SOLVING STATIONARY INCOMPRESSIBLE FLOW PROBLEMS IN STRESS-VELOCITY-PRESSURE FORMULATION WITH LINEAR FINITE ELEMENTS

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Abstract. *A three-field finite element scheme designed for solving systems of partial differential equations governing stationary incompressible flows is studied. It is based on the simulation of a time-dependent behavior. Once a classical time-discretization is performed, the resulting three-field system of equations allows for a stable approximation of velocity, pressure and extra stress tensor, by means of continuous piecewise linear finite elements, in both two- and three-dimension space. This is proved to hold for the linearized form of the system. The main advantage of the new formulation is the fact that it implicitly provides an algorithm for the iterative resolution of system non-linearities. Existence and uniqueness of solution to the discretized version of the stationary three-field system is demonstrated, together with convergence in an appropriate sense applying to the three flow fields.*

Keywords: *Finite elements, Navier-Stokes, Stokes system, three-field formulation Viscoelastic fluids.*

1. INTRODUCTION

The numerical solution of viscous flow problems for classical newtonian or quasi-newtonian fluids is nowadays a well-established technique. This is mainly due to the first contributions in this field carried out from the seventies on. Most of those works were developed for the classical velocity-pressure Galerkin formulation of the flow equations. About two decades ago a new approach, allowing for the use of equal order interpolations of both fields was introduced and exploited by many authors such as Franca, Hughes, Loula and Miranda (1988). The main characteristics of this technique, sometimes called stabilized, is the use of suitable Petrov-Galerkin or Galerkin least-squares formulations. In the framework of the flow of viscoelastic liquids, a parallel evolution took place, initiated by celebrated work by Marchal and Crochet (1987), followed by those of Fortin and collaborators (cf. Fortin Guénette and Pierre (1997)), among other contributions including our own (cf. Ruas, Carneiro de Araujo and Silva Ramos (1993)). Indeed, in this case the incorporation to the numerical model of an additional field, namely, the (extra) stress tensor, is mandatory. Both Galerkin and Galerkin least-squares approaches, including variants incorporating the deformation rate as a fourth unknown field, have been used by specialists in viscoelastic flow simulations since the early nineties. The purpose of this work is to present a new variational formulation of the stationary incompressible flow equations in terms of extra stress tensor, velocity and pressure, discretized by piecewise linear finite elements, depending on a fictitious time step. Several techniques employed in previous work for this class of problems inspired the authors, such as those proposed in Franca, Hughes, Loula and Miranda (1988) and Codina and Zienkiewicz (2002). Moreover the kind of pressure Poisson equation proposed in Goldberg and Ruas (1999) is a key feature of the present approach. While this formulation was first applied to solve viscoelastic flow problems by means of an explicit scheme (cf. Brasil Jr, Carneiro de Araujo and Ruas (To appear)), we show that it can be employed with advantages, to the case of Newtonian or quasi-newtonian fluids. Mathematical results certifying the adequacy of this approach are given. Corresponding numerical examples illustrating its good behavior are presented.

2. MAXWELL FLOW EQUATIONS

In order to motivate our methodology let us first consider its application to the numerical solution of systems governing stationary flow of viscoelastic fluids. Although the technique to be developed hereafter extend in a straightforward manner to the case of a wide spectrum of viscoelastic constitutive laws, for the sake of simplicity we consider as a model the case of Maxwell fluids.

