DETERMINATION OF A DYNAMICAL SPHERE OF INFLUENCE

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Abstract. *In this work we made a study of the sphere of influence taking into account the relative velocity in your definition. We adopted a numerical approach based in integrations of the restricted three-body problem, where we followed the temporal variation of the two-body energy of a particle that suffers a close encounter with a more massive body. The evolution of such energy shows if the particle was significantly affected by the gravitational influence of the secondary body, for some specific initial conditions. This procedure results in a mathematical function to calculate de sphere of influence radius as a function of the relative velocity and of the system's mass ratio.*

Keywords*:sphere of influence,close encounters*

1. INTRODUCTION

In problems that involve close encounters between celestial bodies, it is used a concept of sphere of influence. The sphere of influence of a body can be thought as a spherical surface centered in this body, where its gravitational influence is predominant relative to the gravitational influence of other bodies. The Laplace's sphere (Roy, 1988) and the Hill´s sphere (Hill, 1878) are the most used models in the literature.

Examples of such applications are the determination of orbital stability zones generally determined as a function of Hill's sphere (Hamilton & Burns, 1991), (Domingos, Winter, Yokoyama, 2006), or the studies about formation of giant planets, where the mathematical function used to calculate the growth of protoplanetary core mass depends of the Hill's sphere of the planet (Kornet, Wolf, Rózyczka, 2006). Besides this, the idea of sphere of influence is amply used in studies that involve the gravitational capture as in (Vieira Neto, Winter, Yokoyama, 2004.) for example, and orbital maneuvers as the *swing-by maneuver* (Prado, 2001)

In this work the dynamic effects of a close encounter between two bodies are considered to determine the sphere of influence as a function of the relative velocity of the encounter. This is done with the purpose to show that the gravitational influence of a body over other body less massive is related with the relative velocity between them. In this way, it is possible to obtain a mathematical formulation to calculate the sphere of influence radius as a function of this velocity and of the system's mass ratio.

With this goal, the method adopted consists on numerical integrations of the restricted three-body problem, and on follow the temporal evolution of the two-body energy for a range of specific initial conditions.

The aplication of this method results in a new model of sphere of influence with a variable size, in opposition of the models existing given as a function of the system's mass ratio and of the distance between the two bodies, resulting in a sphere with a fixed size as Laplace's sphere or Hill's radius for example.

2. METHODOLOGY

2.1. The initial conditions

We consider a system with three bodies (M_1, M_2, M_3) as shown at Fig.1.

In this system, M_1 is the most massive body, called central body. M_2 is a body less massive than M_1 called secondary body. M_3 is a particle (P) whose mass relative to the others two mass is so small that can be rejected.

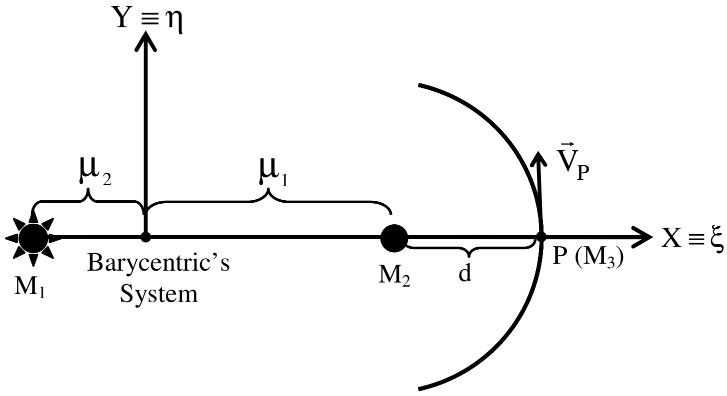

Figure 1 – Initial conditions

When $t = 0$, M_1, M_2 e M_3 are on the same line. At this moment, the axis of a synodic system (X, Y) coincide with the axis of a sidereal system (ξ, η). The distance between the secondary body (M_2) and the particle at this moment is called *distance parameter* (d), and $\vec{V}_P$ is the particle's velocity.

From this configuration, it is possible to see that the initial position of the particle in the synodic system is given by:

$$X = \mu_1 + d \qquad and \qquad Y = 0 \tag{1}$$

It is also possible to show that the velocity's components in the synodic system will be given by:

$$\dot{X} = 0 \qquad and \qquad \dot{Y} = V_{PS} - nd \tag{2}$$

In this equation, V_{PS} is the particle's velocity relative to the secondary body, given by

$$V_{PS} = V_P - V_S \tag{3}$$

where V_S is the secondary body velocity.

