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THE NEWTON-EULER MULTIBODY EQUATIONS REVISITED

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Abstract. *The equations corresponding to Newton-Euler iterative method for the determination of forces and torques acting on the rigid links of a manipulator are given a new treatment by using aggregated vectors for the representation of both kinematical and dynamical quantities. It is shown that Lagrange´s Equations for the motion of a scleronomic system are easily found from basic linear and angular accelerations of each link and from forces and torques computed directly by Newton-Euler equations for a rigid body.*

Keywords: *Multibody, Newton-Euler, Lagrange, Robots.*

1. INTRODUCTION

In previous works (Cortizo, 1991; Cortizo and Giacaglia, 1993; Kottke, 2004; Giacaglia and Kottke, 2007) it has been shown how to approach the dynamics of connected rigid bodies by the introduction of vectors representing simultaneous kinematical and dynamical quantities, which have been called aggregated vectors. In these works the Newton-Euler Iterative Method for computing forces and torques acting on each link and joint were not given in explicit forms and the final differential equations where left in a complex vector form. In this work, considering an arbitrary geometrical representation for the configuration of the system such as, for example, the Denavit-Hartenberg parameters (Craig, 1986), generalized coordinates are introduced and all kinematical and dynamical vectors are expressed in terms of these coordinates although maintaining their original Newtonian character. Aggregated vectors representing all forces and torques acting on each link and joint are given explicit form. The final differential equations, given in terms of aggregated vectors, are transformed by simple scalar products between these aggregated vectors and it is shown that the resulting differential equations are completely equivalent to the differential equations obtained from the Lagrangian of the system. A different approach, although also based on basic principles of differential geometry, has been developed by Blajer (1995). The method is applicable to rigid multibody systems in the presence of holonomic and nonholonomic constraints, a possibility available through the methodology used in the present work. Blajer (1995) uses the Gram-Schmidt orthogonalization process in order to generate an orthonormal basis of the unconstrained space with respect to the constrained subspace. The method is most fitted for numerical formulation and integration. This method and the one developed in this work are more effective and straightforward than classical methods such as the ones described by Craig (1986) and Shabana (1989).

2. KINEMATICAL EQUATIONS

Let us consider a robotic manipulator as a mechanical system constituted by n rigid links connected sequentially by n constraints. More precisely, link 1 is connected to a fixed base at one hand by means of a rotational or prismatic joint and at the other hand is connected with link 2 by means of a rotational or prismatic constraint. Link 2 is connected to link 3 in the same way and this chain goes on to the last link, number n, which is connected on one hand to link $n-1$ and free at the other hand. These rigid bodies will be called links of the manipulator robot. The center of mass of link number i shall be indicated by C_i for $i = 1, 2,..., n$.

Joint i is the joint connecting link $i-1$ to link i (the fixed base should be considered as the link number 0). The position and the aperture of link i will be described by a reference point J_i, by a directional unit vector $\boldsymbol{e_i}$ and by a scalar parameter of aperture q^i. As an example of the theory described in earlier works, we shall here consider only rotational links and each body is a slender straight bar where rotation about its axis will be neglected. These bodies will be called links or arms of a robot. Since joint i is rotational we call J_i a point chosen on the rotation axis, that is, fixed to links $i-1$ and i, $\boldsymbol{e_i}$ is a unit vector parallel to that axis and q^i is the angle between links $i-1$and i, giving the aperture of this constraint, oriented according to vector $\boldsymbol{e_i}$.

The set of angles $q = (q^1, q^2, ..., q^n)$ can be used as an independent set of generalized coordinates describing the robot configuration at any instant. With a description of the robot geometry, by means of Denavit-Hartenberg parameters or any other equivalent representation, it is possible to compute the position of points J_i and vectors $\boldsymbol{e}_i$ corresponding to $q = (q^1, q^2, ..., q^n)$.

Let $\dot{q} = (\dot{q}^1, ..., \dot{q}^n)$ be the *generalized velocities* and $\ddot{q} = (\ddot{q}^1, ..., \ddot{q}^n)$ the *generalized accelerations* associated to a configuration given by the coordinates q. For a given configuration of the system it is possible to find the n-vectors $\dot{q}$ and $\ddot{q}$. These vectors define in a unique way the coordinates of the centers of mass C_i, as well as their linear velocities. They also determine in a unique way the angular velocities of each link, represented by vectors

$$\begin{cases} \boldsymbol{\upsilon}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^{\alpha} \boldsymbol{\upsilon}_{\alpha i}(q) \\ \boldsymbol{\omega}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^{\alpha} \boldsymbol{\omega}_{\alpha i}(q) \end{cases} \tag{1}$$