Let then Ω be a bounded domain of $\Re^N$, $N = 2$ or 3, with boundary $\partial\Omega$. Under the action of volumetric forces $\mathbf{f}$, we consider the evolution in time t, of the flow in Ω of a viscoelastic liquid obeying a constitutive law of the differential type. Throughout this work we assume that the velocity of the liquid is prescribed on $\partial\Omega$, say $\mathbf{u} = \mathbf{g}$. Moreover without any loss of essential aspects, just to simplify the presentation, we consider a constitutive law of the upper convected type, which relates the extra stress tensor to the velocity in the following manner:

$$\sigma + \lambda \left[\frac{\partial \sigma}{\partial t} + (\mathbf{u} \cdot \nabla)\, \sigma - (\nabla \mathbf{u})\sigma - \sigma(\nabla \mathbf{u})^T \right] = 2\eta D(\mathbf{u}). \tag{1}$$

In (1) λ is the stress relaxation time of the liquid and η is its reference viscosity, both assumed to be constant; ∇ represents the gradient of a scalar or a vector valued function and $D(\mathbf{u})$ denotes the strain rate tensor, i.e., $D(\mathbf{u}) := \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right]$.

Then from a given state at time $t = 0$, that is, given a solenoidal velocity $\mathbf{u}^0$ and an extra stress σ^0, for $t > 0$, in addition to the law (1), the flow is governed by the following system:

$$\left.\begin{array}{rcl} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \nabla \cdot \sigma + \nabla p & = & \mathbf{f} \\ \nabla \cdot \mathbf{u} & = & 0 \end{array}\right\} \text{in } \Omega \times (0, \infty) \tag{2}$$

where the density of the liquid is assumed to be equal to one.

In this work we will be concerned about the search of steady state solutions. Therefore we shall further assume in all the sequel, that both $\mathbf{f}$ and $\mathbf{g}$ are independent of t.

Now we consider the following semi-implicit discretization in time of system (1)-(2). Let $\Delta t > 0$ be a given time step, and $\mathbf{u}^n$, p^n and σ^n denote approximations of $\mathbf{u}(n\Delta t)$, $p(n\Delta t)$ and $\sigma(n\Delta t)$, respectively, for a strictly positive integer n. Starting from $\mathbf{u}^0$ and σ^0, and prescribing $\mathbf{u}^n = \mathbf{g}$ on $\partial\Omega$ for every n, $\mathbf{u}^n$, p^n and σ^n, for $n = 1, 2, \ldots$, are determined as the solution of the following system in Ω:

$$\left\{\begin{array}{rcl} \dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n-1} - \nabla \cdot \sigma^n + \nabla p^n & = & \mathbf{f} \\ \nabla \cdot \mathbf{u}^n & = & 0 \\ \sigma^n + \lambda \left[\dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} + (\mathbf{u}^{n-1} \cdot \nabla)\sigma^{n-1} - (\nabla \mathbf{u}^{n-1})\sigma^{n-1} - \sigma^{n-1}(\nabla \mathbf{u}^{n-1})^T \right] & = & 2\eta D(\mathbf{u}^n) \end{array}\right. \tag{3}$$

As one can readily infer, (3) is a linear problem for every n. Actually assuming moderate velocities and velocity gradients, the non linear terms may be neglected. In this case we can legitimately linearize (1)-(2) into the system governing the very slow flow of a viscoelastic fluid of the Maxwell type. Actually for the sake of conciseness we introduce our methodology in the context of the following generalized Stokes system, derived from the linearization of the equations that govern the flow of a Maxwell viscolelastic liquid (cf. Marchal and Crochet (1987)), namely:

From a given state at time $t = 0$ defined by a given solenoidal velocity $\mathbf{u}^0$ and an extra stress tensor σ^0, for $t > 0$ find $p, \mathbf{u}, \sigma$ that solve the following system, with $\mathbf{u} = \mathbf{g}$ on $\partial\Omega \times (0, \infty)$:

$$\left.\begin{array}{rcl} \dfrac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \sigma + \nabla p & = & \mathbf{f} \\ \nabla \cdot \mathbf{u} & = & 0 \\ \sigma + \lambda \dfrac{\partial \sigma}{\partial t} & = & 2\eta D(\mathbf{u}) \end{array}\right\} \text{in } \Omega \times (0, \infty). \tag{4}$$

3. TIME DISCRETIZATION AND SPLITTING ALGORITHM

In this section we present an algorithm for solving both newtonian and non newtonian flow equations, in the $\mathbf{u}, p, \sigma$ formulation. Although this algorithm and the underlying variational formulation are described here only in the context of problem (4), its adaption to more general cases is straightforward, including for instance the Navier-Stokes equations, or yet turbulent flow with turbulent stress models. Indeed in the latter cases it suffices to take $\lambda = 0$, before incorporating non linear expressions or terms. It seems however that in the context of viscolastic flow the new approach appears to be the most promising, since in this case the use of a three-field formulation is mandatory.

