for collocation points positioned at the interior of the element for usual shape functions [Bonnet (1999)].

The integrals of eq. (9) exhibit singularities of order 1/r and ln(1/r). The improper integral requires the use of the Cauchy principal value sense and the application of the first order finite part results in an analytical expression for the singular term and a numerical integration using the Gauss-Legendre scheme for the regular term. Considering the local parametric co-ordinate ξ defined in the range (-1, 1), the collocation point position ξ' [Portela, Aliabadi and Rooke (1992); Almeida and Palermo (2004)] and displacement components u_j approximated in the local co-ordinate system in terms of nodal values u^n_j, the first order finite-part integral expressed in the local co-ordinate ξ is:

$$\int_{\Gamma e} T_{ij}(x',y).u_j(y).d\Gamma(y) = u_j^n.\left[\int_{-1}^{+1} \frac{f_{ij}^n(\xi) - f_{ij}^n(\xi')}{\xi-\xi'}.d\xi + f_{ij}^n(\xi').\int_{-1}^{+1}\frac{d\xi}{\xi-\xi'}\right] \tag{10}$$

The regular function $f^n_{ij}(\xi)$ is the product of the fundamental solution, the shape function, the Jacobian of the co-ordinate transformation and the term (ξ-ξ'). The first integral of the right hand side of eq. (10) is regular and the second can be integrated analytically [Portela, Aliabadi and Rooke (1992)].

The order of singularities in the traction BIE using TDO (eq. 8) is 1/r in both kernels of the boundary integrals. The improper integrals are similar to that shown in eq. (10) and require the use of the first order finite part.

The integrals of eq. (7) exhibit singularities of order 1/r and $1/r^2$. The strong singularity ($1/r^2$) requires the use of the Cauchy and the Hadamard principal-value integral. The application of the second order finite part results in analytical expressions for singular terms and a numerical integration using the Gauss-Legendre scheme for the regular term. The term containing the strong singularity written in the local coordinate ξ has the following expression:

$$n'_a(x')C_{\text{aki m}}\int_{\Gamma e} T_{ij,m}(x',y)\,u_j(y)\,d\Gamma(y) = u_j^n\int_{-1}^{+1}\frac{g_{kij}^n}{(\xi-\xi')^2}\,d\xi \tag{11}$$

The regular function $g^n_{kij}(\xi)$ is the product of the fundamental solution, the Hooke tensor, the shape function, the Jacobian of the co-ordinate transformation, the direction cosines of the outward normal and the term $(\xi\text{-}\xi')^2$. The integral of the right-hand side of eq. (11) can be transformed with the aid of the first term of Taylor's expansion of the function g^n_{kij} around the collocation point ξ'.

$$\int_{-1}^{+1}\frac{g_{kij}^n(\xi)}{(\xi-\xi')^2}.d\xi = \int_{-1}^{+1}\frac{g_{kij}^n(\xi)-g_{kij}^n(\xi')-g_{kij}^{n(1)}(\xi').(\xi-\xi')}{(\xi-\xi')^2}.d\xi + g_{kij}^n(\xi').\int_{-1}^{+1}\frac{d\xi}{(\xi-\xi')^2} + g_{kij}^{n(1)}(\xi').\int_{-1}^{+1}\frac{d\xi}{\xi-\xi'} \tag{12}$$

$g^{n\,(1)}_{kij}(\xi')$ is the first derivative of g^n_{kij} at ξ'. The first integral of the right hand side of eq. (12) is regular whereas the second and the third require analytical expressions. [Portela, Aliabadi and Rooke (1992)].

3. BOUNDARY ELEMENTS AND COLLOCATION POINTS

The present study employed isoparametric linear boundary elements. Conformal and non-conformal interpolations employed the same shape functions with nodal parameters positioned at the ends of the elements. The nodes of the elements were the positions of collocation points in conformal interpolations with displacement BIEs whereas non-conformal interpolations employed internal points of the element. On the other hand, the internal points of the elements were the positions of collocation points in conformal or non-conformal interpolations with traction BIEs. Three positions of internal collocation points (ξ') in the range (-1, 1) were considered: ξ'=±0.5; ξ'=±0.67 and ξ'=±0.75.

