ANALYSIS OF THIN PLATE SUBJECTED TO AN UNIFORM LOAD AND UNILATERAL CONTACT

Ivan Moura Belo, ivanbelo@gmail.com
Marcelo Krajnc Alves, krajnc@emc.ufsc.br
Universidade Federal de Santa Catarina - UFSC - Campus Universitário Trindade, Dep. Engenharia Mecânica - Bloco A, GMAC - Piso 2, CEP:88040-900, Florianópolis - SC

Marcelo Maldaner, maldaner.eng@gmail.com
Universidade Tecnológica Federal do Paraná, Dep. Engenharia Mecânica, Av. Sete de Setembro, 3165 - Rebouças, CEP: 80.230-901, Curitiba - PR

Abstract. *The problem of a thin plate subjected to an uniform load and unilateral contact is developed. The formulation is based on a Classical Laminated Plate Theory (CLPT), known as the Kirchhoff Model, and on the Equivalent Single-Layer Theory assumptions. The objective of this paper is to delvelop an approximate solution to the displacement which is treated as a nonlinear discretized problem solved by Newton's Method. Hence, in order to solve the variational inequality problem the Exterior Penalty Method is considered. Also, an approximate solution to this problem can be obtained by applying Ritz method. In this case, the approximation finite dimensional space is defined. It is demonstrated through this solution that the discrete problem is nonlinear since, for the given distributed load $f(x)$, we are not able a priori to know if the string will be constrained by the support or not. Therefore, to define the contact area in the contact problem, the load is applied following an incremental technique, based on the response of displacements and forces for the load applied at a previous step. The problem is implemented into a Matlab® code. Finally, to validate this procedure, results are shown considering the unilateral contact problem related to the thin plate clamped in two edges with a rigid obstacle.*

Keywords: *Unilateral Contact Problems, Exterior Penalty Method, Newton's Method, Kirchhoff Plate Model*

1. INTRODUCTION

In this work, the Ritz Method is applied first, to plate bending considering small strains and second, to the unilateral contact analysis, without friction, between a plate and a rigid obstacle. The Kirchhoff's model, which is a refined theory that holds for thin plates, is used. The algorithm and the procudure used to numerical implementation are described.

2. PRELIMINARIES

2.1 Kirchhoff Plate Theory

2.1.1 Kinematics Assumptions

In this paper, the analyses of plates is based on Equivalent single-layer theory (ESL) – (Reddy, 2004), which is derived from 3D elasticity model by suitable assumptions. Thus, the problem is reduced to a 2D. The Kirchhoff Plate Theory is the simplest ESL plate model and it is based on the following displacement field:

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) - z\,\frac{\partial w_0}{\partial x} \\
v(x,y,z,t) &= v_0(x,y,t) - z\,\frac{\partial w_0}{\partial y} \\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned}
\tag{1}
$$

It is assumed that the Kirchhoff Hypothesis holds: (a) straight lines perpendicular to the midsurface before deformation remain straight after deformation; (b) the transverse normals are inextensible and (c) the transverse normals rotate such that they remain perpendicular to the midsurface after deformation.

2.1.2 Constutive Equations

In the classical plate theory, all three transverse strain components are zero by definition, i.e, $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$. For assumed displacement fiel in Eq. (1), the strains reduces to

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} - z \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}
\tag{2}
$$

For generalized plane stress, the strian-stress relations is given by:

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-\nu^2)}(\varepsilon_{xx} + \nu\varepsilon_{yy}) \\ \sigma_{yy} &= \frac{E}{(1-\nu^2)}(\varepsilon_{yy} + \nu\varepsilon_{xx}) \\ \sigma_{xy} &= G\gamma_{xy} \end{aligned} \tag{3}$$

where the constants E and ν represent the Young modulus and the Poisson ratio, respectively.

