Procedings of COBEM 2007
19th International Congress of Mechanical Engineering

November 5-9, 2007, Brasília, DF

APPLICATION OF A FEASIBLE INTERIOR POINT ALGORITHM FOR NONLINEAR COMPLEMENTARITY ON CONTACT PROBLEMS IN 3D LINEAR ELASTICITY

Sandro R. Mazorche, sandro.mazorche@ufjf.edu.br
Mathematics Department - ICE, UFJF, Juiz de Fora, MG, Brazil, Campus Universitário, CEP 36036-330

José Herskovits, jose@optimize.ufrj.br
Alfredo Canelas, acanelas@fing.edu.uy
Gabriel M. Guerra, gguerra@fing.edu.uy
Mechanical Engineering Program, COPPE, UFRJ, Rio de Janeiro, RJ, Brazil, P.O.Box 68503, 21945 970

Abstract. *In this paper we present two methods to solve contact problems in 3D linear elasticity based on a recently presented feasible interior point algorithm for nonlinear complementarity problems (FIPA-NCP). The first method is based on the variational formulation of the elasticity equations and uses the Finite Element Method (FEM) to discretize the continuous problem. The resultant discrete equations define a complementarity problem which is solved using the FIPA-NCP. The second method uses the Boundary Element Method (BEM) to discretize the continuous problem. The discretized equations define a mixed complementarity problem that is solved using a variant of the FIPA-NCP for mixed complementarity problems (FIPA-MNCP). Some examples of contact problems in elasticity are solved showing the applicability of the presented strategies.*

Keywords: *Feasible Interior Point Algorithm, Nonlinear Complementarity Problems, Contact Problems.*

1. INTRODUCTION

Contact Problems in Solid Mechanics appear when contact forces are transmitted between two different bodies through their boundaries. Many fields like forming operations, crashworthiness and biomechanical applications deal with contact problems. The analysis of these problems has been considered for several authors, we can mention the works by Panagiotopoulos (1975), Simo et al. (1985), Klarbring (1986), Kikuchi and Oden (1988), Raous et al. (1988), Bjorkman et al. (1995) and Wriggers and Fisher (2003).

In reason of the nonlinear nature of the contact problems, analytic solutions cannot in general be obtained and we need numerical algorithms to find an approximated solution. Here, we present two methods for the contact problem in linear elasticity (Signorini's problem). Both methods formulate the contact problem in linear elasticity as a complementarity problem. Complementarity Problems arise in many fields of Engineering and Economics (Ferris and Pang, 1997). Several works dealing with such kind of problems have been presented, we can mention the works by Cottle et al. (1980), Chen and Mangasarian (1996) and Mazorche and Herskovits (2005). The main advantage of formulating the contact problem as a complementarity one is to take advantage of the new, fast and robust algoritms that have been recently created for such kind of problems. In this work we investigate the effectiveness of the recently proposed algorithms FIPA-NCP and FIPA-MNCP, for simple and mixed complementarity problems, respectively.

The first proposed method is based on the variational formulation of Signorini's problem and uses the FEM to discretize the continuous problem. The discrete form is a finite dimensional optimization problem with linear constraints where the Karush-Kunh-Tucher optimality conditions can be formulated as a complementarity problem. For solving this problem we use the feasible interior point algorithm FIPA-NCP (Mazorche and Herskovits, 2005). The second method is based on a boundary integral formulation of Signorini's problem and employs the Boundary Element Method to discretize the boundary equations. The boundary conditions when applied to the BEM equations lead to a finite dimensional mixed complementarity problem. This problem is solved using a variant of the FIPA-NCP algorithm for mixed complementarity problems called FIPA-MNCP.

Section 4 describes the nonlinear complementarity problem and presents the iterative algorithm FIPA-NCP for such problems. Section 5 describes the mixed nonlinear complementarity problem and presents the iterative algorithm FIPA-MNCP. Section 6 introduces the contact problem in linear elasticity and shows how to formulate it as a finite dimensional complementarity problem using the FEM or the BEM. Some examples are presented in Section 7. Finally, the conclusions are presented in Section 8.

2. THE NONLINEAR COMPLEMENTARITY PROBLEM (NCP)

Let $\mathbf{S} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear vector function. The nonlinear complementarity problem consists in finding $\mathbf{x} \in \mathbb{R}^n$ such that:

$$\mathbf{x} \geq 0, \quad \mathbf{S}(\mathbf{x}) \geq 0 \quad \text{and} \quad \mathbf{x} \circ \mathbf{S}(\mathbf{x}) = 0 \tag{1}$$

where $\mathbf{x} \geq 0$ means that each component of vector $\mathbf{x}$ is nonnegative, and "$\circ$" denote the entrywise Hadamard product for vectors, given by $(\mathbf{x} \circ \mathbf{y})_i = \mathbf{x}_i\, \mathbf{y}_i$.