From Eqs. (1) and (2), we have the particle's initial conditions given in a synodic system, with $n = 1$:

$$\begin{aligned} X &= \mu_1 + d \qquad & \dot{X} &= 0 \\ Y &= 0 & \dot{Y} &= V_{PS} - d \end{aligned} \tag{4}$$

2.2. The two-body energy

The two-body energy particle-central body (E_{PC}) is given by:

$$E_{PC} = \frac{1}{2}V_{PC}^2 - \frac{\mu_1}{r_1} \tag{5}$$

where:
V_{PC} - is the particle's velocity relative to the central body.
r_1 - is the distance particle-central body.
μ_1 - is the specific mass of the central body, where $\mu = \mu_1 + \mu_2 = 1$

The relative velocity V_{PS} can be written in terms of its components as:

$$V_{PC} = \sqrt{(\dot{\xi} - \dot{\xi}_C)^2 + (\dot{\eta} - \dot{\eta}_C)^2} \tag{6}$$

where the components $(\dot{\xi}, \dot{\eta})$ are given by:

$$\dot{\xi} = (\dot{X} - nY)cos(nt) - (\dot{Y} + nX)sen(nt)$$
$$\dot{\eta} = (\dot{X} - nY)sen(nt) + (\dot{Y} + nX)cos(nt) \quad (7)$$

The components $(\dot{\xi}_C, \dot{\eta}_C)$ are also obtained from this equation, remebering that to the central body, when $t = 0$, $X = -\mu_2$, $Y = 0$ and $\dot{X} = \dot{Y} = 0$. This leads to:

$$\dot{\xi}_C = \mu_2 sen(nt) \qquad and \qquad \dot{\eta}_C = -\mu_2 cos(nt) \quad (8)$$

With these considerations, we are able to calculate the particle's velocity relative to the central body through Eq. 6, and finally the two-body energy given by equation (4).

2.3. The method

The method consist on:

- Fixing the system's mass ratio.
- Fixing the particle's velocity relative to the secondary mass.
- Varying the distance parameter d.
- Numerically integrating the system for a given lenght of time.
- Calculating, for each value of d, the percentual variation of energy due to the encounter.
- Stipulating the value of variation of energy for which is considered that the particle was significantly influenced by the secondary mass.
- Considering the value of d that delimits this condition as the sphere of influence radius to the particle with that fixed velocity.

Here we assumed the value of a significantly variation of energy as $\Delta E\% = 1,0\%$. The exact value of the distance parameter d for which the variation is exactly that, is assumed to be the sphere of influence radius.

2.4. Obtaining the datas

Here is presented an example that helps us to understand the aplication of the method described on section 2.3, and how the datas are obtained.

Here was considered a particle with a relative velocity to the secondary mass equal to V_{PS}=0,0080, in a system with mass ratio of 10^{-7}. The first value of d considered is that one that don not leads the particle to a gravitational capture with these initial conditions (Araujo, Winter, Prado, 2006). Then, the value of d is increased, and for each one of these values the system is numerically integrated for a given lenght of time t (here $t = 2$ orbital periods of the secondary body).

The graphic on Fig.2 shows the variations of energy for fourteen values of d considered. It is possible to see that for smaller values of d, the variation of energy is larger. As this value increases, this variation decreases and become almost constant, indicating that the particle left the sphere of influence of the secondary mass.

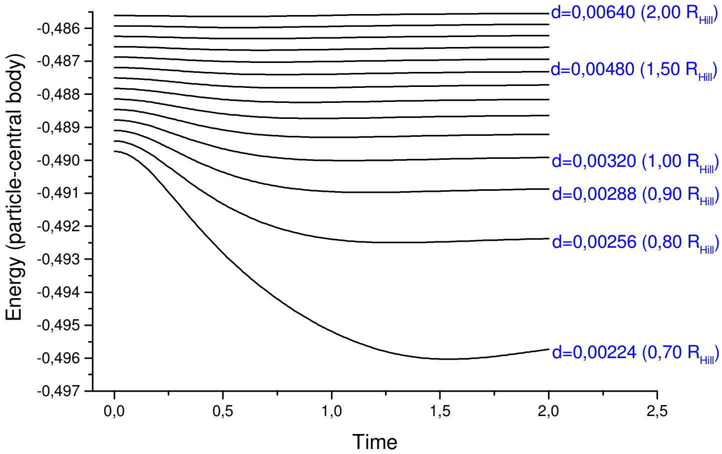

Figure 2 – Variation of the two-body **e**nergy with the time to different values of distance parameter.

