ON THE DERIVATION OF CONSISTENT ELASTOPLASTIC TANGENT OPERATORS FOR DENSITY-DEPENDENT PLASTICITY MODELS

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Abstract. *In this work analytical elastoplastic tangent operators are derived for the class of density-dependent plasticity models, as the elliptical and cap models. The density dependence of some parameters of the model, implies in a correction in the so called consistent tangent operators in finite elastoplasticity in order to achieve better convergences when the Newton-Raphson method is used. The consistent tangent operators are derived inside of the framework of isotropic multiplicative elastoplasticity. A Total Lagrangian formulation is considered and the density-dependent constitutive model is written in terms of the rotated Kirchhoff stress and of its logarithmic strain conjugate measure. In order to simulate compaction processes a contact formulation is used based on the Signorini condition and on the assumption of frictionless condition as well. In order to attest the performance of the proposed tangent modulus derivation, some numerical results are presented in the context of the finite element method, under plane strain assumption.*

Keywords: *Tangent operator, Powder compaction, Geomechanical materials, Density-dependent materials.*

1. INTRODUCTION

The nonlinear problem study in this work is relating to the modeling of the behavior of compressible porous materials in the context of finite deformation, elastoplasticity and the finite element method (FEM). The proposal of this work is to investigate the impact of the relative density on the convergence process in density-dependent finite plasticity models.

In general, the solution of nonlinear problems requires the linearization of the set of nonlinear equations and then the use of an iterative process which shall leads to a convergent sequence to the solution point. The *Newton-Raphson* method is the most popular method used to solve nonlinear problems in continuum mechanics. The correct linearization of the set of nonlinear equations plays an very important role in the iterative solution search. Approximate linearizations may lead to a dramatic increase of the number of iterations or even to a non convergence sequence.

During the linearization of such kind of nonlinear problems, the so called tangent operators arises due to the nonlinear stress-strain relationship. These tangent operators are derived here letting the relative density vary slightly around the solution and then leading to corrections on the standard tangent operators.

To model the kinematics of deformation of the body we assume the *Total Lagrangian* description and to impose the contact with walls we make use of the *Signorini* condition together with the frictionless assumption. Moreover, the constitutive formulation is given in terms of the logarithmic deformation measure and the rotated *Kirchhoff* stress tensor. In addition, the exponential mapping is employed which preserves the return mapping procedure in the same manner as done in infinitesimal strain formulation.

2. FINITE STRAIN DESCRIPTION

2.1 Kinematics of deformation

This paper considers a multiplicative decomposition of the deformation gradient tensor $\mathbf{F}$, $\mathbf{F} = \nabla_{\mathbf{X}} \varphi (\mathbf{X}, t)$, into an elastic deformation gradient tensor, $\mathbf{F}^e$, and a plastic deformation gradient tensor, $\mathbf{F}^p$. Thus,

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \tag{1}$$

The elastic deformation gradient admits the polar decomposition, i.e.,

$$\mathbf{F}^e = \mathbf{R}^e \mathbf{U}^e \tag{2}$$

where $\mathbf{R}^e$ is the elastic rotation tensor and

$$\mathbf{U}^e = \sqrt{\mathbf{C}^e} \qquad \therefore \qquad \mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e \tag{3}$$

where $\mathbf{C}^e$ is the elastic right Cauchy-Green tensor. Here, one assumes the deformation measure to be given by the logarithmic or *Hencky* strain tensor, given by

$$\mathbf{E}^e = \ln\left(\mathbf{U}^e\right). \tag{4}$$

2.2 Conjugate stress measure

In the formulation of constitutive theories, the stress-strain pairs must be such that the rate of the work density remains preserved. Considering the material to be isotropic, one obtains as the conjugate stress, associated with the *Hencky* strain, the rotated *Kirchhoff* stress $\bar{\tau}$, given by

$$\bar{\tau} = \left(\mathbf{R}^e\right)^T \tau \mathbf{R}^e \tag{5}$$

where τ is the *Kirchhoff* stress, $\tau = \det\left(\mathbf{F}\right)\sigma$, with σ denoting the *Cauchy* stress tensor. Thus, based on Eq.(5) and Eq.(4), the rotated *Kirchhoff* stress will be related with the logarithmic or *Hencky* strain tensor by means of

$$\bar{\tau} = \mathbb{D}\mathbf{E}^e. \tag{6}$$

where $\mathbb{D}$ is the standard fourth order elasticity tensor.

3. FINITE DEFORMATION FORMULATION WITH UNILATERAL FRICTIONLESS CONTACT

3.1 Strong form

Let Ω_o be the initial configuration of a body with boundary $\partial\Omega_o$, subjected to: a prescribed body force $\bar{\mathbf{b}}$ defined in Ω_o; a prescribed surface traction $\bar{\mathbf{t}}$defined on Γ_o^t; a prescribed displacement $\bar{\mathbf{u}}$ defined on Γ_o^u; and a contact with friction condition on Γ_o^c, with $\partial\Omega_o = \Gamma_o^t \cup \Gamma_o^u \cup \Gamma_o^c$ and $\Gamma_o^t \cap \Gamma_o^u = \Gamma_o^t \cap \Gamma_o^c = \Gamma_o^u \cap \Gamma_o^c = \emptyset$. The strong form of the quasi-static contact problem may be stated as: Find $\mathbf{u}$, for each t, such that

$$div\mathbf{P} + \rho_o\bar{\mathbf{b}} = \mathbf{0} \quad in \ \Omega_o \tag{7}$$

$$\mathbf{Pm} = \bar{\mathbf{t}} \quad on \ \Gamma_o^t \tag{8}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad on \ \Gamma_o^u \tag{9}$$