The position of collocation points in conformal interpolation with displacement BIEs was the main changing of this study with reference to former papers [Almeida and Palermo (2004); Palermo et alli (2006)]. The crack analyses in those papers have always employed internal positions for collocation points, which were according to positions used in [Portela, Aliabadi and Rooke (1992)] with non-conformal interpolations along the crack. This study recovers the meaning of the BEM formulation with the displacement BIE, which does not require internal points for collocation points in conformal interpolations, and shows that crack analysis using DBEM only requires the basic continuity conditions for the displacement and the traction BIE.

The traction BIE employed collocation points positioned at internal points of the elements as the continuity condition requires, which were not necessarily the same positions used on the opposite surface for collocation points of the displacement BIE. The diagonal terms were functions of the collocation point position and the shape function without using the rigid body motion.

The present analysis considered two computer codes: a code using the well-known traction BIE with the strong singularity and another code with TDO in the traction BIE. The present study used the derivatives of the adopted shape function for displacements (linear functions) in the traction equation as required for the TDO and the tangent derivative has yielded to constant values with opposite signs.

A revision in eq. (8) was required due to the use of non-conformal interpolations. The eq. (8) is rewritten next to include the effect of the discontinuity in case of a mesh with discontinuity in displacements at one point [Palermo et all (2006)]:

$$\begin{aligned}\frac{1}{2}t_k(x') &= n'_a(x')C_{\mathrm{aki\,m}}\int_\Gamma \sigma_{ibj}(x',y)D_{mb}\left[u_j(y)\right]d\Gamma(y) - n'_a(x')\int_\Gamma \sigma_{iak}(x',y)t_j(y)\,d\Gamma(y) + \ldots \\ &\ldots + n'_a(x')C_{\mathrm{aki\,m}}e_{3bm}\sigma_{ibj}(x,y)\left[u_j^B(y) - u_j^F(y)\right]\end{aligned} \qquad (13)$$

The term between brackets of eq. (13) includes the effect of the ends from the integration by parts used to obtain TDO and is a multiplier of displacements uj^B and uj^F, which are at the backward and at the forward side of the discontinuity respectively. It is important to note that uj^B and uj^F have the same geometrical coordinates according to the strategy for non-conformal interpolation in this study.

4. STRESS INTENSITY FACTOR EVALUATION

The analysis of stresses in the crack neighborhood did an assessment of stress intensity factors using the near-tip displacement extrapolation, as explained in [Portela, Aliabadi and Rooke (1992), Almeida and Palermo (2004)]. Considering a polar coordinate system (r, θ) centered at the crack tip, such that the crack surfaces could be defined with $\theta = \pm\pi$. The displacement field on the crack surface has the following expressions considering the first term of William’s expansion:

$$u_2(\theta=\pi) - u_2(\theta=-\pi) = \frac{\kappa+1}{\mu}K_I\sqrt{\frac{r}{2\pi}} \qquad (14)$$

$$u_1(\theta=\pi) - u_1(\theta=-\pi) = \frac{\kappa+1}{\mu}K_{II}\sqrt{\frac{r}{2\pi}} \qquad (15)$$

The stress intensity factors for deformation modes II and I are KII and KI, respectively; the parameter κ is equal to 3-4η; η is equal to ν for plane strain problems and equal to ν/(1+ν) for plane stress problems The near-tip displacement extrapolation works with eq. (14) and (15) to obtain the stress intensity factors when the displacements are known. The situation is shown in Figure 1, where opposite elements share the crack tip at nodes B and C. The length of the linear element is equal to l. The expressions for the stress intensity factors are given by:

$$K_I^{DE} = \left(u_2^D - u_2^E\right)\frac{\mu}{\kappa+1}.\sqrt{2}.\sqrt{\frac{\pi}{l}} \qquad (16)$$

$$K_{II}^{DE} = \left(u_1^D - u_1^E\right)\frac{\mu}{\kappa+1}.\sqrt{2}.\sqrt{\frac{\pi}{l}} \qquad (17)$$