The next step, according to the method is to calculate the percentual energy variation, for each one of this values of d. This is presented at Tab. 1.

Table 1 – Percentual energy variation for diferents values of d. In blue are pointed out the values of d that are in the graphic of the figure 2 in the interval of $0,70 \le d \le 1,00$ Hill's radius

d	d (R_{Hill})	ΔE(%)	d	d (R_{Hill})	ΔE(%)
0,00224	0,70	1,38	**0,00275**	0,86	0,44
0,00230	0,72	1,04	**0,00282**	0,88	0,40
0,00237	0,74	0,88	**0,00288**	0,90	0,36
0,00243	0,76	0,77	**0,00294**	0,92	0,33
0,00250	0,78	0,67	**0,00301**	0,94	0,30
0,00256	0,80	0,60	**0,00307**	0,96	0,28
0,00262	0,82	0,54	**0,00314**	0,98	0,25
0,00269	0,84	0,49	**0,00320**	1,00	0,23

This values give us a graphic as shown at Fig. 3.

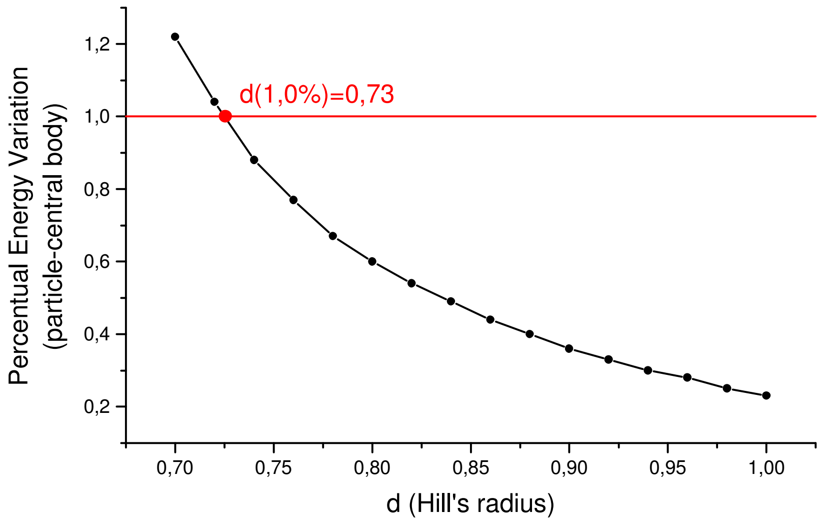

Figure 3 – Percentual energy variation as a function of the distance parameter d for one particle with V_{PS}=0,0080 . The red line indicates when this variation is of 1,0%, and give us the value of d for which this happens. This value is considered the sphere of influence radius for the particle with this velocity in a system with mass ratio of 10^{-7}

This graphic permit us to obtain the values of d for which the energy variation is that one fixed ($\Delta E\% = 1,0\%$). From this example we conclude that one particle with a relative velocity to the secondary body V_{PS}=0,0080, in a system with a mass ratio of 10^{-7}, will be influenced if $d < 0,73$ Hill's radius , and will not be influenced if $d > 0,73$ Hill's radius (considering the criterion of $\Delta E\% = 1,0\%$). Therefore, the value $d = 0,73$ Hill's radius corresponds to the sphere of influence radius to particles with this velocity in a system with this mass ratio.

3. RESULTS

3.1. Initial considerations

The method described above was applied considering the follow criteria:

- Only cases of prograde movement had been considered.
- The distance parameter should always be larger than 0,5 Hill's radius.

The initial idea was to apply the method to twelve mass ratios (from 10^{-1} to 10^{-12}), but with the development of the work, the results showed that this would not be possible.

At first, is known that the definition of sphere the influence consist on consider that the gravitational influence of one body over other body is predominant relative to the attraction of other bodies, and that while this domination exists the problem can be considered a two-body problem.