Defining the feasible set $\Upsilon = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq 0,\, \mathbf{S}(\mathbf{x}) \geq 0\}$, it is easy to see that $\mathbf{x}$ is solution of problem (1) if and only if $\mathbf{x}$ is in the feasible set and $\mathbf{x} \circ \mathbf{S}(\mathbf{x}) = 0$.

FIPA-NCP is an iterative algorithm to find the solution of problem (1). It starts from an initial point in Υ and generates a sequence of points in Υ that converge to the required solution. It first defines a search direction and performs a line search along that direction to find a point with lower value for the potential function $\Phi(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{x}_i\, \mathbf{S}_i(\mathbf{x})$. That point is defined to be the next point of the sequence and the algorithm returns to the first step till a convergence criterion be satisfied. The search direction is based on the Newton's direction for the nonlinear system of equations $\mathbf{x} \circ \mathbf{S}(\mathbf{x}) = 0$. To obtain convergence to the solution, the Newton's direction is modified by a restoration direction like in (Herskovits, 1998). The present approach is supported by strong theoretical studies (see Mazorche and Herskovits, 2005).

The following notation will be employed to describe the algorithm FIPA-NCP: $\mathbf{S}^k = \mathbf{S}(\mathbf{x}^k)$, $\mathbf{M}^k = \nabla(\mathbf{x} \circ \mathbf{S}(\mathbf{x}))$, $\Phi^k = \Phi(\mathbf{x}^k)$, $\nabla\Phi^k = \nabla\Phi(\mathbf{x}^k)$ and $\mu^k = \Phi^k/n$.

FIPA-NCP
Data: $\mathbf{x}^0 \in \text{int}(\Upsilon)$, $k = 0$, $\epsilon > 0$, $\mathbf{E} = [1, ..., 1]^T$, $\nu, \nu_1 \in (0, 1)$, $\alpha \in (0, 1/2)$.

Step 1: Computation of the search direction $\mathbf{d}^k$
Find $\mathbf{d}^k$ solving the following linear system of equations:

$$\mathbf{M}^k\, \mathbf{d}^k = -\mathbf{x}^k \circ \mathbf{S}^k + \alpha\, \mu^k\, \mathbf{E} \tag{2}$$

Step 2: Line search
Set t as the first number in the sequence $\{1, \nu, \nu^2, \nu^3, ...\}$ that satisfies:

$$\begin{aligned} \mathbf{x}^k + t\, \mathbf{d}^k &\geq 0 \\ \mathbf{S}(\mathbf{x}^k + t\, \mathbf{d}^k) &\geq 0 \\ \Phi^k + t\, \nu_1\, (\nabla\Phi^k \cdot \mathbf{d}^k) &\geq \Phi(\mathbf{x}^k + t\, \mathbf{d}^k) \end{aligned}$$

Step 3: Update
Set $\mathbf{x}^{k+1} = \mathbf{x}^k + t\, \mathbf{d}^k$ and $k = k + 1$.

Step 4: Stop criterion
If $\|\mathbf{x}^k \circ \mathbf{S}^k\| \leq \epsilon$ stop, else go to step 1.

In reference (Mazorche and Herskovits, 2005) has been shown that the search direction $\mathbf{d}^k$ is well defined in Υ whether function $\mathbf{S}$ verifies some usual regularity assumptions.

3. THE MIXED NONLINEAR COMPLEMENTARITY PROBLEM (MNCP)

Let $\mathbf{S} : \mathbb{R}^{m+p} \to \mathbb{R}^m$ and $\mathbf{Q} : \mathbb{R}^{m+p} \to \mathbb{R}^p$ be nonlinear vector functions. The mixed nonlinear complementarity problem consist in finding $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+p}$ such that:

$$\mathbf{x} \geq 0\ ,\ \mathbf{S}(\mathbf{x}, \mathbf{y}) \geq 0 \ \text{ and } \ \begin{cases} \mathbf{x} \circ \mathbf{S}(\mathbf{x}, \mathbf{y}) = 0 \\ \mathbf{Q}(\mathbf{x}, \mathbf{y}) = 0 \end{cases} \tag{3}$$

It can be easily shown that this definition is equivalent to the classic definition given for example in (Ferris and Pang, 1997). Let the feasible set be: $\Upsilon_M = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+p} \;/\; \mathbf{x} \geq 0 \;\text{ e }\; \mathbf{S}(\mathbf{x}, \mathbf{y}) \geq 0\}$. Then, a point $(\mathbf{x}, \mathbf{y})$ is a solution of the MNCP if it is in the feasible set and verifies $\mathbf{x} \circ \mathbf{S}(\mathbf{x}, \mathbf{y}) = 0$ and $\mathbf{Q}(\mathbf{x}, \mathbf{y}) = 0$.