In the present case all generalized variables are angles of link i with respect to the previous link i-1 so that the absolute value of vector $\boldsymbol{\upsilon}_{\alpha i}(q)$ is the distance ℓ_i between point C_i and joint J_i. Vectors $\boldsymbol{\omega}_{\alpha i}(q)$ are dimensionless. The linear and angular accelerations are given by

$$\begin{aligned} \frac{d\boldsymbol{\upsilon}_i}{dt}(q,\dot{q},\ddot{q}) &= \frac{d}{dt}\left[\sum_{\alpha=1}^{n} \dot{q}^{\alpha} \boldsymbol{\upsilon}_{\alpha i}(q)\right] \\ &= \sum_{\alpha=1}^{n} \ddot{q}^{\alpha} \boldsymbol{\upsilon}_{\alpha i}(q) + \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \dot{q}^{\beta} \dot{q}^{\gamma} \frac{\partial \boldsymbol{\upsilon}_{\beta i}}{\partial q^{\gamma}}(q) \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{d\boldsymbol{\omega}_i}{dt}(q,\dot{q},\ddot{q}) &= \frac{d}{dt}\left[\sum_{\alpha=1}^{n} \dot{q}^{\alpha} \boldsymbol{\omega}_{\alpha i}(q)\right] \\ &= \sum_{\alpha=1}^{n} \ddot{q}^{\alpha} \boldsymbol{\omega}_{\alpha i}(q) + \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \dot{q}^{\beta} \dot{q}^{\gamma} \frac{\partial \boldsymbol{\omega}_{\beta i}}{\partial q^{\gamma}}(q) \end{aligned} \tag{3}$$

3. ITERATIVE METHOD

Consider now the *distributions of actions* on each link. The pair $(\mathbf{F}_i, \mathbf{N}_i)$ is the distribution of all actions (forces and torques) acting on link i, using as pole their centers of mass C_i $(i = 1,...,n)$. The first part of the iterative method of Newton-Euler leads to the computation of these forces $\mathbf{F}_i$ and torques $\mathbf{N}_i$ corresponding to a given $3n$ dimensional set of generalized coordinates, velocities and accelerations $(q, \dot{q}, \ddot{q})$. This calculation is well known and starts from the fixed base of the manipulator up to its free end. Using elementary rigid body kinematics and taking into account the geometry and mass distribution of each link.

The total torque acting on link i is given by

$$\begin{aligned} \mathbf{N}_i(q,\dot{q},\ddot{q}) &= I_i\left(\frac{d\boldsymbol{\omega}_i}{dt}\right) + \boldsymbol{\omega}_i \times I_i(\boldsymbol{\omega}_i) = \\ &= I_i\left[\sum_{\alpha=1}^{n} \ddot{q}^{\alpha} \boldsymbol{\omega}_{\alpha i}(q) + \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \dot{q}^{\beta} \dot{q}^{\gamma} \frac{\partial \boldsymbol{\omega}_{\beta i}}{\partial q^{\gamma}}(q)\right] + \\ &+ \left[\sum_{\beta=1}^{n} \dot{q}^{\beta} \boldsymbol{\omega}_{\beta i}(q)\right] \times I_i\left[\sum_{\gamma=1}^{n} \dot{q}^{\lambda} \boldsymbol{\omega}_{\gamma i}(q)\right] = \\ &= \sum_{\alpha=1}^{n} \ddot{q}^{\alpha} \left\{I_i\left[\boldsymbol{\omega}_{\alpha i}(q)\right]\right\} + \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \dot{q}^{\beta} \dot{q}^{\gamma} \left\{\left[\frac{\partial (I_i \boldsymbol{\omega}_{\beta i})}{\partial q^{\gamma}}(q)\right] + \left[\boldsymbol{\omega}_{\beta i}(q)\right] \times I_i\left[\boldsymbol{\omega}_{\gamma i}(q)\right]\right\} \end{aligned} \tag{4}$$

The force acting on the center of mass C_i of link i is given by

$$\begin{aligned}\mathbf{F}_i(q,\dot{q},\ddot{q}) &= m_i\left(\frac{d\mathbf{v}_i}{dt}\right) = m_i\left[\sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\mathbf{v}_{\alpha i}(q)+\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\dot{q}^{\gamma}\frac{\partial\mathbf{v}_{\beta i}}{\partial q^{\gamma}}(q)\right] \\ &= \sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\left[m_i\mathbf{v}_{\alpha i}(q)\right]+\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\dot{q}^{\gamma}\left[m_i\frac{\partial\mathbf{v}_{\beta i}}{\partial q^{\gamma}}(q)\right]\end{aligned} \tag{5}$$

The results of the above evaluations corresponding to the *3n* set $(q,\dot{q},\ddot{q})$ will be simply indicated by $\mathbf{F}_i(q,\dot{q},\ddot{q})$ and $\mathbf{N}_i(q,\dot{q},\ddot{q})$ for *(i=1,...,n)*.