where $\mathbf{P}$ is the first *Piola-Kirchhoff* stress tensor. For the treatment of frictionless contact it is sufficient impose the non-penetration condition, which can be stated in the following form, see Wriggers (2002),

$$Q_\nu^c \geq 0, \quad g_\nu\left(\mathbf{u}\right) \leq 0, \quad Q_\nu^c g_\nu\left(\mathbf{u}\right) = 0 \tag{10}$$

which is also known as the *Hertz-Signorini-Moreau* condition. Here, $\mathbf{Q}^c = \mathbf{Pm}$ is the surface traction on Γ_o^c and $\mathbf{g}$ is the gap vector function. Moreover, $\mathbf{Q}^c$ and $\mathbf{g}$ are decomposed as:

$$\mathbf{Q}^c = \mathbf{Q}_T^c + Q_\nu^c \nu \tag{11}$$

and

$$\mathbf{g} = \mathbf{g}_T + g_\nu \nu \tag{12}$$

where

$$Q_\nu^c = \mathbf{Q}^c \cdot \nu \qquad \therefore \qquad g_\nu = \mathbf{g} \cdot \nu \tag{13}$$

$$\mathbf{Q}_T^c = \left(\mathbf{I} - \nu \otimes \nu\right)\mathbf{Q}^c \qquad \therefore \qquad \mathbf{g}_T = \left(\mathbf{I} - \nu \otimes \nu\right)\mathbf{g} \tag{14}$$

in which ν is the outer normal to the rigid obstacle at the contact point. With the assumption of frictionless contact only the terms related with the normal contact, Q_ν^c e g_ν, are enough to impose the contact. In the case of friction contact the terms related with the tangential quantities must be also taken into account.

3.2 Weak form

In order to solve the Total Lagrangian description of the unilateral contact frictionless problem, in its incremental form, one applies the Augmented Lagrangian method. As a result, the solution to the contact with friction problem is determined by solving the following sequence of problems: Given $\epsilon_\nu > 0$ and $\lambda_{N_{n+1}}^o \geq 0$, find $\mathbf{u}_{n+1}$

$$\mathbf{u}_{n+1} = \lim_{k\to\infty} \mathbf{u}_{n+1}^k, \tag{15}$$

where $\mathbf{u}_{n+1}$ is the solution of: Determine $\mathbf{u}_{n+1}^k \in \mathcal{W}_p^1(\Omega_o)$ so that

$$\mathcal{G}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) = 0 \qquad \forall\, \hat{\mathbf{u}} \in \mathcal{W}_p^1(\Omega_o). \tag{16}$$

Here ϵ_ν and λ_ν are penalties parameters and the Lagrange multiplier associated with the imposition of the and the impenetrability condition. The term $\mathcal{G}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right)$ can be decomposed as follows

$$\mathcal{G}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) = \mathcal{G}_{int}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) + \mathcal{G}_{ext}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) + \mathcal{G}_{cont}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right), \tag{17}$$

where

$$\mathcal{G}_{int}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) = \int_{\Omega_o} \mathbf{P}\left(\mathbf{u}_{n+1}^k\right) \cdot \nabla_{\mathbf{X}} \hat{\mathbf{u}}\, d\Omega_o \tag{18a}$$

$$\mathcal{G}_{ext}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) = -\int_{\Omega_o} \rho_o \bar{\mathbf{b}} \cdot \hat{\mathbf{u}}\, d\Omega_o - \int_{\Gamma_o^t} \bar{\mathbf{t}}_{n+1} \cdot \hat{\mathbf{u}}\, d\Gamma_o^t \tag{18b}$$

and

$$\mathcal{G}_{cont}\left(\mathbf{u}_{n+1}^k, \hat{\mathbf{u}}\right) = -\int_{\Gamma_o^c} \mathbf{Q}^c\left(\mathbf{u}_{n+1}^k, \epsilon_\nu, \lambda_{\nu_{n+1}}^k\right) \cdot \hat{\mathbf{u}}\, d\Gamma_o^c. \tag{19}$$

In this iterative process, the Lagrange multipliers are updated as, see Simo and Laursen (1992),

$$\lambda_{\nu_{n+1}}^{k+1} = \left\langle \lambda_{\nu_{n+1}}^k + \epsilon_\nu g_\nu\left(\mathbf{u}_{n+1}^k\right)\right\rangle, \tag{20}$$

where $\langle\cdot\rangle$ is the Macauley bracket, defined as $\langle x\rangle = \frac{1}{2}(x + |x|)$.

Notice that, Eq.(18a), Eq.(18b) and Eq.(19) may be seen respectively as the virtual work done by the internal forces, external forces and by the surface tractions $\mathbf{Q}^c$ which are associated with the contact condition.

In order to determine $\mathbf{u}_{n+1}^k$ in Eq.(17), one applies *Newton-Raphson's* method, reducing Eq.(17) to the solution of a sequence of linear problems, defined as: Given the initial guess $\mathbf{u}_{n+1}^{k(o)} = \mathbf{u}_n$, $\mathbf{u}_n$ denoting the converged solution at t_n , find $\mathbf{u}_{n+1}^k$,

$$\mathbf{u}_{n+1}^k = \lim_{j\to\infty} \mathbf{u}_{n+1}^{k(j)}, \tag{21}$$

where $\mathbf{u}_{n+1}^{k(j)}$ is computed, from the previous iteration, by

$$\mathbf{u}_{n+1}^{k(j)} = \mathbf{u}_{n+1}^{k(j-1)} + \Delta\mathbf{u}_{n+1}^{k(j-1)}. \tag{22}$$