2.2 Variational Method

The virtual work and variational principles can be used to obtain governing differential equations and associated boundary conditions, (Deus, 2004). So, a special case of the principle of virtual displacements that deals with linear as well as nonlinear elastic bodies is known as the principle of minimum total potencial energy and it is applied to solve the variational problem. Let $\mathbf{u}_0$ be the solution to the problem. Then,

$$\pi(\mathbf{u}_0) \leq \pi(\mathbf{u}) \qquad \forall\, \mathbf{u} \in \mathbb{K} \tag{4}$$

thus, $\mathbf{u}_0$ minimize $\pi(\mathbf{u})$, where:

$$\pi(\mathbf{u}) = \frac{1}{2}\int_\Omega \mathbf{D}\varepsilon \cdot \varepsilon \, d\Omega - \int_\Omega b \cdot \mathbf{u} \, d\Omega - \int_{\Gamma_T} t \cdot \mathbf{u} \, d\Gamma_T \tag{5}$$

Notice that,

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \\ \sigma &= \mathbf{D} \cdot \varepsilon \end{aligned} \tag{6}$$

replacing Eq. (6) into Eq. (5), the first integral is equal to

$$\pi_1(\mathbf{u}) = \frac{1}{2}\int_\Omega \mathbf{D}\varepsilon \cdot \varepsilon \, d\Omega = \frac{1}{2}\int_\Omega (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\gamma_{xy}) \, d\Omega \tag{7}$$

substituting the relations of Eq. (3) implies that,

$$\pi_1(\mathbf{u}) = \frac{1}{2}\int_\Omega \left[\frac{E}{1-\nu^2}(\varepsilon_{xx}^2 + \nu\varepsilon_{xx}\varepsilon_{yy}) + \frac{E}{1-\nu^2}(\varepsilon_{yy}^2 + \nu\varepsilon_{xx}\varepsilon_{yy}) + G\gamma_{xy}^2\right] d\Omega \tag{8}$$

thus,

$$\pi_1(\mathbf{u}) = \frac{1}{2}\int_\Omega \left[\frac{E}{1-\nu^2}(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + 2\nu\varepsilon_{xx}\varepsilon_{yy}) + G\gamma_{xy}^2\right] d\Omega \tag{9}$$

Denoting,

$$\begin{aligned} A &= \varepsilon_{xx}^2 + \varepsilon_{yy}^2 + 2\nu\varepsilon_{xx}\varepsilon_{yy} \\ B &= \gamma_{xy}^2 \end{aligned} \tag{10}$$

and substituting the relations of Eq. (1) in A and B gives,

$$\begin{aligned} A &= \left[(u_{,x} - zw_{,xx})^2 + (v_{,y} - zw_{,yy})^2 + 2\nu(u_{,x} - zw_{,xx})(v_{,y} - zw_{,yy})\right] \\ B &= \left[(u_{,y} + v_{,x}) - 2zw_{,xy}\right]^2 \end{aligned} \tag{11}$$

hence,

$$\pi_1(\mathbf{u}) = \frac{1}{2}\int_\Omega \left[\frac{E}{1-\nu^2}(A) + G(B)\right] d\Omega \tag{12}$$

Let $\Omega = \Lambda \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, where h is the thickness and Λ is the area of the plate. Integrating trough the thickness $\left(z \in \left[-\frac{h}{2}, \frac{h}{2}\right]\right)$, i.e. dz,

$$\int_{-h/2}^{h/2} 1 \, dz = h \qquad \int_{-h/2}^{h/2} z \, dz = 0 \qquad \int_{-h/2}^{h/2} z^2 \, dz = \frac{h^3}{12} \tag{13}$$

the functional is given by,

$$\pi_1(\mathbf{u}) = \frac{1}{2}\int_\Lambda \left\{ \frac{Eh}{1-\nu^2}(A_1) + \frac{Eh^3}{12(1-\nu^2)}(B_1) + Gh(u_{,y}+v_{,x})^2 - \frac{4Gh^3}{12} w_{,xy}^2 \right\} d\Lambda \tag{14}$$

with,

$$\begin{aligned} A_1 &= \left[u_{,x}^2 + v_{,y}^2 + 2\nu u_{,x} v_{,y}\right] \\ B_1 &= \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy}\right] \end{aligned} \tag{15}$$

Defining,

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad G = \frac{E}{2(1+\nu)} \tag{16}$$

this implies that,

$$\frac{4Gh^3}{12} = 2D(1-\nu) \tag{17}$$

so,

$$\frac{Eh}{(1-\nu^2)} 2(1-\nu) u_{,x} v_{,y} = 4Gh\, u_{,x} v_{,y} \tag{18}$$

Substituting all relations and considering the static bending in the absence of in-plane forces, the functional takes the form,

$$\pi_1(w) = \frac{1}{2}\int_\Lambda D\left\{(w_{,xx}+w_{,yy})^2 + 2(1-\nu)[w_{,xy}^2 - w_{,xx}w_{,yy}]\right\} d\Lambda \tag{19}$$