In this case, the potential function is defined as: $\Phi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \mathbf{x}_i\, \mathbf{S}_i(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{m} \mathbf{Q}_j(\mathbf{x}, \mathbf{y})^2$.

The following notation will be employed to describe the algorithm FIPA-MNCP: $\mathbf{S}^k = \mathbf{S}(\mathbf{x}^k, \mathbf{y}^k)$, $\mathbf{Q}^k =$

$\mathbf{Q}(\mathbf{x}^k, \mathbf{y}^k)$, $\mathbf{M}^k = \nabla[\mathbf{x}^k \circ \mathbf{S}(\mathbf{x}^k, \mathbf{y}^k), \mathbf{Q}(\mathbf{x}^k, \mathbf{y}^k)]$, $\mathbf{d}^k = [\mathbf{d}^k_\mathbf{x}, \mathbf{d}^k_\mathbf{y}]^T$, $\Phi^k = \Phi(\mathbf{x}^k, \mathbf{y}^k)$, $\nabla\Phi^k = \nabla\Phi(\mathbf{x}^k, \mathbf{y}^k)$ and

$$\mu^k = \begin{cases} (\Phi^k + c_1^k)/n & \text{if} \quad n + c_2^k \leq 0 \\ (\Phi^k + c_1^k)/(n + c_2^k) & \text{if} \quad n + c_2^k > 0 \end{cases} \quad \text{with } c_1^k = \sum_{j=1}^m \mathbf{Q}_j(\mathbf{x}^k, \mathbf{y}^k)^2 \ \text{ and } c_2^k = 2\sum_{j=1}^m \mathbf{Q}_j(\mathbf{x}^k, \mathbf{y}^k)$$

FIPA-MNCP

Data $(\mathbf{x}^0, \mathbf{y}^0) \in \text{int}(\Upsilon_\text{M})$, $k = 0$, $\epsilon > 0$, $\mathbf{E} = [1, ..., 1]^T$, $\nu, \nu_1 \in (0, 1)$, $\alpha \in (0, 1/2)$.

Step 1: Computation of the search direction $\mathbf{d}^k$

Find $\mathbf{d}^k$ solving the linear system of equations:

$$\mathbf{M}^k\, \mathbf{d}^k = -[\mathbf{x}^k \circ \mathbf{S}^k, \mathbf{Q}^k]^T + \alpha\, \mu^k\, \mathbf{E} \tag{4}$$

Step 2: Line search

Set t as the first number in the sequence $\{1, \nu, \nu^2, \nu^3, ...\}$ that satisfies:

$$\begin{aligned} \mathbf{x}^k + t\,\mathbf{d}^k_\mathbf{x} &\geq 0 \\ \mathbf{S}(\mathbf{x}^k + t\,\mathbf{d}^k_\mathbf{x}, \mathbf{y}^k + t\,\mathbf{d}^k_\mathbf{y}) &\geq 0 \\ \Phi^k + t\,\nu_1\,(\nabla\boldsymbol{\Phi}^k \cdot \mathbf{d}^k) &\geq \Phi(\mathbf{x}^k + t\,\mathbf{d}^k_\mathbf{x}, \mathbf{y}^k + t\,\mathbf{d}^k_\mathbf{y}) \end{aligned}$$

Step 3: Update

Set $\mathbf{x}^{k+1} = \mathbf{x}^k + t\,\mathbf{d}^k_\mathbf{x}$, $\mathbf{y}^{k+1} = \mathbf{y}^k + t\,\mathbf{d}^k_\mathbf{y}$ and $k = k + 1$.

Step 4: Stop criterion

If $\|[\mathbf{x}^k \circ \mathbf{S}^k, \mathbf{Q}^k]\| \leq \epsilon$ stop, else go to step 1.