The displacement increment $\Delta\mathbf{u}_{n+1}^{k(j-1)}$ is determined by solving the linear problem:

$$D\mathcal{G}\left(\mathbf{u}_{n+1}^{k(j-1)}, \hat{\mathbf{u}}\right)\left[\Delta\mathbf{u}_{n+1}^{k(j-1)}\right] = -\mathcal{G}\left(\mathbf{u}_{n+1}^{k(j-1)}, \hat{\mathbf{u}}\right) \qquad \forall\, \hat{\mathbf{u}} \in \mathcal{W}_p^1(\Omega_o) \tag{23}$$

in which,

$$D\mathcal{G}\left(\mathbf{u}_{n+1}^{k(j-1)}, \hat{\mathbf{u}}\right)\left[\Delta\mathbf{u}_{n+1}^{k(j-1)}\right] = \frac{d}{d\epsilon}\, \mathcal{G}\left(\mathbf{u}_{n+1}^{k(j-1)} + \epsilon\Delta\mathbf{u}_{n+1}^{k(j-1)}, \hat{\mathbf{u}}\right)\Big|_{\epsilon=0}. \tag{24}$$

The linearization of the weak form presented in Eq.(24) leads to the linearization of the term associated with the internal virtual work, i.e.,

$$\mathcal{G}_{int}\left(\mathbf{u}_{n+1}^{k(j-1)}, \hat{\mathbf{u}}\right) = \int_{\Omega_o} \mathbf{P}\left(\mathbf{u}_{n+1}^{k(j-1)}\right) \cdot \nabla_{\mathbf{X}} \hat{\mathbf{u}}\, d\Omega_o. \tag{25}$$

which can be expressed, after a straightforward manipulation, as

$$D\mathcal{G}_{int}\left(\mathbf{u}_{n+1}^{k(j-1)}, \hat{\mathbf{u}}\right)\left[\Delta\mathbf{u}_{n+1}^{k(j-1)}\right] = \int_{\Omega_o} \mathbb{A}\left(\mathbf{u}_{n+1}^{k(j-1)}\right) \nabla_{\mathbf{X}}\left(\mathbf{u}_{n+1}^{k(j-1)}\right) \cdot \nabla_{\mathbf{X}} \hat{\mathbf{u}}\, d\Omega_o \tag{26}$$

where $\mathbb{A}$ is the called tangent modulus given by

$$\left[\mathbb{A}\left(\mathbf{u}_{n+1}^{k(j-1)}\right)\right]_{ijkl} = \frac{\partial P_{ij}}{\partial F_{kl}}\Big|_{\mathbf{u}_{n+1}^{k(j-1)}} = \frac{\partial \tau_{ip}}{\partial F_{kl}} F_{jp}^{-1} - \tau_{ip} F_{jk}^{-1} F_{lp}^{-1}. \tag{27}$$

3.3 Notes on the determination of the tangent modulus $\mathbb{A}$

The determination of the tangent modulus $\mathbb{A}$ requires the computation of the derivative of the *Kirchhoff* stress tensor with relation to the deformation gradient tensor, as seem in Eq.(27). In addition, it is possible to relate *Kirchhoff* stress tensor with the rotated *Kirchhoff* stress tensor by Eq.(5) and therefore write τ as a function of the $\bar{\tau}$. In other words, the computation of $\frac{\partial \tau}{\partial \mathbf{F}}$ requires the computation of $\frac{\partial \bar{\tau}}{\partial \mathbf{F}}$. The derivative of the rotated *Kirchhoff* stress tensor with relation to the deformation gradient tensor can be evaluated realizing that

$$\bar{\tau}_{n+1} = \hat{\bar{\tau}}_{n+1}\left(\mathbf{E}_{n+1}^{e^{trial}}, (\cdot)_n\right). \tag{28}$$

Now, by applying the chain rule we have

$$\hat{\mathbb{D}} = \frac{\partial \bar{\tau}_{n+1}}{\partial \mathbf{F}_{n+1}} = \frac{\partial \bar{\tau}_{n+1}}{\partial \mathbf{E}_{n+1}^{e^{trial}}} \frac{\partial \mathbf{E}_{n+1}^{e^{trial}}}{\partial \mathbf{C}_{n+1}^{e^{trial}}} \frac{\partial \mathbf{C}_{n+1}^{e^{trial}}}{\partial \mathbf{F}_{n+1}}. \tag{29}$$

Denoting now

$$\tilde{\mathbb{D}} = \frac{\partial \bar{\tau}_{n+1}}{\partial \mathbf{E}_{n+1}^{e^{trial}}}, \quad \mathbb{G} = \frac{\partial \mathbf{E}_{n+1}^{e^{trial}}}{\partial \mathbf{C}_{n+1}^{e^{trial}}} \quad and \quad \mathbb{H} = \frac{\partial \mathbf{C}_{n+1}^{e^{trial}}}{\partial \mathbf{F}_{n+1}} \tag{30}$$

it is possible rewrite the Eq.(29) as

$$\hat{\mathbb{D}} = \tilde{\mathbb{D}}\mathbb{G}\mathbb{H}. \tag{31}$$

Lets now describe some important points about the determination of each one of these three tensors. Knowing that $\mathbf{C}_{n+1}^{e^{trial}} = \left(\mathbf{F}_{n+1}^{e^{trial}}\right)^T \mathbf{F}_{n+1}^{e^{trial}}$ and after a straightforward manipulation, the $\mathbb{H}$ term can be expressed as

$$[\mathbb{H}]_{ijkl} = \left[\mathbf{F}_n^{p^{-1}}\right]_{li} F_{kj_{n+1}}^{e^{trial}} + F_{ki_{n+1}}^{e^{trial}} \left[\mathbf{F}_n^{p^{-1}}\right]_{lj}. \tag{32}$$