Now, supposing that the plate is subjected to a uniform load q, the second part of the functional reduces to:

$$\pi_2(w) = \int_\Omega b\cdot\mathbf{u}\, d\Omega + \int_{\Gamma_T} t\cdot\mathbf{u}\, d\Gamma_T = \int_\Lambda q\cdot w\, d\Lambda \tag{20}$$

Therefore, the problem is reduces to determinate $w(x,y)$ that minimize the following functional:

$$\pi(w) = \frac{1}{2}\int_\Lambda D\left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1+\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} d\Lambda - \int_\Lambda q\cdot w\, d\Lambda \tag{21}$$

2.3 Exterior Penalty Method

Consider again the string problem with an obstacle. The problem consists in the minimization of the functional $\pi(w)$ subjected to the constraint $w \in \mathbb{K}$, i.e., in the determination of $u \in \mathbb{K}$ such that

$$\pi(u) = \min \pi(w), \qquad \forall\, w \in \mathbb{K} \tag{22}$$

where $\mathbb{K} = \{w \in \mathbf{W} \,|\, w - g \leq 0 \,\text{at}\, (x,y) \in (0,L)\times(0,L)\}$.

With the introduction of the indicator of the convex set $\mathbb{K}$, defined as

$$I_{\mathbb{K}}(w) = \left\{ \begin{array}{lll} 0, & \text{if} & w \in \mathbb{K} \\ \infty, & \text{if} & w \notin \mathbb{K} \end{array} \right. \tag{23}$$

The application of the exterior penalty method determines the solution of Eq. (22) by solving a sequence of unconstrained problems, formulated as: Find $u \in \mathbb{K}$ that

$$u = \lim_{\epsilon \to 0} u_\epsilon \tag{24}$$

where u_ϵ is the solution of: Given $\epsilon > 0$, determine $u_\epsilon \in \mathbb{W}$ solution of:

$$u_\epsilon = \arg\min_{w\in\mathbb{W}} \pi_\epsilon(w) \tag{25}$$

in which

$$\pi_\epsilon(w) = \pi(w) + \frac{1}{2\epsilon}P(w) \tag{26}$$

The functional $P(w)$ must satisfy the following conditions:

1. $P(w) = 0$, if $w \in \mathbb{W}$
2. $P(w) \geq 0$ and $P(w) \to \infty$, for $\|w\| \to \infty$, $w \notin \mathbb{K}$

Differents forms to construction of the functional $P(w)$ are presented in the literature. In this paper, will be considered the functionals differentiables, representing the restriction of the unilateral contact, so the contact in one point is given by $P(w) = [\langle w(x^*, y^*) - g\rangle^+]^2$. Where,

$$\langle w - g\rangle^+ = \begin{cases} w - g, & \text{if} \quad w - g > 0 \\ 0, & \text{if} \quad w - g \leq 0 \end{cases}$$

Consequently, the extended functional considered just in one point of contact yields:

$$\begin{aligned} \pi_\epsilon(w) = \frac{1}{2}\int_\Lambda D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1+\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} d\Lambda \\ - \int_\Lambda q \cdot w \, d\Lambda + \frac{1}{2\epsilon}[\langle w(x^*, y^*) - g\rangle^+]^2 \end{aligned} \tag{27}$$

2.4 Approximate Numerical Solution – The Ritz Method

An approximate solution to this problem can be obtained by applying Ritz method, (Glowinski, 1984). In this case, it is defined by

$$w^{\text{aprox}}(x, y) = \sum_{j=1}^{N} a_j \phi_j(x, y) \tag{28}$$

where ϕ_j are linear independent function given by $\phi_j(x, y) = x^{1+m} y^{1+n}$ with $m, n = 1, ..., N$.
Thus, replacing Eq.(28) into Eq.(27) yields,

$$\pi_\epsilon(w) = \frac{1}{2}\int_\Lambda D\left\{[A_3]^2 + 2(1+\nu)[B_3]\right\} d\Lambda - \int_\Lambda q \cdot [C_3] \, d\Lambda + \frac{1}{2\epsilon}[D_3]^2 \tag{29}$$

where

$$\begin{aligned} A_3 &= \sum_{j=1}^{N} a_j \frac{\partial^2 \phi_j}{\partial x^2} + \sum_{j=1}^{N} a_j \frac{\partial^2 \phi_j}{\partial y^2} \\ B_3 &= \left(\sum_{j=1}^{N} a_j \frac{\partial^2 \phi_j}{\partial x \partial y}\right)^2 + \left(\sum_{j=1}^{N} a_j \frac{\partial^2 \phi_j}{\partial x^2}\right)\left(\sum_{j=1}^{N} a_j \frac{\partial^2 \phi_j}{\partial y^2}\right) \\ C_3 &= \sum_{j=1}^{N} a_j \phi_j(x, y) \\ D_3 &= \left(\sum_{j=1}^{N} a_j \phi_j(x^*, y^*) - g\right)^+ \end{aligned} \tag{30}$$