4. SIGNORINI'S PROBLEM

The classical form of the Signorini's problem in linear elasticity reads as follow:

$$\begin{array}{lrcll} \text{a)} & -\nabla\cdot\boldsymbol{\sigma} & = & \bar{\mathbf{f}} & \text{on } \Omega \\ \text{b)} & \mathbf{u} & = & \bar{\mathbf{u}} & \text{in } \Gamma^D \\ \text{c)} & \mathbf{p} & = & \bar{\mathbf{p}} & \text{in } \Gamma^N \\ & & & & \\ \text{d)} & \mathbf{u}\cdot\bar{\mathbf{n}} + \bar{s} & \geq & 0 & \\ \text{e)} & \mathbf{p}\cdot\bar{\mathbf{n}} & \geq & 0 & \text{in } \Gamma^C \\ \text{f)} & (\mathbf{u}\cdot\bar{\mathbf{n}} + \bar{s})\cdot(\mathbf{p}\cdot\bar{\mathbf{n}}) & = & 0 & \end{array} \tag{5}$$

where Ω is the open domain occupied by the solid and $\Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^C$ its boundary, $\mathbf{u}$ is the displacement function, the Cauchy stress tensor $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon} = \nabla^S\mathbf{u}$ with $\mathbb{C}$ the elasticity tensor and $\nabla^S\mathbf{u} = 1/2(\nabla\mathbf{u} + \nabla^T\mathbf{u})$. Function $\mathbf{p} = \boldsymbol{\sigma}\mathbf{n}$ with $\mathbf{n}$ the outward unit normal vector of Γ. Functions $\bar{\mathbf{f}}$, $\bar{\mathbf{u}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{n}}$ and $\bar{s}$ are given (see Fig. 1).

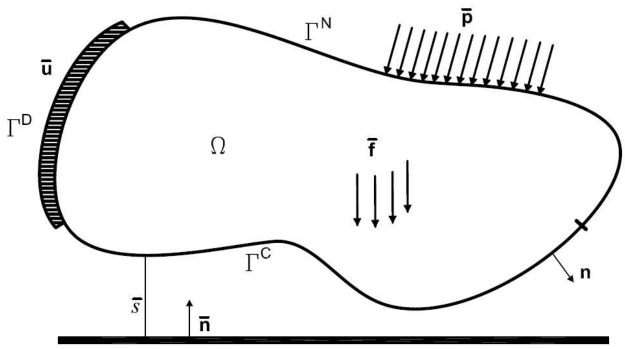

Figure 1. Contact problem in linear elasticity

4.1 Modeling the contact problem with the FEM

The variational formulation of the Signorini's problem (5) is:

$$\begin{array}{ll} \text{Minimize} & \Pi(\mathbf{u}) \\ \text{Subject to:} & \mathbf{u} \in \Sigma \end{array} \tag{6}$$

with

$$\Pi(\mathbf{u}) = \frac{1}{2}\int_{\Omega} (\mathbb{C}\boldsymbol{\epsilon})\cdot\boldsymbol{\epsilon}\,\mathrm{d}\Omega - \int_{\Omega} \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\Omega - \int_{\Gamma^N} \bar{\mathbf{p}}\cdot\mathbf{u}\,\mathrm{d}\Gamma$$

and

$$\Sigma = \left\{\mathbf{u} \in U \ \mid \ \mathbf{u} = \bar{\mathbf{u}} \text{ in } \Gamma^D \text{ and } \mathbf{u}\cdot\bar{\mathbf{n}} + \bar{s} \geq 0 \text{ in } \Gamma^C\right\}$$

Using the finite element discretization we obtain the discrete problem:

$$\begin{array}{ll} \text{Minimize} & \Pi_h(\mathbf{u}) \\ \text{Subject to:} & \mathbf{u} \in \Sigma_h \end{array} \tag{7}$$

where $\Pi_h(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{F}^T\mathbf{u}$, with $\mathbf{K}$ the global stiffness matrix and $\mathbf{F}$ the vector of nodal forces. Σ_h can be represented as: $\Sigma_h = \{\mathbf{u} \in U_h \ \mid \ \mathbf{A}\mathbf{u} + \bar{\mathbf{s}} \geq 0\}$ where $\mathbf{A}$ is the matrix describing the constraints.