Following the Newton- Euler method one computes the actions exchanged between consecutives link of the robot, starting from pairs $(\mathbf{F}_i,\mathbf{N}_i)$, describing the distribution of all actions acting on link *i*, taking as the reference pole the center of mass of each link. From these pairs, one computes the pairs of actions $(\mathbf{f}_i,\mathbf{n}_i)$ describing the distribution of actions applied by link *i* - 1 on link *i* and taking as reference pole the connecting points (joints) J_i (i = 1,...n). The distribution of actions applied by link *i* on link *i – 1* for *(I = 1, 2, ...,n)* is obviously given by Newton´s Third Law of Action and Reaction and given by $(-\mathbf{f}_i,-\mathbf{n}_i)$.

The classical Newton-Euler Iterative method computes these pairs starting from the free and of the robot and going toward the fixed base. In order to do this one performs the balance of efforts applied to a specific link taking into account the fact that the distribution of all actions applied to link i can be decomposed in two other: the distribution of actions applied on this link by the preceding link and the distribution of actions applied on this same link by the following link, that is

$$\begin{cases}\mathbf{F}_i = \mathbf{f}_i - \mathbf{f}_{i+1} \\ \mathbf{N}_i = [\mathbf{n}_i + \mathbf{C}_i\mathbf{J}_i\times\mathbf{f}_i]+[(-\mathbf{n}_{i+1})+\mathbf{C}_i\mathbf{J}_{i+1}\times(-\mathbf{f}_{I+1})]\end{cases} \qquad (i = 1,\ldots,\ n-1)$$
$$\{\mathbf{F}_n = \mathbf{f}_n, \quad \mathbf{N}_n = [\mathbf{n}_n + \mathbf{C}_n\mathbf{J}_n\times\mathbf{f}_n] \tag{6}$$

It is important to interpret correctly these equations for the torque $\mathbf{N}_i$: terms between brackets represent simply a “change of pole” of pairs $(\mathbf{f}_i,\mathbf{n}_i)$ and $(-\mathbf{f}_i+1,-\mathbf{n}_{i+1})$, done in order to match its poles with those of $(\mathbf{F}_i,\mathbf{N}_i)$, allowing this way that the above summation to be written correctly. In other words, pair $(\mathbf{f}_i,\mathbf{n}_i)$ gives a distribution of actions taking joint J_i, as the pole while pair $(\mathbf{f}_i,\mathbf{n}_i+\mathbf{C}_i\mathbf{J}_i\times\mathbf{f}_i)$ gives exactly the same distribution of actions with respect to pole C_i, which is the pole of pair $(\mathbf{F}_i,\mathbf{N}_i)$. Like way pairs $(-\mathbf{f}_{i+1},-\mathbf{n}_{i+1})$ and $(-\mathbf{f}_{i+1}\ ,\ -\mathbf{n}_{i+1}+\mathbf{C}_i\mathbf{J}_{i+1}\times(-\mathbf{f}_{i+1}))$ give the same distribution of actions with respect to different poles: the first uses as pole the joint J_{j+1}, while the second uses the mass center C_i of link *i*. Therefore the above equations are exact even if points J_i have been chosen arbitrarily apart from the joints between links. This subject will be given a full development after the concept of aggregated vectors is introduced.

At this point aggregated vectors, representing in a single 6n – dimensional vector any kinematical and dynamical quantity involved, are introduced.

The first aggregated vector to be introduced is represented by $\mathbf{v\omega}$ and is given by the

$$\mathbf{v\omega} = (\mathbf{v}_1,\boldsymbol{\omega}_1,\mathbf{v}_2,\boldsymbol{\omega}_2,\ldots,\mathbf{v}_n,\boldsymbol{\omega}_n) \tag{7}$$

and from the above expressions for the linear and angular velocities it is found that

$$\mathbf{v\omega} = \sum_{\alpha=1}^{n}\dot{q}^{\alpha}\mathbf{v\omega}_{\alpha} \tag{8}$$

where

$$\mathbf{v\omega}_{\alpha} = (\mathbf{v}_{\alpha 1},\boldsymbol{\omega}_{\alpha 1},\mathbf{v}_{\alpha 2},\boldsymbol{\omega}_{\alpha 2},\ldots,\mathbf{v}_{\alpha n},\boldsymbol{\omega}_{\alpha n}) \tag{9}$$

with components defined as above; The 6n – dimensional vector $\mathbf{v\omega}$ is the vector of virtual velocities associated to the state $(q,\dot{q})$ of the system.

The pair force-torque will be represented by $(\mathbf{F}_i,\mathbf{N}_i)(q,\dot{q},\ddot{q})$ where the components vectors are given by $\mathbf{F}_i(q,\dot{q},\ddot{q})$ and

$\mathbf{N}_i(q,\dot{q},\ddot{q})$ ($i = 1,..., n$) and corresponding to the 3n – dimensional set $(q,\dot{q},\ddot{q})$.