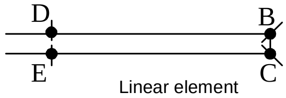
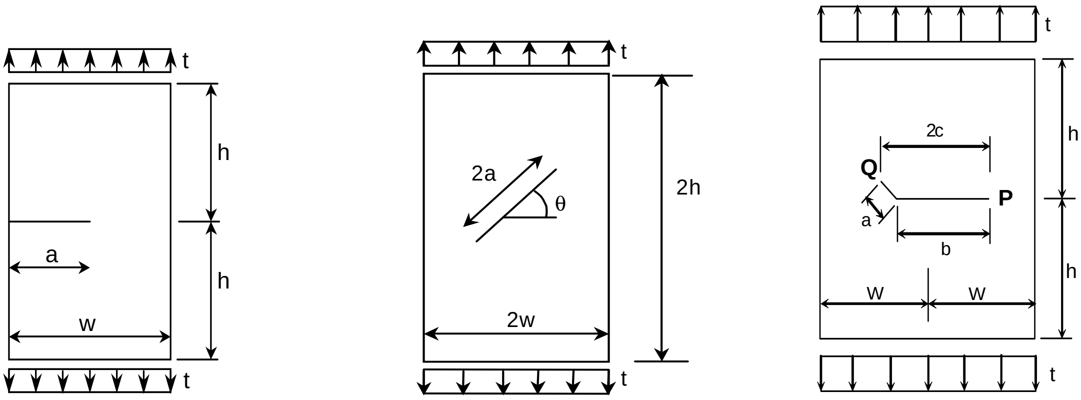


Figure 1: Crack tip at points B and C.

5. NUMERICAL EXAMPLES

The obtained results with both computer codes (one using the well-known DBEM formulation, eq. (7) and (9), and other using TDO, eq. (8) and (9)) did not present significant differences as noted in [Palermo et all (2006)]. The differences in the obtained results from each code have disappeared when the round off was introduced to match the precision of this paper. Nevertheless, different positions of internal collocation points carried to different results. The traction BIEs and double nodes required the use of internal position for collocation points. Three cases were studied and a conformal interpolation was applied along the crack surfaces with double nodes introduced at the corners and at the crack tips.

A rectangular plate containing a single horizontal edge crack shown in Figure 2a used a mesh with 48 linear elements plus 8 elements on each crack surface (64 B.E.). The crack length is a, the plate width is w and the height is 2h. A uniform traction in the height direction was symmetrically applied at the ends. Results obtained for the ratio h/w equals to 0.5 are shown in Table 1. Three ratios a/w were considered: 0.2, 0.4, and 0.6. The stress intensity factor was obtained with eq. (16).



a) Single edge crack. b) Central Slant Crack (θ=45°). c) Central Kinked Crack.

Figure 2: Analyzed cracks.

Table 1: Results for the single edge crack with 64 linear elements: $K_I /(t\sqrt{\pi a})$

a/w	Civilek and Erdogan (1982)	ξ'=0.5	ξ'=0.67	ξ'=0.75
0.2	1.488	1.637	1.642	**1.418**
0.4	2.324	2.601	**2.328**	2.221
0.6	4.152	4.759	**4.209**	4.513

A rectangular plate containing a central slant crack shown in Figure 2b had a mesh with 48 linear elements plus 12 elements on each crack surface (72 B.E.). The crack length is 2a, the plate width is 2w and the height is 2h. A uniform traction in the height direction was symmetrically applied at the ends. Results obtained for the h/w ratio equal to 2 are shown in Table 2. Three a/w ratios were considered: 0.2, 0.4, and 0.6. The stress intensity factors were obtained with eq. (16) and eq. (17).

Table 2. Results with 72 linear elements.

a/w	Mode I: $K_I /(t\sqrt{\pi a})$				Mode II: $K_{II} /(t\sqrt{\pi a})$			
	Murakami (1987)	ξ'=0.5	ξ'=0.67	ξ'=0.75	Murakami (1987)	ξ'=0.5	ξ'=0.67	ξ'=0.75
0.2	0.518	0.559	**0.512**	0.478	0.507	0.546	**0.501**	0.468
0.4	0.572	0.620	**0.566**	0.525	0.529	0.570	**0.522**	0.486
0.6	0.661	0.730	**0.658**	0.605	0.567	0.610	**0.557**	0.518

A rectangular plate containing an internal kinked crack shown in Figure 2c had a mesh with 96 linear elements plus 10 elements on each horizontal crack surface and 8 elements on each inclined crack surface (132 B.E.). One of the segments of the crack is horizontal with length a while the other segment makes an angle of 45 degrees with the horizontal and has length b; the horizontal projection of the total crack is given by 2c=b+a(√2)/2. The kink of the crack is at the center of the plate, the plate width is 2w and the height is 2h. Three a/b ratios were considered: 0.2, 0.4 and 0.6. The results obtained for b/w equal to 0.1 are shown in Tables 3 and 4. Stress intensity factors were obtained with eq. (16) and eq. (17).