At this point, the problem is to find $a \in \mathbb{R}^N$ that $a = \arg\min[\pi_\epsilon(a)]$ and the discretized functional can be written as,

$$\pi_\epsilon(a) = \pi(a) + \frac{1}{2\epsilon}P(a) \tag{31}$$

From the necessary optimality criteria for $a \in \mathbb{R}^N$ to be the minimun of $\pi_\epsilon(a)$:

$$\left.\frac{\partial \pi_\epsilon^N}{\partial a_i}\right|_a = 0 \qquad \pi_\epsilon^N : \mathbb{R}^N \to \mathbb{R} \tag{32}$$

Note that the discrete problem is nonlinear since, for the given distributed load, we are not able *a priori* to know if the plate will be constrained by the support or not.
The necessary optimality criterion establishes that

$$\nabla \pi_\epsilon|_a = 0 \qquad \Rightarrow \qquad \nabla \pi|_a + \frac{1}{2\epsilon}\nabla P(a) = 0 \tag{33}$$

$\pi(a)$ is a quadratic function in a. Also, $\nabla\pi|_a$ is given by $[K_1]a - F$, where $[K_1]$ is a constant matrix $N \times N$ and F is a constant vector. So, in this case:

$$P(a) = \begin{cases} \left[\sum_{j=1}^{N} a_j\phi_j(x^*, y^*) - g\right]^2, & \text{if} \quad \sum_{j=1}^{N} a_j\phi_j(x^*, y^*) \geq g \\ 0, & \text{if} \quad \sum_{j=1}^{N} a_j\phi_j(x^*, y^*) \leq g \end{cases} \tag{34}$$

$\nabla P(a)|_a$ is given by $[K_2(a)]a$. Thus,

$$[K_1]a + \frac{1}{\epsilon}[K_2(a)]a = F \qquad \Rightarrow \qquad \left[K_1 + \frac{1}{\epsilon}K_2(a)\right]a = F \tag{35}$$

denoting

$$[\mathbf{K}(a)] = K_1 + \frac{1}{\epsilon}K_2(a) \tag{36}$$

Finally, from the optimality criterion that a must satisfy the following set of nonlinear equations:

$$[\mathbf{K}(a)]a = F \tag{37}$$

For the solution to the set of nonlinear equations, in Eq. (37), Newton's method is applied.

2.5 Newton's Method

Let $R(a^k) = F - [\mathbf{K}(a^k)]a^k$ the residual vector, the problem consists in finding an approximate solution a^k such that $\|R(a^k)\| < \text{tol} = 10^{-6}$. The algorithm may be described as:

1. initialize a^k, error = 1, tol = 10^{-6} and set k = 0
2. while (error > tol) do:
   - compute the corrector Δa^k by solving the following linear system

     $$[\mathbf{K}(a^k)]\Delta a^k = -R(a^k) \tag{38}$$

   - compute the new trial solution a^{k+1}

     $$a^{k+1} = a^k + \Delta a^k \tag{39}$$

   - compute the error measure

     $$\text{error} = \|R(a^{k+1})\| \tag{40}$$

   - perform the update procedure

     $$k = k + 1, \qquad a^k \leftarrow a^{k+1} \tag{41}$$

3. end while

By the way, the iterative procedure associated with Newton's method is obtained.

3. NUMERICAL APPLICATION

3.1 Problem Analysed

Consider the unilateral contact problem related to the thin plate problem, shown in Fig. 1. The problem is a square plate clamped in two edges, subjected to a uniform load. The mechanical properties and dimensions of the laminae are the following: $E = 210$ GPa, $\nu = 0.3$, $h = 5$ mm, $a = 2$m and $b = 2$m. The definition of the numbers of base functions used in the aproximation procedure is $N = 3$. Also, the gap or penalty factor is $\Delta = 10$mm.