The fourth order tensor $\mathbb{G}$ is computed by the following expression

$$\mathbb{G} = \frac{\partial}{\partial \mathbf{C}_{n+1}^{e^{trial}}} \ln\left(\mathbf{U}_{n+1}^{e^{trial}}\right) = \frac{1}{2} \frac{\partial}{\partial \mathbf{C}_{n+1}^{e^{trial}}} \ln\left(\mathbf{C}_{n+1}^{e^{trial}}\right). \tag{33}$$

Note that in the $\mathbb{G}$ determination we need to compute a derivative that involves $\frac{\partial \ln(\mathbf{X})}{\partial \mathbf{X}}$, that is a derivative of the isotropic function $\ln(\mathbf{X})$. This class of functions and their derivatives are investigated in details in the works presented by Souza Neto *et al.* (1998) and Ortiz *et al.* (2001). In the Eq.(31) the fourth order tensor $\tilde{\mathbb{D}}$ is the term that involves the material constitutive relationship. The other two are related with geometric portion of the tangent modulus. In fact, the derivation of $\tilde{\mathbb{D}}$ will depend on the type of material being modeled, i.e., in the case of a material that exhibits elastic and inelastic behavior, if the yield function $f \leq 0$ then $\tilde{\mathbb{D}}$ is taken as the elastic modulus $\mathbb{D}$,see Eq.(6), otherwise if $f > 0$ then $\tilde{\mathbb{D}}$ will be the elastoplastic tangent operator $\mathbb{D}^{ep}$.

4. DENSITY-DEPENDENT FINITE PLASTICITY MODELS

In the development of plasticity models for compressible porous materials it is necessary to establish a yield criterion and a flow rule from which the stress-strain relations can be derived. However, the yielding of porous materials is much more complex than the yielding of fully dense materials, mainly due to the fact that the yielding is not only influenced by the deviatoric part of the stress, but also by its hydrostatic part, see reference [1].

Examples of compressible materiais are soils, powders and foams. Each one of these materials has its particularity, which influences its modeling. In the specialized literature, it is possible find many models and dozens of their variations. In general soils are modeled through *Cap* models, powders can be modeled by *Cap* models, for lower relative density values, and by the called elliptical models, a variation of the *von Mises* that incorporates the hydrostatic part of the stress, and foams are in general modeled through the usage of a specific model.

Despite of the functional form of the mathematical model used in the modeling of density-dependent finite plasticity models they, in most cases, have some parameters that are dependent on the evolution of the mass density ρ

$$\rho = \frac{\rho_o}{\det(\mathbf{F})}, \tag{34}$$

where ρ_o is the initial mass density, or in terms of relative density η, which is defined by

$$\eta = \frac{\rho}{\rho_m} = \frac{\eta_o}{\det(\mathbf{F})} \tag{35}$$

where ρ_m is the mass density of fully dense material and $\eta_o = \frac{\rho_o}{\rho_m}$. Therefore, in the modeling of density-dependent finite plasticity models, using the approach earlier presented, the yield function will be written as

$$f = f\left(\bar{\tau}, \cdots, \eta\right). \tag{36}$$

4.1 Return mapping

The return mapping class of algorithms are the most employed in the solution of the equations that arise from the enforcement of the inelastic behavior of the material. Commonly, based on a given load history, this enforcement comprises the elastic relation, the plastic flow rule, the evolution of the internal variables and the satisfaction of the yield function. By assuming and associative flow[1] rules these

$$f\left(\bar{\tau}, \cdots, \eta\right) = 0 \tag{37a}$$

$$\bar{\mathbf{D}}^p = \dot{\lambda}\frac{\partial f}{\partial \bar{\tau}} = \dot{\lambda}\mathbf{N}_{\bar{\tau}} \tag{37b}$$

$$\dot{\alpha} = \dot{\lambda}\frac{\partial f}{\partial \beta} = \dot{\lambda}\mathbf{N}_{\beta} \tag{37c}$$

$$\bar{\tau} = \mathbb{D}\left(\eta\right)\mathbf{E}^e \tag{37d}$$

In equation Eq.(37b) and Eq.(37c) $\dot{\lambda}$ is the plastic multiplier, which is determined by the satisfaction of the *Karush-Kuhn-Tucker* conditions

$$f \le 0 \qquad \dot{\lambda} \ge 0 \qquad \dot{\lambda} f = 0, \tag{38}$$

and $\bar{\mathbf{D}}^p$ is the modified plastic evolution, $\dot{\alpha}$ plays the role of the internal variables evolution vector, β denotes the vector of internal variables that conjugates to α. $\mathbf{N}_{\bar{\tau}}$ and $\mathbf{N}_{\beta}$ plays the role of the "normals" with respect to the yield function f.

4.2 Aproximation via the *operator split* technique

The use of the aproximation via the *operator split* technique leads to an algorithm based on two main steps, which are:

1. Elastic prediction: the problem is assumed to be purely elastic between t_n e t_{n+1};
2. Plastic correction: by the enforcement of the elastic relation, plastic flow rule, the evolution of hardening variables (internal variables) and the satisfaction of the *Karush-Kuhn-Tucker* conditions.