A value of this 3n-dimensional set $(q,\dot{q},\ddot{q})$ defines the vector positions, velocities and accelerations of all the n rigid bodies constituting the system. Newton-Euler equations give the resulting forces $\mathbf{F}_i(q,\dot{q},\ddot{q})$ and resulting torques $\mathbf{N}_i(q,\dot{q},\ddot{q})$ acting on link i (i =1, ..., n) of the robot and corresponding to that $3n$-dimensional set $(q,\dot{q},\ddot{q})$. These $2n$ three dimensional vectors can be aggregated in only one 6n – dimensional vector $\mathbf{FN}(q,\dot{q},\ddot{q})$**,** given by

$$\mathbf{FN}(q,\dot{q},\ddot{q}) = \left(\mathbf{F}_1(q,\dot{q},\ddot{q}), \mathbf{N}_i(q,\dot{q},\ddot{q}),..., \mathbf{F}_n(q,\dot{q},\ddot{q}), \mathbf{N}_n(q,\dot{q},\ddot{q})\right) \tag{10}$$

The aggregated actions actually accelerate the Newton-Euler iterative method. In fact, consider the aggregated vectors $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2,..., \mathbf{F}_n)$ and $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2,..., \mathbf{N}_n)$ and the corresponding pairs $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2,..., \mathbf{f}_n)$ and $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2,..., \mathbf{n}_n)$ connected by Eq. (6). It is easily found that

$$\mathbf{f} = (\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n, \mathbf{F}_2 + \mathbf{F}_3 \cdots + \mathbf{F}_n,..., \mathbf{F}_{n-1} + \mathbf{F}_n, \mathbf{F}_n) \tag{11}$$

and

$$\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2,..., \mathbf{n}_n) \tag{12}$$

where the component vectors are found backward from the last joint J_n to the first joint J_1 by recurrence as follows:

$$\mathbf{n}_n = \mathbf{N}_n - \mathbf{C}_n\mathbf{J}_n \times \mathbf{F}_n \tag{13}$$

$$\mathbf{n}_k = \mathbf{n}_{k+1} + \mathbf{N}_k - \mathbf{C}_k\mathbf{J}_k \times \mathbf{F}_k + \mathbf{C}_k\mathbf{J}_{k-1} \times \left(\mathbf{F}_{k+1} + \mathbf{F}_{k+2} + \cdots + \mathbf{F}_n\right) \tag{14}$$

for $k = n-1, n-2,...,2,1$.

Using the above relations it is possible to compute the aggregate force-torque vectors acting on each center of mass of the robots links and on the joints between links here represented by Eq. (10) and

$$\mathbf{fn} = (\mathbf{f}_1, \mathbf{n}_1, \mathbf{f}_2, \mathbf{n}_2,..., \mathbf{f}_n, \mathbf{n}_n) \tag{15}$$

4. DYNAMICS

The expressions derived above (Eqs. (4) and (5)) for $\mathbf{F}_i(q,\dot{q},\ddot{q})$ and $\mathbf{N}_i(q,\dot{q},\ddot{q})$ can be condensed in a unique 6n – dimensional vector

$$\mathbf{FN}(q,\dot{q},\ddot{q}) = \sum_{\alpha=1}^{n} \ddot{q}^{\alpha}\mathbf{PL}_{\alpha}(q) + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} \dot{q}^{\beta}\dot{q}^{\gamma}\mathbf{XY}_{\beta\gamma}(q) \tag{16}$$

where

$$\mathbf{PL}_{\alpha} = \left(\mathbf{P}_{\alpha 1}, \mathbf{L}_{\alpha 1}, \mathbf{P}_{\alpha 2}, \mathbf{L}_{\alpha 2}, \dots, \mathbf{P}_{\alpha n}, \mathbf{L}_{\alpha n}\right) \tag{17}$$

and

$$\mathbf{XY}_{\beta\gamma} = \left(\mathbf{X}_{\beta\gamma 1}, \mathbf{Y}_{\beta\gamma 1}, \mathbf{X}_{\beta\gamma 2}, \mathbf{Y}_{\beta\gamma 2}, \dots, \mathbf{X}_{\beta\gamma n}, \mathbf{Y}_{\beta\gamma n}\right) \qquad (\alpha,\beta,\gamma = 1,...,n) \tag{18}$$

are given by

$$\mathbf{P}_{\alpha i}(q) = m_i \boldsymbol{\upsilon}_{\alpha i}(q), \qquad \mathbf{L}_{\alpha i}(q) = I_i\left[\boldsymbol{\omega}_{\alpha i}(q)\right] \tag{19}$$

and

$$\mathbf{X}_{\beta\gamma i}(q) = m_i \frac{\partial \boldsymbol{\upsilon}_{\beta i}}{\partial q^{\gamma}}(q) \qquad \mathbf{Y}_{\beta\gamma i}(q) = I_i\left[\frac{\partial \boldsymbol{\omega}_{\beta i}}{\partial q^{\gamma}}(q)\right] + [\boldsymbol{\omega}_{\beta i}(q)] \times I_i[\boldsymbol{\omega}_{\gamma i}(q)] \qquad \text{for } i = 1,\ldots,n. \tag{20}$$