The Karush-Kuhn-Tucker conditions for problem (7) are:

$$\begin{aligned} \mathbf{K}\mathbf{u} - \mathbf{F} - \mathbf{A}^T\boldsymbol{\lambda} &= 0 \\ (\mathbf{A}\mathbf{u} + \bar{\mathbf{s}}) \circ \boldsymbol{\lambda} &= 0 \\ \mathbf{A}\mathbf{u} + \bar{\mathbf{s}} &\geq 0 \\ \boldsymbol{\lambda} &\geq 0 \end{aligned} \tag{8}$$

An equivalent complementarity problem for the variable $\boldsymbol{\lambda}$ can be obtained if we use the first equation to define $\mathbf{u}$ depending on $\boldsymbol{\lambda}$. Replacing the obtained expression for $\mathbf{u}$ in the next equations, we obtain:

$$\begin{aligned} \mathbf{S}(\boldsymbol{\lambda}) \circ \boldsymbol{\lambda} &= 0 \\ \mathbf{S}(\boldsymbol{\lambda}) &\geq 0 \\ \boldsymbol{\lambda} &\geq 0 \end{aligned} \tag{9}$$

where the product of the first line in Eq. (9) is a Hadamard product and function $\mathbf{S}$ is defined as: $\mathbf{S}(\boldsymbol{\lambda}) = (\mathbf{A}\mathbf{K}_h^{-1}\mathbf{A}_h^T)\boldsymbol{\lambda} - \mathbf{A}\mathbf{K}_h^{-1}\mathbf{F} + \bar{\mathbf{s}}$.

4.2 Modeling the contact problem with the BEM

The boundary integral equation for linear elasticity with $\bar{\mathbf{f}} = 0$ in Eq. (5) is (Brebbia et al., 1984; Beer and Watson, 1992; París and Cañas, 1998):

$$\mathbf{c}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \int_{\Gamma} \mathbf{u}^*(\boldsymbol{\xi},\mathbf{x})\mathbf{p}(\mathbf{x})\,\mathrm{d}\Gamma - \int_{\Gamma} \mathbf{p}^*(\boldsymbol{\xi},\mathbf{x})\mathbf{u}(\mathbf{x})\,\mathrm{d}\Gamma \tag{10}$$

where function $\mathbf{u}^*$ is the fundamental solution for the linear elasticity problem and $\mathbf{p}^*$ is its correspondent fundamental surface traction. Matrix $\mathbf{c}(\boldsymbol{\xi})$ depends on the local geometry of boundary Γ at point $\boldsymbol{\xi}$ and the second integral on the right is defined in the Cauchy principal value sense (Brebbia et al., 1984; Beer and Watson, 1992; París and Cañas, 1998).

Applying the BEM method, the discrete form of the boundary integral equation result:

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{p} = 0 \tag{11}$$

where now, the vectors $\mathbf{u}$ and $\mathbf{p}$ define the displacements and traction forces on the boundary Γ and $\mathbf{H}$ and $\mathbf{G}$ are the BEM matrices.

Applying the boundary conditions Eqs. (5.b) and (5.c) to Eq. (11), and denoting $\mathbf{x}$ the vector of unknowns related to the normal tractions in Γ^C and $\mathbf{y}$ the vector of remaining unknowns, we obtain:

$$\mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{y} = \mathbf{q} \tag{12}$$

The boundary conditions in Γ^C, Eqs. (5.d) to (5.f), can be written as:

$$\begin{aligned} \mathbf{S}(\mathbf{y}) &\geq 0 \\ \mathbf{x} &\geq 0 \\ \mathbf{S}(\mathbf{y}) \circ \mathbf{x} &= 0 \end{aligned}$$

Finally, defining $\mathbf{Q}(\mathbf{x},\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{q}$, we have the following mixed complementarity problem:

$$\begin{aligned} \mathbf{S}(\mathbf{y}) &\geq 0 \\ \mathbf{x} &\geq 0 \\ \mathbf{S}(\mathbf{y}) \circ \mathbf{x} &= 0 \\ \mathbf{Q}(\mathbf{x},\mathbf{y}) &= 0 \end{aligned} \tag{13}$$

5. NUMERICAL EXAMPLES

This section presents some examples showing the efficacy of the proposed methods. For the FEM analysis the commercial package ABAQUS was employed (ABAQUS, 2003).

5.1 Cylinder

This example consist of a cylinder in contact with a rigid plane as shown by Fig. 2. We use a two-dimensional BEM model.

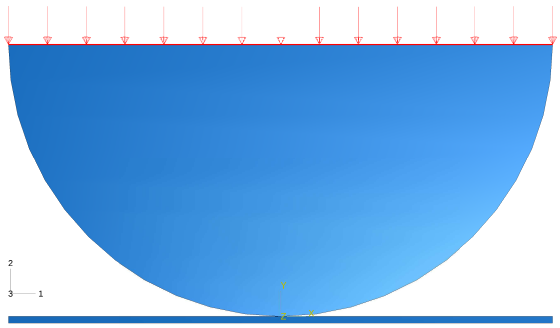

Figure 2. Cylinder in contact with a rigid plane.