We obtained that to mass ratios larger than $2{,}0\text{x}10^{-6}$ the aproximation of the problem as a two-body problem cannot be considered, that is, the secondary body will always affect the movement of the particle, even when it is distant of it. Therefore, the problem has to be always considered as a three-body problem. Because of this behavior we cannot think about the concept of sphere of influence when such mass ratios are considered. Besides this, the results showed that to systems with mass ratio smaller than $2{,}0\text{x}10^{-8}$ the variation of 1,0% only would be reached if $d < 0{,}5$ Hill's radius. According to the method adopted this value is an acceptable limit distance between the particle and the secondary mass, so d should always be larger than this value.

Because of the reasons listed above, here we present the results of simulations made to mass ratios from $2{,}0\text{x}10^{-8}$ to $2{,}0\text{x}10^{-6}$.

3.2. Integration's results.

It has been done integrations for eleven mass ratios and, for each one of them was obtained one graphic as showed on Fig.5

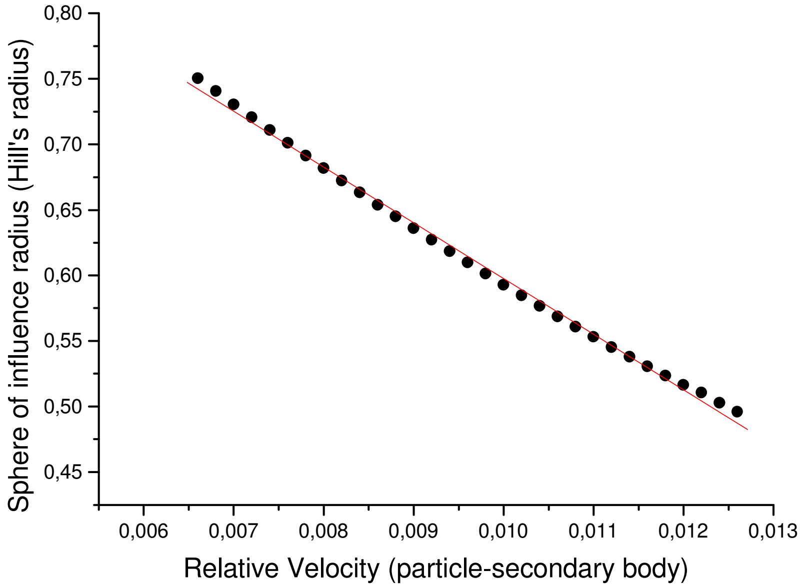

Figure 5 – Graphic of the sphere of influence as a function of the particle's velocity relative to the secondary body considering $\Delta E\% = 1{,}0\%$. The red line represents the linear fit that was done.

Each one of the points of this graphic was obtained by following the method presented on sections 2.3 and 2.4. A linear fit in this curve (red line) give us the mathematical function to calculate the sphere of influence radius as a function of the relative velocity V_{PS} (Eq. 9), which is given in Hill's radius.

$$R(V_{PS}) = 1{,}0227 - 42{,}4970(V_{PS}) \quad (8)$$

The respective values of Hill's radius to the eleven mass ratios considered were calculated through:

$$R_{Hill} = \left(\frac{\mu_2}{3}\right)^{1/3} \quad (9)$$

and can be found at Tab.2.

Table 2 – Calculated values of Hill's radius to the eleven mass ratios considered

Mass ratios	Hill's radius	Mass ratios	Hill's radius
$2,0 \times 10^{-8}$	0,00188	**$4,0 \times 10^{-7}$**	0,00511
$4,0 \times 10^{-8}$	0,00237	**$6,0 \times 10^{-7}$**	0,00585
$6,0 \times 10^{-8}$	0,00271	**$8,0 \times 10^{-7}$**	0,00644
$8,0 \times 10^{-8}$	0,00299	**$1,0 \times 10^{-6}$**	0,00693
$1,0 \times 10^{-7}$	0,00322	**$2,0 \times 10^{-6}$**	0,00874
$2,0 \times 10^{-7}$	0,00405		

The curves made for all the mass ratios are showed on the graphics on Figs. 6 and 7.