4.3 Elastic prediction

In the elastic prediction it is assumed that

$$\dot{\mathbf{F}}^p = \mathbf{0} \tag{39}$$

$$\dot{\alpha} = \mathbf{0}. \tag{40}$$

As the solution is former assumed as elastic then

$$\mathbf{F}^{p^{trial}}_{n+1} = \mathbf{F}^p_n \tag{41}$$

$$\alpha^{trial}_{n+1} = \alpha_n. \tag{42}$$

The called *trial elastic state* is obtained through

$$\mathbf{F}^{e^{trial}}_{n+1} = \mathbf{F}_{n+1}\left(\mathbf{F}^p_n\right)^{-1}. \tag{43}$$

This implies that the logaritmic strain measure is computed by

$$\mathbf{E}^{e^{trial}}_{n+1} = \frac{1}{2}\ln\left(\mathbf{C}^{e^{trial}}_{n+1}\right) \tag{44}$$

with $\mathbf{C}^{e^{trial}}_{n+1} = \left(\mathbf{F}^{e^{trial}}_{n+1}\right)^T \mathbf{F}^{e^{trial}}_{n+1}$.

Since that $\mathbf{E}^{e^{trial}}_{n+1}$ is determined, then it is possible determine the trial rotated *Kirchhoff* stress tensor by the use of the elastic relation,i.e,

$$\bar{\tau}^{trial}_{n+1} = 2\mu\left(\eta_{n+1}\right)\mathbf{E}^{e^{trial}}_{n+1} + \left(\kappa\left(\eta_{n+1}\right) - \frac{2}{3}\mu\left(\eta_{n+1}\right)\right) tr\left(\mathbf{E}^{e^{trial}}_{n+1}\right)\mathbf{I} \tag{45}$$

[1]Non associative functions could also be used.

where we assume the standard fourth order elasticity tensor being also dependent on the relative density, that means

$$\mathbb{D} = 2\mu(\eta)\,\mathbb{I} + \left(\kappa(\eta) - \frac{2}{3}\mu(\eta)\right)\mathbf{I}\otimes\mathbf{I}. \quad (46)$$

4.4 Plastic correction

The plastic correction must be performed if $f\left(\bar{\tau}^{trial}_{n+1}, \alpha^{trial}_{n+1}, \eta_{n+1}\right) > 0$. The procedure adopted to perform the plastic correction belongs to the return mapping algorithms, extensively explored in literature. In this work, as proposed by Eterovic and Bathe (1990) and Weber and Anand (1990), the exponential mapping is used. The combination of the logarithmic strain measure and the exponential mapping integration scheme leads to the same return mappings algorithms found in the small-strain theory, see Simo (1992) and Péric and Owen (1998) for more details.

4.4.1 Exponential return mapping

At this point the evolution laws are approximated. The plastic evolution

$$\dot{\mathbf{F}}^p = \bar{\mathbf{D}}^p\mathbf{F}^p, \quad (47)$$

with $\bar{\mathbf{D}}^p = \dot{\lambda}\mathbf{N}_{\bar{\tau}}$, is approximated based on the backward exponential approximation resulting in

$$\mathbf{F}^p_{n+1} = \exp\left(\Delta\lambda\mathbf{N}_{\bar{\tau}_{n+1}}\right)\mathbf{F}^p_n. \quad (48)$$

In addition, the evolution of the internal variables are approximated based on the backward Euler, i.e.,

$$\alpha_{k_{n+1}} = \alpha_{k_n} + \Delta\lambda\mathbf{N}_{\beta_{k_{n+1}}}. \quad (49)$$

Moreover, after a straightforward manipulation, Eq.(48) reduces to

$$\mathbf{E}^e_{n+1} = \mathbf{E}^{e^{trial}}_{n+1} - \Delta\lambda\mathbf{N}_{\bar{\tau}_{n+1}}. \quad (50)$$

Also, it can be shown that $\mathbf{R}^e_{n+1} = \mathbf{R}^{e^{trial}}_{n+1}$. As a result, the return mapping algorithm comprises the solution of the following non-linear system of equations

$$\left\{\begin{array}{l} \mathbf{E}^e_{n+1} - \mathbf{E}^{e^{trial}}_{n+1} + \Delta\lambda\mathbf{N}_{\bar{\tau}_{n+1}} \\ \alpha_{k_{n+1}} - \alpha_{k_n} - \Delta\lambda\mathbf{N}_{\beta_{k_{n+1}}} \\ f\left(\bar{\tau}_{n+1}, \beta_{k_{n+1}}, \eta_{n+1}\right) \end{array}\right. = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ 0 \end{array}\right] \quad (51)$$

for $\Delta\lambda$, $\alpha_{k_{n+1}}$ and $\mathbf{E}^e_{n+1}$.

Remark: Notice that based on a fixed incremental displacement at the instant t_{n+1}, that is $\mathbf{u}_{n+1}$, the deformation gradient $\mathbf{F}_{n+1}$ is computed and so the relative density η_{n+1}. Therefore, the relative density is fixed, not a variable, in the context of the return mapping algorithm, see Pérez-Foguet *et al.* (2001).

4.5 Derivation of consistent elastoplastic tangent operator

The relative density model dependence impose a correction in the tangent operator. In fact, the linearization of the return mapping equations must consider the elastic trial strain also as a variable. This implies that $d\mathbf{E}^{e^{trial}}_{n+1} \rightarrow d\eta_{n+1}$. This means that a coupled relation among $d\mathbf{E}^{e^{trial}}_{n+1}$ and $d\eta_{n+1}$ must be derived.

The relation among $d\mathbf{E}^{e^{trial}}_{n+1}$ and $d\eta_{n+1}$ should be consistent with the algorithm used. In the elastic prediction phase we state that

$$\mathbf{F}_{n+1} = \mathbf{F}^{e^{trial}}_{n+1}\mathbf{F}^p_n. \quad (52)$$

Based on the trial elastic state assumption and on the Eq.(44) and reminded that

$$\mathbf{C}^{e^{trial}}_{n+1} = \exp\left(2\mathbf{E}^{e^{trial}}_{n+1}\right) \quad (53)$$

it is possible show, after a straightforward manipulation, that

$$\det\left(\mathbf{F}^{e^{trial}}_{n+1}\right) = \exp\left(E^{e^{trial}}_{v_{n+1}}\right). \quad (54)$$

Thus, substituting Eq.(54) and Eq.(52) into Eq.(35) yields

$$\eta_{n+1} = \hat{\eta}\exp\left(-E^{e^{trial}}_{v_{n+1}}\right) \quad (55)$$

where $E^{e^{trial}}_{v_{n+1}} = tr\left(\mathbf{E}^{e^{trial}}_{n+1}\right)$ and $\hat{\eta} = \frac{\eta_o}{\det(\mathbf{F}^p_n)}$.