Table 3. Results at P with 132 linear elements.

a/b	Mode I: $K_I/(t\sqrt{\pi c})$				Mode II: $K_{II}/(t\sqrt{\pi c})$			
	Murakami (1987)	ξ'=0.5	ξ'=0.67	ξ'=0.75	Murakami (1987)	ξ'=0.5	ξ'=0.67	ξ'=0.75
0.2	0.995	1.074	**0.988**	0.923	0.028	0.032	**0.028**	0.024
0.4	0.990	1.068	**0.984**	0.921	0.033	0.038	**0.035**	0.033
0.6	0.986	1.066	**0.982**	0.918	0.030	0.035	**0.032**	0.031

Table 4. Results at Q with 132 linear elements.

a/b	Mode I: $K_I/(t\sqrt{\pi c})$				Mode II: $K_{II}/(t\sqrt{\pi c})$			
	Murakami (1987)	ξ'=0.5	ξ'=0.67	ξ'=0.75	Murakami (1987)	ξ'=0.5	ξ'=0.67	ξ'=0.75
0.2	0.598	0.708	0.630	**0.558**	0.557	0.661	0.591	**0.537**
0.4	0.574	0.671	**0.603**	0.546	0.607	0.711	**0.635**	0.571
0.6	0.568	0.661	**0.595**	0.539	0.627	0.734	**0.655**	0.588

The obtained values closer to the literature were bolded in Tables 1 to 4. The best internal position (ξ') in the range (-1, 1) for the collocation point was ξ'=±0.67. The adopted meshes were the similar to those employed in [Portela, Aliabadi and Rooke (1992)], [Almeida and Palermo (2004)] and [Palermo et all (2006)]. The obtained position is according to the best position obtained in [Almeida and Palermo (2004) in spite of the changing of the position of the collocation point in conformal interpolations when the displacement BIE was applied. It is important to mention on the behavior of the obtained results with the traction BIE using the TDO, which had similar values to those obtained with the strong singularity even when the position of the collocation point was changed.

6. CONCLUSION

The numerical implementation of the present paper employed the displacement BIE in the same way currently used in the boundary element method, collocation points positioned at nodes in conformal interpolations. On the other hand, interior positions in case of double nodes or in case of the traction BIE are the basic procedures to satisfy the continuity requirements. These features simplify the DBEM with a reduced number of internal collocation points and the possibility to apply conformal interpolation along the crack surface. It is important to note that the traction BIE was the main equation to define the collocation point position and it was independent of the introduction of the TDO, as shown from the analysis of the best position. Furthermore, this study strongly suggests the use of the TDO due to the simplifications to treat the singularity (the treatment of the improper integrals only used the first order finite part). Finally, the use of low order elements did not penalty the precision of the DBEM.

The stress intensity factor were obtained by near-tip displacement extrapolation and it is important to note that better values for stress intensity factors can be obtained with J-integral technique. The benefit of the J-integral technique was shown in [Portela, Aliabadi and Rooke (1992)].

7. REFERENCES

Almeida, LPCPF, Palermo Jr., L, On the Implementation of the Two Dimensional Dual Boundary Element Method for Crack Problems, "5th International Conference on Boundary Elements Techniques" Lisboa, Portugal, 2004.

Chen, WH, Chen, TC, An efficient dual boundary element technique for a two-dimensional fracture problem with multiple cracks. International Journal of Numerical Methods in Engineering, 38, 1739-1756, 1995.

Civelek, M. B., Erdogan, F., 1982, Crack problems for a rectangular sheet and an infinite strip, International Journal of Fracture, vol.19, pp.139-159.

Hong, H, Chen, J, 1988, Derivations of integral equations of elasticity, Journal of Engineering Mechanics ASCE, 114, pp. 1028-1044.

Murakami, Y., 1987, Stress Intensity Factors Handbook, Pergamon Press, Oxford.

Palermo Jr., L, Almeida, LPCPF, Gonçalves, PC, The Use of the Tangential Differential Operator in the Dual Boundary Element Equation, Structural Durability & Health Monitoring, vol.2, no.2, pp.123-130, Tech Science Press, 2006.

Portela, A, Aliabadi, MH, Rooke, DP, The dual boundary element method: Effective implementation for crack problems, International Journal of Numerical Methods in Engineering, 33, 1269-1287, 1992.

Telles, JCF, Castor, GS, Guimarães, S, Hypersingular Green´s function generation and fracture mechanics problems, Boundary Element Method XVI, Brebbia, CA, Computational Mechanics Publications, 443-453, 1994.

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