To compare the displacements before and after the plate reach a rigid obstacle, four loads are choosen, namely: $q_{0_1} = 2.5$ kN/m^2, $q_{0_2} = 5$ kN/m^2, $q_{0_3} = 50$ kN/m^2 and $q_{0_4} = 500$ kN/m^2. In the first case, the obstacle is not hitted. In the second load, the plate is tangent with the rigid obstacle. The third and fourth loads reach the obstacle. The problem is solved using an uniform mesh (20×20).

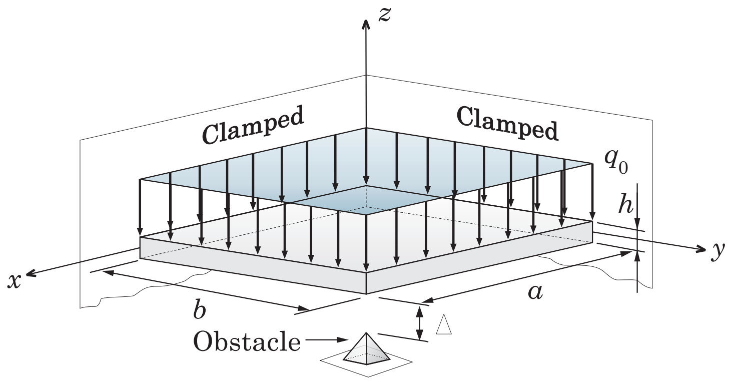
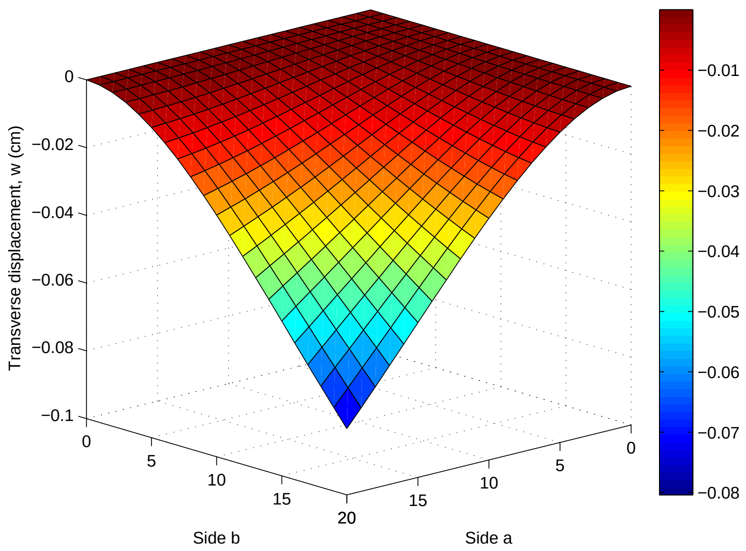
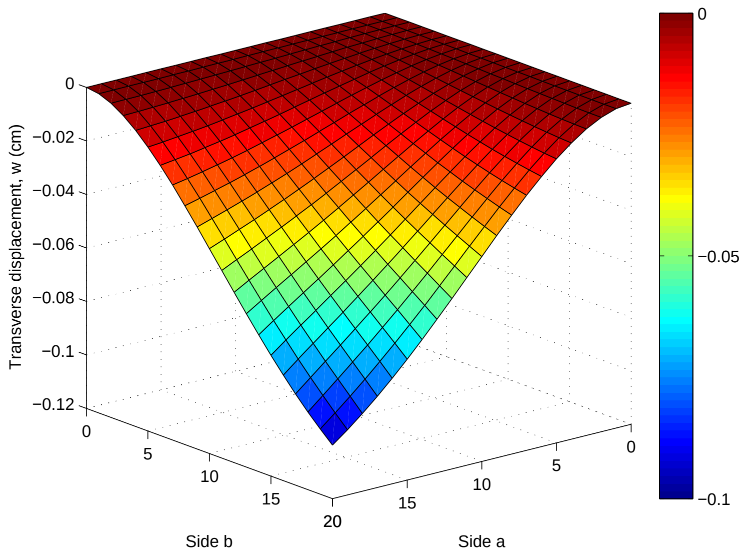


Figure 1. The plate subjected to an uniform load, q_0, and a rigid obstacle.

The displacement plots in Fig. 2(a) through Fig. 3(b) help to demonstrate the behavior of the solutions.