It should be noted that the scalar product of two aggregated vectors must satisfy the physical meaning of such product. As an example, consider the power associated to the aggregated vector

$$\mathbf{FN}(q,\dot{q},\ddot{q}) = (\mathbf{F}_1(q,\dot{q},\ddot{q}), \mathbf{N}_1(q,\dot{q},\ddot{q}),\ldots, \mathbf{F}_n(q,\dot{q},\ddot{q}), \mathbf{N}_n(q,\dot{q},\ddot{q})). \tag{21}$$

The scalar product of this force – torque action by the linear – angular velocity aggregated vector $\boldsymbol{\upsilon\omega} = (\boldsymbol{\upsilon}_1, \boldsymbol{\omega}_1, \boldsymbol{\upsilon}_2, \boldsymbol{\omega}_2,\ldots, \boldsymbol{\upsilon}_n, \boldsymbol{\omega}_n)$ is given by the obvious equation

$$\mathbf{FN}(q,\dot{q},\ddot{q}) \circ \boldsymbol{\upsilon\omega}(q,\dot{q}) = \sum_{i=1}^{n}[\mathbf{F}_i \circ \mathbf{v}_i + \mathbf{N}_i \circ \boldsymbol{\omega}_i] \tag{22}$$

and this quantity is clearly the total power of the forces and torque acting on the system.

Vectors $\boldsymbol{\upsilon}_{\alpha i}$ and $\boldsymbol{\omega}_{\alpha i}$ defined in Eq. (1) are the linear and angular velocities of body i ($i = 1,\ldots, n$) in the state where all generalized velocities are zero except the one (α) which is unitary. Therefore vectors $\mathbf{P}_{\alpha i}$ and $\mathbf{L}_{\alpha i}$ above defined represent the linear and angular momenta of body i in this state of the system (only the α velocity is different from zero and unitary, all other being zero). This way it follows that the aggregated vector $\mathbf{PL}_{\alpha}$ is simply the aggregated torque corresponding to a motion of the system where all generalized coordinates are constant except the α coordinate which moves with unitary velocity.

5. EQUATIONS OF MOTION

From Eq. (16) it is very simple to obtain the differential equations of motion of the whole system obtaining an equivalent result as that given by the explicit form of Lagrange´s Equations (Giacaglia, 1978).

In fact consider the aggregated vectors, for $\alpha = 1,2,\ldots, \mathrm{n}$,

$$\mathbf{v\omega}_{\alpha} = (\mathbf{v}_{\alpha 1}, \boldsymbol{\omega}_{\alpha 1},\ldots, \mathbf{v}_{\alpha n}, \boldsymbol{\omega}_{\alpha n}) \tag{23}$$

where vectors $\mathbf{v}_{\alpha i}, \boldsymbol{\omega}_{\alpha i}$ have been defined by Eq. (1),

$$\begin{cases} \boldsymbol{\upsilon}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^{\alpha} \boldsymbol{\upsilon}_{\alpha i}(q) \\ \boldsymbol{\omega}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^{\alpha} \boldsymbol{\omega}_{\alpha i}(q) \end{cases} \tag{24}$$

Because of the dimension and assumed value of vector $\mathbf{v}_{\alpha i}$ the absolute value of vector $\mathbf{v}_{\alpha i}$ is given by ℓ_i, the distance from the center of mass C_i to joint J_i and vectors $\boldsymbol{\omega}_{\alpha i}$ are dimensionless. Consider Eq. (16) and the scalar product

$$\begin{aligned} \mathbf{PL}_{\alpha}(q) \circ \mathbf{v\omega}_{\gamma}(q) &= \sum_{i=1}^{n}(\mathbf{P}_{\alpha i} \circ \mathbf{v}_{\gamma i} + \mathbf{L}_{\alpha i} \circ \boldsymbol{\omega}_{\gamma i}) = \\ &= \sum_{i=1}^{n}[m_i(\mathbf{v}_{\alpha i} \circ \mathbf{v}_{\gamma i}) + I_i(\boldsymbol{\omega}_{\alpha i}) \circ \boldsymbol{\omega}_{\gamma i}] = a_{\gamma\alpha}(q) \end{aligned} \tag{25}$$