This example has an analytic solution for the contact pressure given by the formula: $p(x) = p_m\sqrt{1-(x/b)^2}$, where p_m is the maximum value of the contact force and b is the width of the contact region. They are given by: $p_m = (2P)/(\pi b)$ and $b = \sqrt{(2PD)(1-\nu_m^2)/(\pi E_m)}$, where P is the applied force per unit length, D is the diameter of the cross section and E_m and ν_m are, respectively, the Young's modulus and Poisson's ratio of the elastic material.

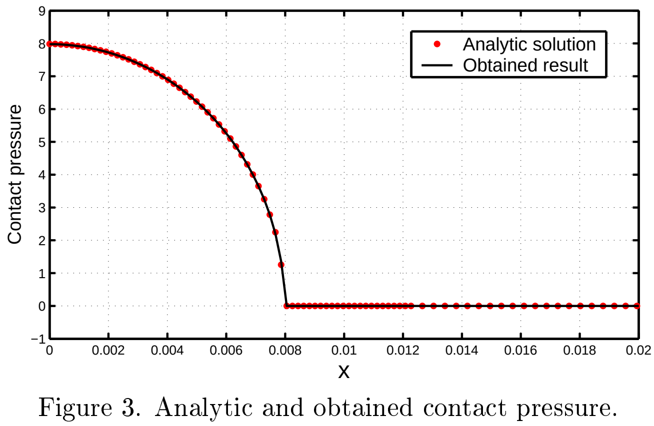

Figure 3. Analytic and obtained contact pressure.

5.2 Curved beam

This example consist of a curved beam in contact with a rigid plane as shown by Fig. 4. This three-dimensional problem was solved using the BEM.

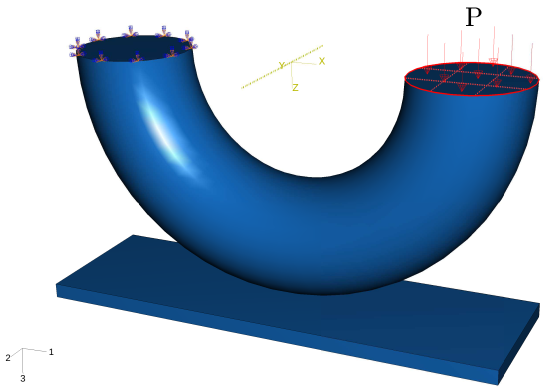

Figure 4. Curved beam.

Figure 5 shows the obtained contact region. The points with positive contact force are enhanced in red color showing the characteristic elliptic shape of the contact region.

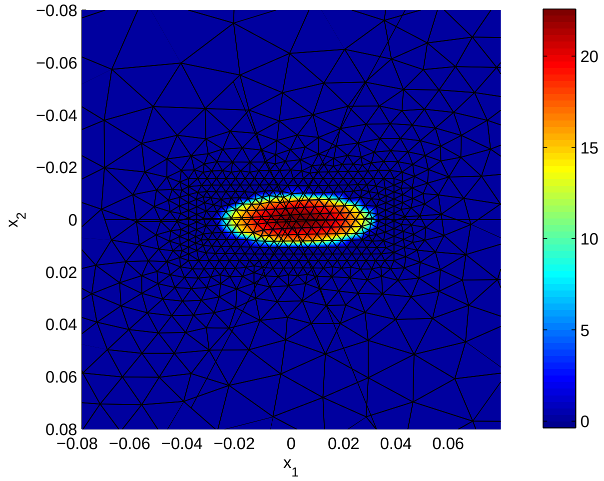

Figure 5. Pressures in the contact region of the curved beam.

5.3 Micro-gripper

In this case we present an elastic three-dimensional self-contact problem. The micro gripper mechanism is loaded by opposite pressures acting on its lateral surfaces as shown by Fig. 6. The non-penetrability condition was taken into account employing conforming meshes. One linear constraint was considered for each pair of opposite nodes. A three-dimensional FEM model was employed.