We have mainly dealt with an explicit splitting algorithm for the time integration or the iterative solution of system (4). However before presenting it we consider the underlying implicit discretization in time of (4).

Let $\Delta t > 0$ be a given time step. Then starting from $\mathbf{u}^0$ and σ^0, for $n = 1, 2, \ldots$, and prescribing $\mathbf{u}^n = \mathbf{g}$ on $\partial\Omega$ for every n, we determine approximations of $p(n\Delta t)$, $\mathbf{u}(n\Delta t)$ and $\sigma(n\Delta t)$, denoted by p^n, $\mathbf{u}^n$ and σ^n

respectively, as the solution of the following problem:

$$\left.\begin{array}{rcl}\dfrac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\Delta t}-\nabla\cdot\sigma^n+\nabla p^n &=& \mathbf{f}\\ \nabla\cdot\mathbf{u}^n &=& 0\\ \sigma^n+\lambda\left(\dfrac{\sigma^n-\sigma^{n-1}}{\Delta t}\right) &=& 2\eta D(\mathbf{u}^n)\end{array}\right\}\text{ in }\Omega. \tag{5}$$

For the sake of simplicity we assume that Ω is connected and has suitable regularity properties. We further assume the following mininum data regularity $\mathbf{f}\in L^2(\Omega)^N$, $\mathbf{g}\in H^{3/2}(\partial\Omega)^N$, $\mathbf{u}^0\in H^1(\Omega)^N$ and $\sigma^0\in H^1(\Omega)^{N\times N}$ (cf. Adams (1975)). Let also $<\cdot,\cdot>_{1/2,\partial\Omega}$ denote the duality product between $H^{1/2}(\partial\Omega)^N$ and $H^{-1/2}(\partial\Omega)^N$, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the standard L^2-inner product and associated norm, respectively. In the sequel $\vec{\nu}$ represents the unit outer normal vector on $\partial\Omega$.

We shall also use the following notations (cf. Girault and Raviart (1986))

$$\begin{array}{rcl}Q &:=& H^1(\Omega)\cap L^2_0(\Omega);\\ \mathbf{V}^g &:=& \{\mathbf{v}\in\mathbf{V},\ \mathbf{v}=\mathbf{g}\text{ on }\partial\Omega\}\text{ with }\mathbf{V}:=H^1(\Omega)^N;\\ \mathbf{V}^0 &:=& H^1_0(\Omega)^N;\\ \Sigma &:=& \{\sigma,\ \sigma\in H(\text{div},\Omega)^N\text{ and }\sigma=\sigma^T\}.\end{array}$$

Theorem 3.1 *(cf. Brasil Jr, Carneiro de Araujo and Ruas (To appear)) For every $\lambda>0$, for every Δt and for every n problem (5) has a unique solution. Moreover as n goes to ∞ the solution of (5) converges in norm of $L^2(\Omega)\times L^2(\Omega)^N\times L^2(\Omega)^{N\times N}$ to the solution $(\bar{p},\bar{\mathbf{u}},\bar{\sigma})$ of the stationary counterpart of (4), with $\bar{\mathbf{u}}=\mathbf{V}^g$, namely*

$$\left\{\begin{array}{rcl}-\nabla\cdot\bar{\sigma}+\nabla\bar{p} &=& \mathbf{f}\\ \nabla\cdot\bar{\mathbf{u}} &=& 0\\ \bar{\sigma} &=& 2\eta D(\bar{\mathbf{u}}).\end{array}\right\}\ \textit{in }\Omega. \tag{6}$$