The coefficients $a_{\gamma\alpha}(q)$ are the coefficients of the quadratic terms $\dot{q}^{\gamma}\dot{q}^{\alpha}$ in the total kinetic energy of the system, that is

$$T = \frac{1}{2}\sum_{\alpha}\sum_{\beta} a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} \tag{26}$$

This is obvious because the first part of Eq. (25) on the right hand side is the linear kinetic energy (except for a factor $\dot{q}^{\alpha}\dot{q}^{\gamma}/2$) of link i because the velocity of the center of mass C_i of this link associated to the generalized coordinate (angle) q^{α} is precisely $\mathbf{v}_{\alpha i}$ except for a factor $\dot{q}^{\alpha}$. The second part is clearly the rotational kinetic energy of this same link except for the same factor $\dot{q}^{\alpha}\dot{q}^{\gamma}/2$.

The same scalar product is now applied to the second term on the right–hand side of Eq. (16), giving

$$\begin{aligned}&\mathbf{XY}_{\alpha\beta}\circ\mathbf{v\omega}_{\gamma}(q)=\\&=\sum_{i=1}^{n}m_i\frac{\partial\mathbf{v}_{\alpha i}}{\partial q^{\beta}}\circ\mathbf{v}_{\gamma i}+\sum_{i=1}^{n}\left[\frac{\partial I_i(\boldsymbol{\omega}_{\alpha i})}{\partial q^{\beta}}+\boldsymbol{\omega}_{\alpha i}\times I_i(\boldsymbol{\omega}_{\beta i})\right]\circ\boldsymbol{\omega}_{\gamma i}=a^{\gamma}_{\beta\alpha}(q)\end{aligned} \tag{27}$$

It is seen that the coefficients $a^{\gamma}_{\beta\alpha}(q)$ correspond to the Christoffel Brackets $\begin{bmatrix}\gamma\\ \beta\alpha\end{bmatrix}$ resulting from the quadratic part of Lagrange´s Equations when the constraints are time independent and considering unitary linear and angular velocities.

Remembering the definition of the coefficients $a_{\gamma\alpha}(q)$ given by Eq. (25) and comparing Eq. (27) with the classical definition (Giacaglia, 1978)

$$\begin{bmatrix}i\\ \beta\alpha\end{bmatrix}=\frac{\partial a_{i\alpha}}{\partial q^{\beta}}+\frac{\partial a_{i\beta}}{\partial q^{\alpha}}-\frac{\partial a_{\alpha\beta}}{\partial q^{i}} \tag{28}$$

it is seen that there is an exact correspondence between the two definitions.

The same scalar product applied to the force-torque aggregated vector given by Eq. (21), gives

$$\mathbf{FN}(q,\dot{q},\ddot{q})\circ\mathbf{v\omega}_{\gamma}(q)=\sum_{i=1}^{n}\mathbf{F}_i\circ\mathbf{v}_{\gamma i}+\sum_{i=1}^{n}\mathbf{N}_i\circ\boldsymbol{\omega}_{\gamma i}=Q_{\gamma} \tag{29}$$

The quantities Q_{γ} are the generalized forces in the Lagrange Equations and give a direct physical interpretation of these quantities. Note that from Eq. (29) the dimension of the generalized forces associated to an angular generalized coordinate q^{α} is ML^2T^{-2}, that is, the torque of the applied force with respect to joint J_i.

With the above definitions, the equations of motion for the generalized coordinates assume the form

$$\sum_{\alpha}a_{\gamma\alpha}\ddot{q}^{\alpha}+\sum_{\alpha}\sum_{\beta}\begin{bmatrix}\gamma\\ \beta\alpha\end{bmatrix}\dot{q}^{\alpha}\dot{q}^{\beta}=Q_{\gamma} \tag{30}$$

As was to be expected Eq. (30) is exactly the explicit form of Lagrange´s Equations for a system with constraints not time dependent (scleronomic constraints).

Following the classical procedure, let $b_{\gamma\alpha}$ be the elements of the inverse matrix $\mathbf{A}=(a_{\gamma\alpha})$ and define the Christoffel Parentheses

$$\begin{pmatrix}p\\ \beta\alpha\end{pmatrix}=\sum_{\gamma}\begin{bmatrix}\gamma\\ \beta\alpha\end{bmatrix}b_{p\gamma} \tag{31}$$

and the transformed generalized force corresponding to the action on the generalized coordinate q^s given by

$$Q^s=\sum_{\alpha}b_{s\alpha}Q_{\alpha} \tag{32}$$

The final differential equations of motion are then given by

$$\ddot{q}^s+\sum_{\alpha}\sum_{\beta}\begin{bmatrix}s\\ \beta\alpha\end{bmatrix}\dot{q}^{\alpha}\dot{q}^{\beta}=Q^s \tag{33}$$

for $s=1,2,\ldots,n$.