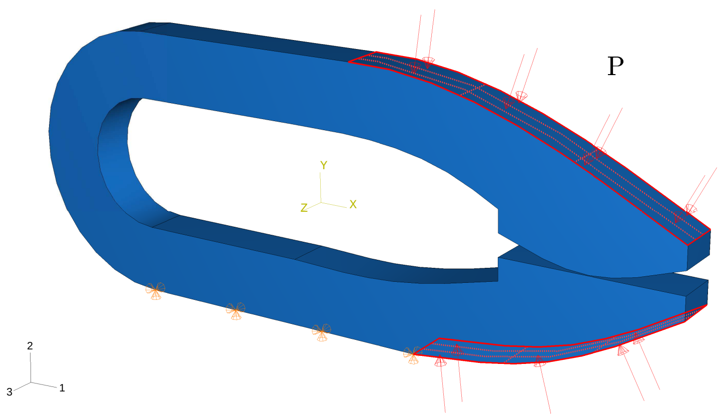

Figure 6. Micro-gripper model with boundary conditions.

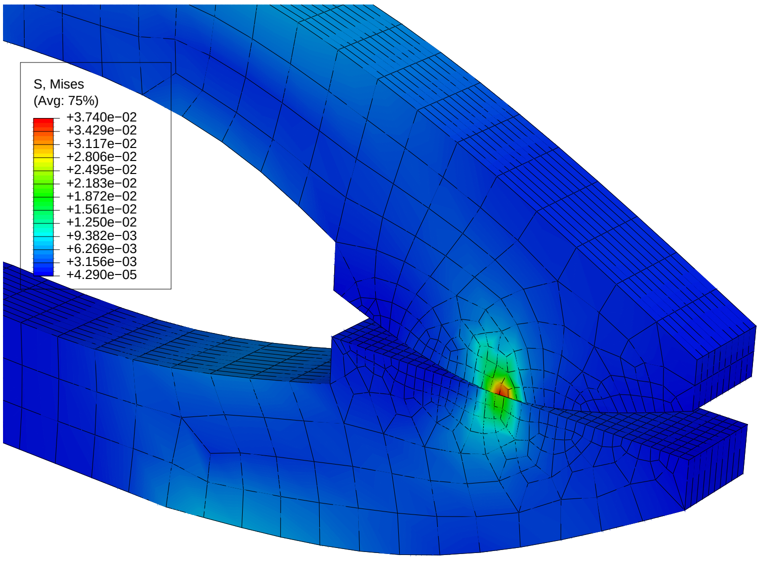

Figure 7. Von Mises stresses in the micro-gripper.

5.4 Rack and pinion

This two-dimensional example presents a contact problem between a rack and a pinion. Like previous example we have used conforming meshes. The structural analysis was carried out using FEM.

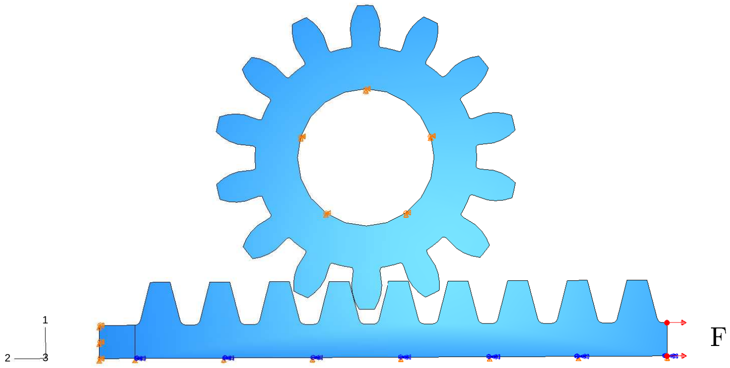

Figure 8. Rack and pinion mechanism with boundary conditions.

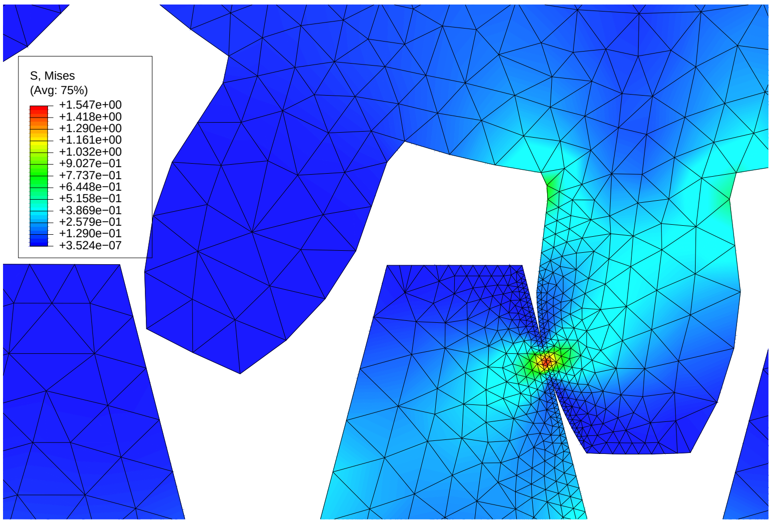

Figure 9. Von Mises stresses in the rack and pinion mechanism.