4.6 Elastoplastic consistent tangent operator determination

Now, for the correct determination of the elastoplastic tangent operator

$$\mathbb{D}^{ep} = \frac{d\bar{\tau}_{n+1}}{d\mathbf{E}^{e^{trial}}_{n+1}}, \tag{56}$$

one must impose Eq.(55) together with the system of equations shown in Eq.(51). To perform this important task one can call for the linearization of Eq.(55) and Eq.(51). Such linearization leads to following set of equations

$$\left\{ \begin{array}{l} d\mathbf{E}^{e}_{n+1} - d\mathbf{E}^{e^{trial}}_{n+1} + d\left(\Delta\lambda\right)\mathbf{N}_{\bar{\tau}_{n+1}} + \Delta\lambda \mathbf{N}_{\bar{\tau}_{n+1}} \\ d\alpha_{k_{n+1}} - d\left(\Delta\lambda\right)\mathbf{N}_{\beta_k}\big|_{n+1} - \Delta\lambda\, d\mathbf{N}_{\beta_k}\big|_{n+1} \\ df\left(\bar{\tau}_{n+1}, \beta_{k_{n+1}}, \eta_{n+1}\right) \\ d\eta_{n+1} + \hat{\eta}\exp\left(-E^{e^{trial}}_{v_{n+1}}\right) dE^{e^{trial}}_{v} \end{array} \right. = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ 0 \\ 0 \end{array} \right] \tag{57}$$

where

$$d\mathbf{N}_{\bar{\tau}_{n+1}} = \frac{\partial \mathbf{N}_{\bar{\tau}_{n+1}}}{\partial \bar{\tau}_{n+1}} d\bar{\tau}_{n+1} + \frac{\partial \mathbf{N}_{\bar{\tau}_{n+1}}}{\partial \alpha_{n+1}} d\alpha_{n+1} + \frac{\partial \mathbf{N}_{\bar{\tau}_{n+1}}}{\partial \eta_{n+1}} d\eta_{n+1} \tag{58a}$$

$$d\mathbf{N}_{\beta_{k_{n+1}}} = \frac{\partial \mathbf{N}_{\beta_{k_{n+1}}}}{\partial \bar{\tau}_{n+1}} d\bar{\tau}_{n+1} + \frac{\partial \mathbf{N}_{\beta_{k_{n+1}}}}{\partial \alpha_{n+1}} d\alpha_{n+1} + \frac{\partial \mathbf{N}_{\beta_{k_{n+1}}}}{\partial \eta_{n+1}} d\eta_{n+1} \tag{58b}$$

$$df\left(\bar{\tau}_{n+1}, \beta_{k_{n+1}}, \eta_{n+1}\right) = \mathbf{N}_{\bar{\tau}_{n+1}} d\bar{\tau}_{n+1} + \frac{\partial f}{\partial \alpha_{n+1}} d\alpha_{n+1} + \frac{\partial f}{\partial \eta_{n+1}} d\eta_{n+1}. \tag{58c}$$

and by assuming that some elastic parameter could depend on the relative density, $\bar{\tau} = \mathbb{D}\left(\eta\right)\mathbf{E}^{e}$, we also impose that

$$d\mathbf{E}^{e}_{n+1} = \mathbb{D}^{-1}\left(\eta_{n+1}\right) d\bar{\tau}_{n+1} + \frac{\partial \mathbb{D}^{-1}\left(\eta_{n+1}\right)}{\partial \eta_{n+1}} \bar{\tau}_{n+1} d\eta_{n+1}. \tag{59}$$

5. MODEL CASE - ELLIPTICAL OR POROUS MATERIAL MODELS

Since the work presented by Doraivelu (1984) many contributions have been made regarding this class of model. Some authors state that the use of the elliptical should be used only when the relative densities are superior to $0,7$, but others authors advocates that its use can be extended to lower values of relative densities. Despite the discussion about the proper use of such kind of model, this model will be used here to illustrate the derivation of consistent elastoplastic tangent operator. Elliptical or porous material models are described by the following yield function functional form

$$F = AJ_2 + BI_1^2 = \sigma_\eta^2. \tag{60}$$

In this equation A and B are scalars that are, in many cases, dependent on the relative density and σ_η is the apparent yield stress. J_2 and I_1 are, respectively, the second invariant of the stress tensor in the deviatoric space and the first invariant of the stress tensor. In general

$$\sigma_\eta^2 = \gamma\sigma_y^2 \tag{61}$$

where σ_y is the initial yield stress of the fully dense material.

Doraivelu *et al.* (1984) showed that the values of A and B are not arbitrary. However, there are a great variety of proposals for A and B in the literature. See table 1.

Zhdanovich (1971) proposes that a Poisson dependence on the relative density, such that

$$\nu = \frac{1}{2}\eta^n. \tag{62}$$

The exponent $n \simeq 2$ has been used to describe such dependence. The γ multiplier is known as the geometric hardening and can be also dependent on the relative density, as shown in table 1. When $\gamma = 1$ the material must behaves as a fully dense material and for some value between 0 and 1 the material should presents no mechanical strength. This value, represented by η_C or η_{PC} in the table 1, can vary for each author but is about the called *tap density*.