6. EXAMPLE

A simple example will be used in order to apply the above developed formalism without complicating the physical system. We consider a two-articulated links moving in a fixed vertical plane OA and AB articulated to a fixed base at point O while A is the free end of the system.. These links are homogeneous rigid bars under the effect of gravity. A torque N_1 is acting on link OA and torque N_2 is acting on link AB and both torques are acting along axes normal to the plane of motion and considered to be positive when acting in the counterclockwise sense.. These torques might represent the total effect of friction in each joint or externally applies torques resulting from motors. Internal forces satisfying Newton Principle of action and reaction play no role in the system , as well as forces not producing work, such as the reaction at the fixed base O of the system.

The rotational joint O is articulated to a fixed support. The rotational joint A is connecting links OA and AB whose lengths are $2L_1$ and $2L_2$. The generalized coordinates are the angles $\theta_1 = q^1$ e $\theta_2 = q^2$, that links OA and AB make with a fixed horizontal line, so that its time derivatives are the absolute angular velocities of each link. This is done instead of defining the angle $\theta = \theta_2 - \theta_1$ between links AB and OA, as it is done in usual robots kinematics. For the purpose of this work there is no point in adopting this notation. The positions of the centers of mass C_1 (link OA) and C_2 (link AB) are given by

$$C_1 - O = L_1(c_1\mathbf{e}_1 + s_1\mathbf{e}_2) \tag{34}$$

and

$$C_2 - O = 2L_1(c_1\mathbf{e}_1 + s_1\mathbf{e}_2) + L_2(c_2\mathbf{e}_1 + s_2\mathbf{e}_2) \tag{35}$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is an inertial base on the fixed plane of motion and $c_1 = \cos q^1$, $s_1 = \sin q^1$, $c_2 = \cos q^2$ and $s_2 = \sin q^2$.

The linear velocities of these points are

$$\mathbf{v}_1 = \dot{q}^1\mathbf{v}_{11} + \dot{q}^2\mathbf{v}_{21},\ \mathbf{v}_{11} = L_1(-s_1\mathbf{e}_1 + c_1\mathbf{e}_2),\ \mathbf{v}_{21} = \mathbf{0} \tag{36}$$

$$\mathbf{v}_2 = \dot{q}^1\mathbf{v}_{21} + \dot{q}^2\mathbf{v}_{22},\ \mathbf{v}_{12} = 2L_1(-s_1\mathbf{e}_1 + c_1\mathbf{e}_2),\ \mathbf{v}_{22} = L_2(-s_2\mathbf{e}_1 + c_2\mathbf{e}_2) \tag{37}$$

The angular velocities of each link are

$$\boldsymbol{\omega}_1 = \dot{q}^1\boldsymbol{\omega}_{11},\ \boldsymbol{\omega}_2 = \dot{q}^2\boldsymbol{\omega}_{22} \tag{38}$$

where

$$\boldsymbol{\omega}_{11} = \boldsymbol{\omega}_{22} = \mathbf{e}_3 \tag{39}$$

because the angles were chosen to be with respect to an inertial line.

The accelerations of these points are also easily found to be

$$\begin{aligned} \frac{d\mathbf{v}_1}{dt} &= \ddot{q}^1\mathbf{v}_{11} + \dot{q}^1\dot{q}^1\mathbf{a}_{11} \\ \mathbf{a}_{11} &= -L_1(c_1\mathbf{e}_1 + s_1\mathbf{e}_2) \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{d\mathbf{v}_2}{dt} &= \ddot{q}^1\mathbf{v}_{12} + \ddot{q}^2\mathbf{v}_{22} + \dot{q}^1\dot{q}^1\mathbf{a}_{12} + \dot{q}^2\dot{q}^2\mathbf{a}_{22} \\ \mathbf{a}_{12} &= -2L_1(c_1\mathbf{e}_1 + s_1\mathbf{e}_2),\ \mathbf{a}_{22} = -L_2(c_2\mathbf{e}_1 + s_2\mathbf{e}_2), \end{aligned} \tag{41}$$

$$\dot{\boldsymbol{\omega}}_1 = \ddot{q}^1\mathbf{e}_3,\ \dot{\boldsymbol{\omega}}_2 = \ddot{q}^2\mathbf{e}_3\,. \tag{42}$$

For the present example, the forces and torques producing work on the system are

$$\mathbf{F}_1(q,\dot{q},\ddot{q}) = -m_1 g\mathbf{e}_2\,,\ \mathbf{F}_2(q,\dot{q},\ddot{q}) = -m_2 g\mathbf{e}_2\,,\ \mathbf{N}_1(q,\dot{q},\ddot{q}) = N_1\mathbf{e}_3\,,\ \mathbf{N}_2(q,\dot{q},\ddot{q}) = N_2\mathbf{e}_3\,. \tag{43}$$