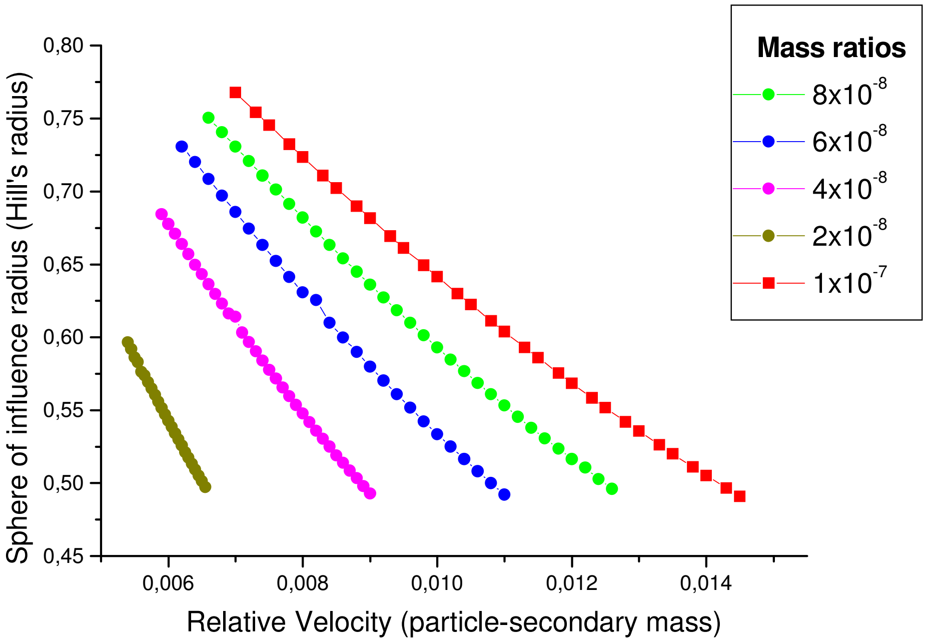

Figure 6 – Sphere of influence radius as a function of the relative velocity to mass ratios from 8×10^{-8} to $1,0 \times 10^{-7}$.

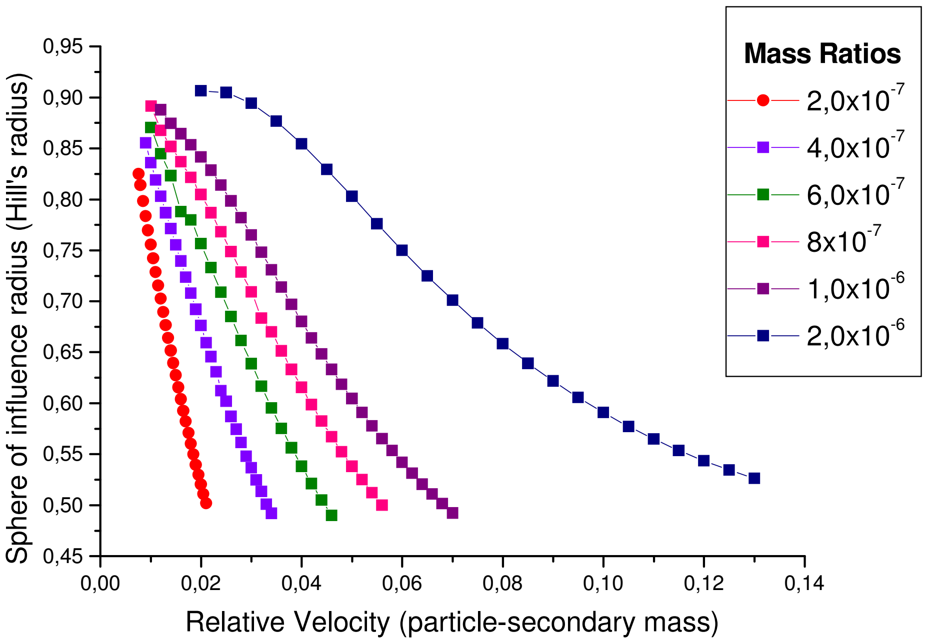

Figure 7 – Sphere of influence radius as a function of the relative velocity to mass ratios from 2×10^{-7} to $2,0 \times 10^{-6}$.

Linear fits on each one of these curves give us one equation with the form:

$$R(V_{PS}) = A - B(V_{PS}) \tag{10}$$

where the coeficients A and B are given by:

Table 3 – Coeficients A e B of the equation 10

Mass ratios	Coeficient A	Coeficient B
2,0x10^{-8}	1,06	85,93
4,0x10^{-8}	1,05	62,16
6,0x10^{-8}	1,04	50,10
8,0x10^{-8}	1,02	42,50
1,0x10^{-7}	1,02	36,92
2,0x10^{-7}	1,00	24,19
4,0x10^{-7}	0,98	14,68
6,0x10^{-7}	0,96	10,74
8,0x10^{-7}	0,97	8,75
1,0x10^{-6}	0,98	7,24
2,0x10^{-6}	0,97	3,46

A single equation to calculate the sphere of influence radius as a function of the relative velocity and of the mass ratio is obtained with the datas of the Tab.3.