The solution of (5) is rather costly, since it is an implicit system at every iteration. That is why we employed an splitting algorithm for solving the corresponding system explicitly. It is based on the computation of approximations $p^{n,s}\in Q$, $\mathbf{u}^{n,s}\in\mathbf{V}^g$ and $\sigma^{n,s}\in\Sigma$ of p^n, $\mathbf{u}^n$ and σ^n, by setting for every $n\geq 0$ $\sigma^{n,0}=\sigma^{n-1}$, and then solving for $s=1,2,\ldots$ successively:

$$(\nabla p^{n,s},\nabla q) = (\mathbf{f},\nabla q)+\left(\nabla\cdot\sigma^{n,s-1},\nabla q\right)\quad\forall q\in Q, \tag{7}$$

$$(\mathbf{u}^{n,s},\mathbf{v}) = \Delta t\left(\mathbf{f}+\nabla\cdot\sigma^{n,s-1}-\nabla p^{n,s},\mathbf{v}\right)+\left(\mathbf{u}^{n-1},\mathbf{v}\right)\quad\forall\mathbf{v}\in V^0, \tag{8}$$

$$\begin{aligned}\frac{\lambda+\Delta t}{2\eta}(\sigma^{n,s},\tau) = {}& \frac{\lambda}{2\eta}\left(\sigma^{n,s-1},\tau\right)-\Delta t^2\left(\mathbf{f}+\nabla\cdot\sigma^{n,s-1}-\nabla p^{n,s},\nabla\cdot\tau\right)-\\ & \Delta t\left[\left(\mathbf{u}^{n-1},\nabla\cdot\tau\right)-<\mathbf{g},\tau\vec{\nu}>_{1/2,\partial\Omega}\right]\quad\forall\tau\in\Sigma,\end{aligned} \tag{9}$$

The above iterative procedure is unlikely to generate converging sequence of approximations as s goes to infinity However, here we applied it in the framework of discrete counterparts of (7), (8), (9) defined by replacing Q, $\mathbf{V}^0$ (resp. $\mathbf{V}^g$) and Σ by finite dimensional spaces (resp. manifold) Q_h, $\mathbf{V}^0_h$ (resp. $\mathbf{V}^g_h$) and Σ_h specified in the following section. In Ruas and Brasil Jr (To appear) the convergence of this procedure in the context of finite element analogues of (6) is demonstrated.

To conclude this section we rewrite (6) for the later convenience in the following equivalent weak form (cf. Brasil Jr, Carneiro de Araujo and Ruas (To appear)). First we define for every n, $n=0,1,\cdots$

$$\mathbf{w}^n=\mathbf{u}^n-\mathbf{u}^0\in\mathbf{V}^0$$

Then instead of $(p^n,\mathbf{u}^n,\sigma^n)\in Q\times\mathbf{V}^g\times\Sigma$ we search for $(p^n,\mathbf{w}^n,\sigma^n)\in Q\times\mathbf{V}^o\times\Sigma$ such that

$$a\left((p^n,\mathbf{w}^n,\sigma^n),(q,\mathbf{v},\tau)\right)=L\left((q,\mathbf{v},\tau)\right)\quad\forall(q,\mathbf{v},\tau)\in Q\times\mathbf{V}^0\times\Sigma \tag{10}$$

where a is the bilinear form on $(Q\times\mathbf{V}\times\Sigma)\times(Q\times\mathbf{V}\times\Sigma)$ given by

$$\begin{aligned}a\left((p,\mathbf{u},\sigma),(q,\mathbf{v},\tau)\right) := {}& \Delta t^2(\nabla p-\nabla\cdot\sigma,\nabla q)+(\mathbf{u},\mathbf{v})+\\ & \Delta t(\nabla p-\nabla\cdot\sigma,\mathbf{v})+\frac{\lambda+\Delta t}{2\eta}(\sigma,\tau)+\Delta t^2(\nabla\cdot\sigma-\nabla p,\nabla\cdot\tau)\end{aligned} \tag{11}$$