5.5 Summary of results

Table 1 shows the obtained maximal values for the contact pressures. The reference value is given by the analytic formula, in the case of the cylinder, or by the ABAQUS, for the other examples.

Table 1. Maximal contact pressures in the presented examples

Example	Obtained value	Reference result	Error (%)
Cylinder	7.98	7.98 *	0
Curved beam	22.58	22.55 **	0.13
Micro-gripper	40.63×10^{-3}	41.86×10^{-3} **	-2.87
Rack and pinion	1.57	1.65 **	-4.85

(*) Analytic, (**) ABAQUS.

6. CONCLUSIONS

We have presented two strategies for solving frictionless contact problems in linear elasticity. The first method makes use of the FEM to define a complementarity problem that is solved employing the FIPA-NCP algorithm. The second one uses the BEM to define a mixed complementarity problem that is solved by the FIPA-MNCP algorithm.

The main advantage of the presented methods is the integration of fast and robust algorithms for complementarity problems with standard tools for finite and boundary elements analysis. This characteristic makes the proposed methods suitable for large-scale applications in contact mechanics.

Both methods have been shown effective in the solution of the presented academic examples and some large-scale real applications. These results are encouraging and motivate further investigation about the efficiency of the proposed methods for large-scale applications.

7. REFERENCES

ABAQUS, 2003, *6.4 Analisys User's Guide*, ©ABAQUS, Inc.

Beer, G. and Watson, J. O., 1992, *Introduction to Finite and Boundary Element Methods for Engineers*, Wiley.

Bjorkman, G., Klarbring, A. and Sjodin, B., 1995, "Sequential quadratic programming for non-linear elastic contact problems", *Int. J. Numer. Methods Eng.* **38**, 137–165.

Brebbia, C. A., Telles, J. C. F. and Wrobel, L. C., 1984, *Boundary Element Technique: Theory and Applications in Engineering*, Springer - Verlag.

Chen, C. and Mangasarian, O. L., 1996, "A class of smoothing functions for nonlinear and mixed complementarity problems", *Comput. Optim. Appl.* **5**, 97–138.

Cottle, R. W., Giannessi, F. and Lions, J. L., 1980, *Variational Inequalities and Complementariy Problems*, John Wiley & Sons.

Ferris, M. C. and Pang, J. S., 1997), "Engineering and economic applications of complementarity problems", *SIAM Review* **39**, 669–713.

Herskovits, J., 1998, "A feasible directions interior point technique for nonlinear optimization", *JOTA - Journal of Optimization Theory and Applications* **99**(1), 121–146.

Kikuchi, N. and Oden, J. T., 1988, *Contact Problems in Elasticity: A Study of Variational Inequalities on Finite Element Methods*, SIAM Studies in Applied Mathematics, Philadelphia.

Klarbring, A., 1986, "A mathematical programming approach to three-dimensional contact problems with friction", *Comp. Methods Appl. Mech. Eng.* **58**, 175–200.

Mazorche, S. R. and Herskovits, J., 2005, A new interior point algorithm for nonlinear complementarity problems, *in* "Sixth World Congress on Structural and Multidisciplinary Optimization - CD Proceedings", Rio de Janeiro, Brazil, pp. 13–14.

Panagiotopoulos, P. D., 1975, "A nonlinear programming approach to the unilateral and friction boundary value problem in the theory of elasticity", *Ingenieur Archiv* **44**, 421–432.

París, F. and Cañas, J., 1998, *Boundary Element Method - Fundamentals and Applications*, Oxford University Press.

Raous, M., Chabrand, P. and Lebon, F., 1988, "Numerical methods for solving unilateral contact problem with friction", *J. Theor. Appl. Mech.* **7**, 111–128.

Simo, J. C., Wriggers, P. and Taylor, R. L., 1985, "A perturbed lagrangian formulation for the finite element solution of contact problems", *Comp. Methods Appl. Mech. Eng.* **50**, 163–180.

Wriggers, P. and Fisher, K., 2003, Recent new developments in contact mechanics, *in* "4^{th} European LS-DYNA Users Conference".

8. Responsibility notice

The author(s) is (are) the only responsible for the printed material included in this paper.