For Eq. (16) we find that

$$\mathbf{PL}_1 = \left[m_1\mathbf{v}_{11}\left(q^1\right), I_1\mathbf{e}_3, m_2\mathbf{v}_{12}\left(q^1\right), \mathbf{0}\right],\ \mathbf{PL}_2 = \left[m_2\mathbf{v}_{21}\left(q^1\right), \mathbf{0}, m_2\mathbf{v}_{22}\left(q^1\right), I_2\mathbf{e}_3\right] \tag{44}$$

$$\mathbf{XY}_{11}(q) = \left[m_1\mathbf{a}_{11}\left(q^1\right), \mathbf{0}, m_2\mathbf{a}_{12}\left(q^1\right), \mathbf{0}\right],\ \mathbf{XY}_{22}(q) = \left[\mathbf{0}, \mathbf{0}, m_2\mathbf{a}_{22}\left(q^2\right), \mathbf{0}\right] \tag{45}$$

all other aggregated vectors being null vectors.

According to Eq. (16), the differential equation of motion is

$$\mathbf{FN}(\mathbf{F}_1,\mathbf{N}_1,\mathbf{F}_2,\mathbf{N}_2) = \ddot{q}^1\mathbf{PL}_1 + \ddot{q}^2\mathbf{PL}_2 + \dot{q}^1\dot{q}^1\mathbf{XY}_{11} + \dot{q}^2\dot{q}^2\mathbf{XY}_{22}$$

with the above definitions for the coefficients of this equation.

We now consider the aggregated vectors

$$\mathbf{v\omega}_1 = \left[\mathbf{v}_{11}\left(q^1\right), \mathbf{e}_3, \mathbf{v}_{12}\left(q^1\right), \mathbf{0}\right] \tag{50}$$

and

$$\mathbf{v\omega}_2 = \left[\mathbf{v}_{21}\left(q^1\right), \mathbf{0}, \mathbf{v}_{22}\left(q^1\right), \mathbf{e}_3\right] \tag{51}$$

It is easily found that

$$\mathbf{FN}\circ\mathbf{v\omega}_1 = -(m_1 + 2m_2)gl_1c_1 + N_1 = Q_1 \tag{52}$$

$$\mathbf{FN}\circ\mathbf{v\omega}_2 = -m_2gl_2c_2 + N_2 = Q_2 \tag{53}$$

Moreover it is found that

$$\mathbf{PL}_1\circ\mathbf{v\omega}_1 = (m_1 + 4m_2)l_1^2 + I_1 = a_{11}(q) \tag{54}$$

$$\mathbf{PL}_1\circ\mathbf{v\omega}_2 = 2l_1l_2m_2c_{12}\,,\ c_{12} = cos\left(q^1 - q^2\right) = a_{21} \tag{55}$$

$$\mathbf{PL}_2\circ\mathbf{v\omega}_1 = 2m_2l_1l_2c_{12} = a_{12} \tag{56}$$

$$\mathbf{PL}_2\circ\mathbf{v\omega}_2 = m_2l_2^2 + I_2 = a_{22}\,. \tag{57}$$

From the definition of the Christoffel Brackets, in this particular example it is found that the only non-zero bracket is

$$a_{22}^1 = 2m_2l_1l_2s_{12}\,,\ s_{12} = sin\left(q^1 - q^2\right).$$

The differential equations are given by

$$\begin{aligned}&\left[(m_1 + 4m_2)l_1^2 + I_1\right]\ddot{q}^1 + \left[2m_2l_1l_2c_{12}\right]\ddot{q}^2 + \left[2m_2l_1l_2s_{12}\right]\dot{q}^2\dot{q}^2 = \\ &= (m_1 + 2m_2)gl_1c_1 + N_1\end{aligned} \tag{58}$$

$$\left[2l_1l_2m_2c_{12}\right]\ddot{q}^1 + \left[m_2l_2^2 + I_2\right]\ddot{q}^2 = -m_2gl_2c_2 + N_2 \tag{59}$$

In order to have explicit equations for $\ddot{q}^1$ and for $\ddot{q}^2$ it is a simple matter of solving the above system for these two quantities. In this simple example, expressing $\ddot{q}^1$ in terms of all other terms in Eq. (35) and substituting into Eq. (34) we obtain a differential equation containing only the first and second derivatives of q^2 with coefficients functions of both variables q^1 and q^2. Obviously this is the well known problem of a double pendulum with the additional complication of the presence of torques applied to the two arms of the pendulum. The integration of this system if not the scope of the present work.

7. CONCLUSIONS

The above development shows how the use of aggregated vectors leads to Lagrange´s Equations of motion of a system composed of connected rigid bodies. The computation of all quantities involved in Eq. (30) is straightforward from the computation of forces and torques acting on the system, as obtained directly by the application of Newton-Euler iterative process to the force-torque aggregated vector.

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