We can see in this table that the coeficient A is almost constant around of the value 1, and for this reason, this will be the value considered to A.

At the same table it is also possible to see that the coeficient B is varying considerably as the mass ratio increases. Therefore, it is necessary to express this variation in function of the mass ratio. With this goal we considered a graphic as showed on Fig. 8. It was done with the datas of the columns 1 and 3 of the Tab.3.

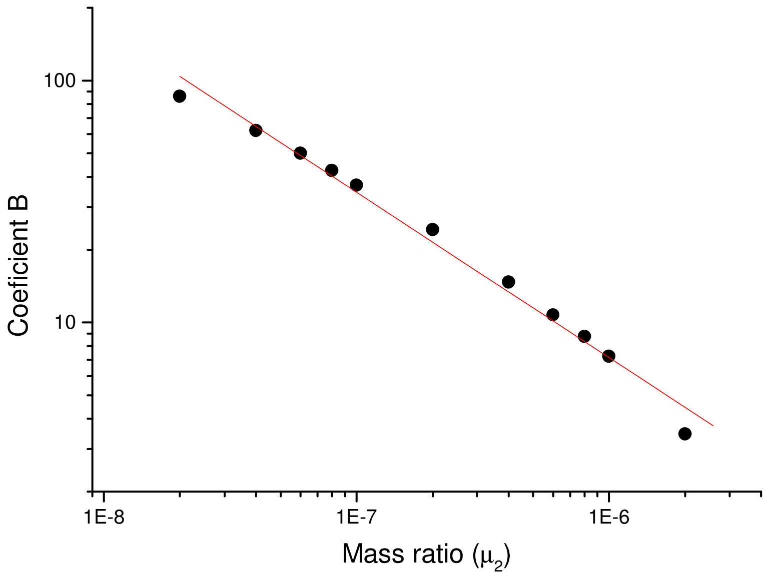

Figure 8 – Graphic of the coeficient B as a function of the mass ratio (μ) given in a logarithmic scale.

In this graphic a logarithmic scale was adopted on the two axis, resulting on a straight line. Such behavior suggests a relation given by :

$$B(\mu) = a\,\mu^{b} \tag{11}$$

The coeficients a and b are obtained through a fit in this curve, represented by the red line, which results in:

Table 4 – Coeficients a and b of equation 11

Coeficients	Values
a	0,00057
b	-0,6835

Therefore, the coeficient B as a function of the mass ratio will be given by:

$$B(\mu) \approx 0{,}0006\, \mu^{-0{,}68} \quad (12)$$

Finally, with this considerations, we found that the function to calculate the sphere of influence radius as a function of the mass ratio and of the relative velocity is given by:

$$R_C(V_{PS}, \mu_2) \approx 1 - 0{,}0006\, \mu_2^{-0{,}68}(V_{PS}) \quad (13)$$

4. CONCLUSION

The main purpose of this work was to obtain a new model to the concept of sphere of influence with a variable radius, and to express the variation of the size of this radius as a function of the particle's relative velocity to the secondary body and of the system's mass ratio.

Such study was done through numerical integrations of the restricted three-body problem and through analysis of the two-body energy for mass ratios from $2{,}0\text{x}10^{-8}$ to $2{,}0\text{x}10^{-6}$. As a result of this procedure we obtained a single mathematical function that allows us to calculate the sphere of influence as a function of the particle's relative velocity to the secondary body and of the system's mass ratio. We concluded that this radius increases as the relative velocity decreases, and that it increases as the mass ratio also increases.

With the development of this study we concluded that for mass ratios larger than $2{,}0\text{x}10^{-6}$ the aproximation of the two-body cannot be considered and that in these situations the problem has always to be considered as a three body problem. We also concluded that for systems with a mass ratio smaller than $2{,}0\text{x}10^{-8}$ a variation of 1,0% in the two-body energy will only be possible with the distance between the particle and the secondary body were smaller than 0,5 Hill's radius.

5. REFERENCE

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