5.1 Proposal model

Let us propose now that the porous material model could experience an isotropic hardening k in its dense matrix. So, the proposal material model can be represented by an yield function as

$$F = AJ_2 + BI_1^2 = \gamma\left(k + \sigma_y\right)^2. \tag{63}$$

Table 1. Some values for A, B and γ.

Authors	A	B	γ
Kuhn & Downey (1971)	$2+\eta^2$	$\frac{1-\eta^2}{3}$	1
Shima & Oyane (1976)	3	$\frac{1}{9a^2(1-\eta)^{2m}}$	η^{2n}
Gurson (1977)	$\frac{12}{5+\eta}$	$\frac{1-\eta}{5-\eta}$	$\frac{4\eta^2}{5-\eta}$
Doraivelu (1984)	$2+\eta^2$	$\frac{1-\eta^2}{3}$	$\frac{\eta^2-\eta_C^2}{1-\eta_C^2}$
Lee & Kim (1992)	$2+\eta^2$	$\frac{1-\eta^2}{3}$	$\left(\frac{\eta-\eta_C}{1-\eta_C}\right)^2$
Park (1999)	$2+\eta^2$	$\frac{1-\eta^2}{3}$	$\left(\frac{\eta-\eta_{PC}}{1-\eta_{PC}}\right)^m$
Pérez-Foguet (2003)	2	$\frac{1}{3}\begin{cases}\left(\frac{1-\eta^2}{2+\eta^2}\right)^{n_1} & \eta<1\\ 0 & \eta>1\end{cases}$	$\frac{2}{3}\begin{cases}\left(\frac{0.02\eta_o}{1-0.98\eta_o}\right)^{n_2} & \eta\le\eta_o\\ \left(\frac{\eta-0.98\eta_o}{1-0.98\eta_o}\right)^{n_2} & \eta>\eta_o\end{cases}$

Taking the square root in both sides of the Eq.(63) this function can be rewritten as

$$f = S_{eq} - \gamma^{\frac{1}{2}}\left(k+\sigma_y\right) \tag{64}$$

where

$$S_{eq} = \sqrt{AJ_2 + BI_1^2} \tag{65}$$

plays the role of equivalent stress. Let us assume that isotropic hardening k of the dense material matrix can be represented by

$$k\left(\alpha\right) = H\alpha + \left(\sigma_\infty - \sigma_y\right)\left(1-e^{-\delta\alpha}\right) \tag{66}$$

where H, σ_∞ and δ are material parameters not dependent on the relative density.

5.2 Tangent operator

Since the model had been described in the previously section, it is possible now to identify the elastoplastic tangent operator for the proposal model. This identification comes from the linearization presented in Eq.(56). After a straightforward algebra manipulation it is possible to show that

$$\tilde{\mathbb{D}} = \mathbb{D}^{ep} = \frac{d\bar{\tau}_{n+1}}{d\mathbf{E}^{etrial}_{n+1}} = \left[\mathbb{T}^{ep}\right]^{-1}\mathbb{T}^{\eta} \tag{67}$$

where

$$\mathbb{T}^{ep} = \mathbb{D}^{-1}\left(\eta_{n+1}\right) + \Delta\lambda\frac{\partial\mathbf{N}_{\bar{\tau}_{n+1}}}{\partial\bar{\tau}_{n+1}} - \frac{1}{\frac{\partial f}{\partial\alpha_{n+1}}\sqrt{\gamma_{n+1}}}\left(\mathbf{N}_{\bar{\tau}_{n+1}}\otimes\mathbf{N}_{\bar{\tau}_{n+1}}\right) \tag{68}$$

$$\mathbb{T}^{\eta} = \mathbb{I} - \eta_{n+1}\left[\mathbb{D}^{-1}\frac{\partial\mathbb{D}(\eta_{n+1})}{\partial\eta_{n+1}}\mathbf{E}^e_{n+1} + f_c\mathbf{N}_{\bar{\tau}_{n+1}} - \Delta\lambda\frac{\partial\mathbf{N}_{\bar{\tau}_{n+1}}}{\partial\eta_{n+1}}\right]\otimes\mathbf{I} \tag{69}$$

$$f_c = \frac{\frac{\partial f}{\partial\eta_{n+1}}}{\frac{\partial f}{\partial\alpha_{n+1}}\sqrt{\gamma_{n+1}}} + \frac{\Delta\lambda}{2\gamma_{n+1}}\frac{\partial\gamma(\eta_{n+1})}{\partial\eta_{n+1}}. \tag{70}$$

6. RESULTS

In order to attest the effect of the correction $\mathbb{D}^{ep}_{\eta_{cor}}$ into the standard elastoplastic tangent operator $\mathbb{D}^{ep}_{std}$ it is proposed in this example the simulation of a simple compression of an unitary body under plane strain assumption. The discretization of the body as well as its boundary conditions are displayed in figure 1. In this example we consider a material model as described by Eq.(64) and Eq.(66) with A, B and γ given by the Doraivelu *et al.* (1984) model, table 1. The simulation consist in the compression of the body by a vertical displacement of $u_y = -0.3mm$. The vertical displacement is imposed by 20 equal steps. The initial relative density is assumed to be $\eta_o = 0.7$. The parameter is assumed to be $\eta_C = 0.4$, and the other material parameters for this hyphotetical material are $E = 10000MPa$, $\nu = 0.1$, $H = 130MPa$, $\delta = 17$, $\sigma_\infty = 715MPa$ and $\sigma_y = 100MPa$.