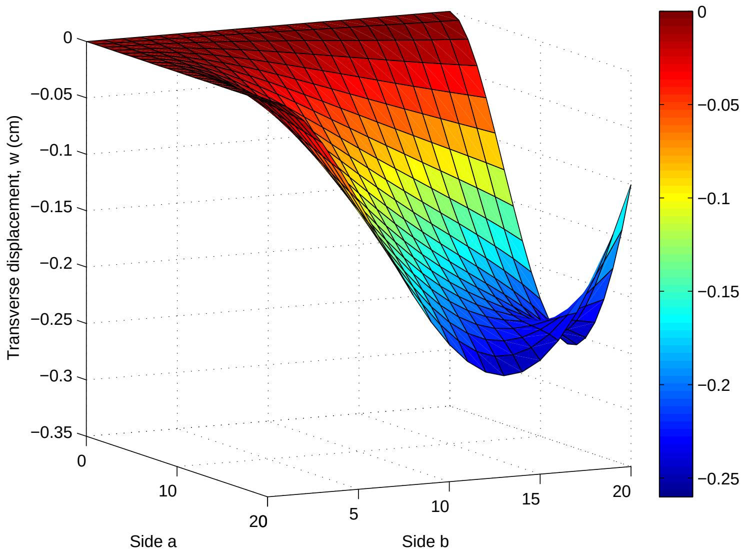


(a) Case 1 – $q_{0_1} = 2.5$ kN/m^2

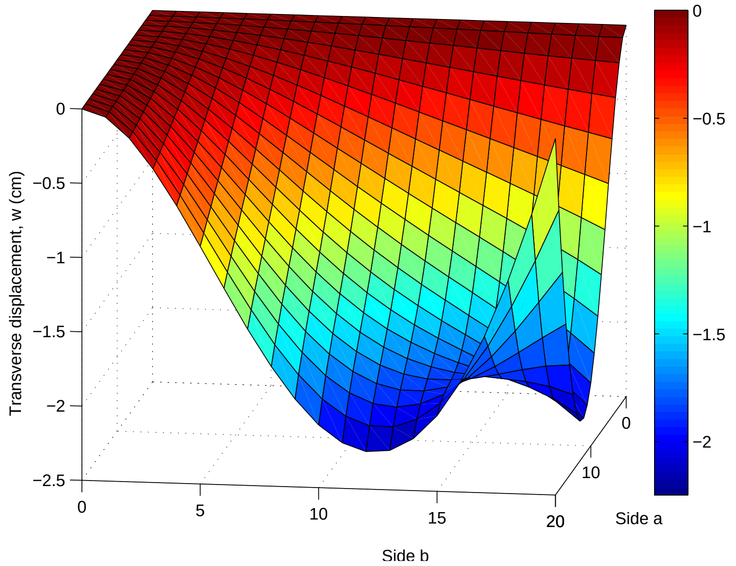


(b) Case 2 – $q_{0_2} = 5$ kN/m^2

Figure 2. Transverse displacements, w.



(a) Case 3 – $q_{0_3} = 50$ kN/m^2



(b) Case 4 – $q_{0_4} = 500$ kN/m^2

Figure 3. Transverse displacements, w.

The effects of the error through the loads is shown in Tab. 1. To solve the first case of the problem two iterations are needed.

Table 1. Error evolution.

	Error			
Iteration	q_{0_1}	q_{0_2}	q_{0_3}	q_{0_4}
1	0.534	1.068	10.681	1.068E+2
2	2.654E-13	1.824E+5	4.524E+6	4.794E+7
3	—	0.006	2.919	2.278
4	—	3.695E-9	1.548E-6	1.593E-6
5	—	—	2.740E-9	1.127E-7

4. SUMMARY AND CONCLUSIONS

This paper is concerned with the analysis of plate employing the Ritz method. To solve the problem of the nonlinear equations, the Newton's method is employed. Also, the algorithm is described. The paper focuses on the plate's displacements and how it affects the good behavior of a numerical solution. Using the transparency of the continuum mechanics, the problem of the thin plate subjected to an uniform load and unilateral contact is solved. Four cases of load were analyzed using the model.

It can be concluded that it is advantageous to use Newton's method to solve a set of nonlinear equations and the Matlab® code used is efficient.

5. ACKNOWLEDGEMENTS

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7. RESPONSIBILITY NOTICE