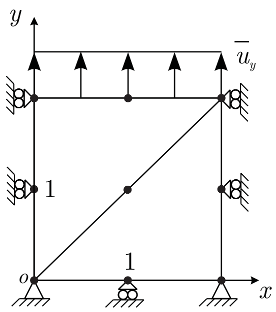

Figure 1. Model problem - Isostatic compaction

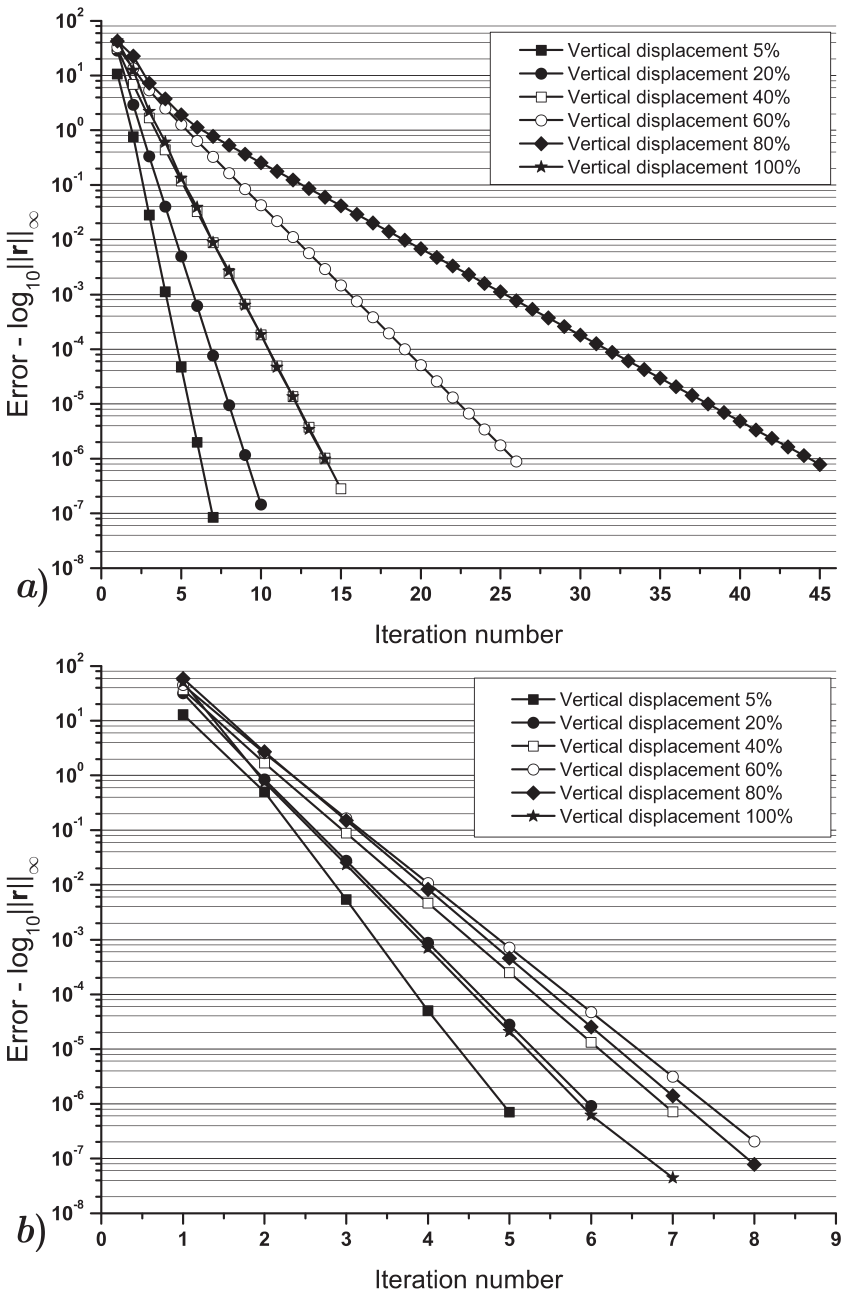

Figure 2. Convergence analysis - a) without correction - b) with correction

The figure 2 shows the convergence results achieved when using only the standard elastoplastic tangent operator $\mathbb{D}^{ep}_{std}$, figure 2(a), and when using the $\mathbb{D}^{ep} = \left[\mathbb{D}^{ep}_{std}\right]^{-1} \mathbb{D}^{ep}_{\eta_{cor}}$, figure 2(b). The convergence results presented in this figure are based in the residue norm $\|\mathbf{r}_{n+1}\|_{\infty} \times n_{iter}$, where n_{iter} is the total number of iterations to achieve the admissible error, $\|\mathbf{r}_{n+1}\|^{adm}_{\infty} \leq 10^{-6}$. In both cases it is employed a *Newton-Raphson* method with line search.

By observing the figure 2 one can note two facts regarding the convergence. The first one is that the convergence deteriorates, in a dramatically manner when no correction is used and in a slightly manner when it is employed. The second, is that the convergence starts with certain rate, this rate decrease, in a first stage, when the relative density increases but, in a second stage, when the relative density tends to be $\eta \approx 1.0$ (fully dense material), the convergence rate tends to increase.

7. CONCLUSION

This paper deals with the derivation of the consistent tangent operators for density-dependent finite plasticity models in the framework of the *Total Lagrangian* formulation, multiplicative finite strain plasticity, logarithmic strains and the exponential return mapping algorithms. It is clearly shown that the density dependence of the material model implies in corrections on the standard elastoplastic tangent operators. Moreover, based on the results shown it is possible note that if no density correction is performed on standard elastoplastic tangent operators then the convergence is affected, increasing dramatically the number the iterations to reach the specified admissible error, or even leading to non convergence state. On the other hand, if the correction is taken into account then the convergence is slightly affected during the simulation.

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