and L is the linear form defined on $(Q\times\mathbf{V}\times\Sigma)$ by

$$\begin{aligned}L\left((q,\mathbf{v},\tau)\right) = {}& \Delta t^2(\mathbf{f},\nabla q-\nabla\cdot\tau)+\Delta t<\mathbf{g},(\tau-Iq)\vec{\nu}>_{1/2,\partial\Omega}+\Delta t(\mathbf{f},\mathbf{v})+\\ & \left(\mathbf{u}^{n-1},\mathbf{v}\right)+\Delta t\left(\mathbf{u}^{n-1},\nabla q-\nabla\cdot\tau\right)+\frac{\lambda}{2\eta}\left(\sigma^{n-1},\tau\right)-\left(\mathbf{u}^0,\mathbf{v}\right).\end{aligned} \tag{12}$$

Remark 31 *By inspection one easily finds out that the terms of (11) and (12) containing $\nabla \cdot \tau$ result from the balance of momentum tested with this quantity. This adds positiveness to the system, which plays a stabilizing role in the fully discretized counterpart of (10) studied in the following Section, similarly to previous works like Franca, Hughes, Loula and Miranda (1988).*

Remark 32 *(10) also incorporates the momentum equation tested with ∇q except for the term $\mathbf{u}^n$. Actually this implies that $\nabla \cdot \mathbf{u}^n = 0$ for every n (cf. Brasil Jr, Carneiro de Araujo and Ruas (To appear)).*

4. SPACE DISCRETIZATION

Now we consider the following discrete analogue of (5). Henceforth we assume that Ω has regularity properties compatible with the regularity of the unknown fields required in the theorems that follow.

Let then T_h be a partition of Ω into N-simplices with maximum edge length equal to h. We assume that T_h satisfies the usual compatibility conditions for finite element meshes, and that it belongs to a quasi-uniform family of partitions. For every $K \in T_h$ we further denote by $P_1(K)$ the space of polynomials of degree less than or equal to one defined in K. In so doing we introduce the following spaces or manifolds associated with T_h:

$$S_h := \left\{ v \mid \; v \in C^0(\bar{\Omega}) \text{ and } v|_K \in P_1(K), \; \forall K \in T_h \right\},$$

$$\mathbf{V}_h := \{ \mathbf{v} \mid \forall i \; v_i \in S_h \}, \quad \mathbf{V}_h^0 := \mathbf{V}_h \cap H_0^1(\Omega)^N,$$

$$\mathbf{V}_h^{\mathbf{g}} := \{ \mathbf{v} \in \mathbf{V}_h \mid \mathbf{v}(P) = \mathbf{g}(P) \; \forall \text{ vertex } P \text{ of } T_h \text{ on } \partial\Omega \},$$

$$Q_h := S_h \cap L_0^2(\Omega),$$

$$\Sigma_h := \left\{ \tau \mid \tau \in [S_h]^{N \times N}, \quad \tau = \tau^T \right\}.$$

We further define $\mathbf{u}_h^0$ be the field of $\mathbf{V}_h^g$ satisfying $\mathbf{u}_h^0(P) = \mathbf{u}^0(P)$, and σ_h^0 be the tensor of Σ_h satisfying $\sigma_h^0(P) = \sigma^0(P)$, for every vertex P of T_h, and set for every $n, n = 0, 1, 2, \ldots$

$$\mathbf{u}_h^n = \mathbf{w}_h^n + \mathbf{u}_h^0$$

where $\mathbf{u}_h^n$ is the approximation of $\mathbf{u}^n$ in $\mathbf{V}_h^g$.

Finally defining the discrete counterpart L_h on $Q_h \times \mathbf{V}_h \times \Sigma_h$ of linear form L by:

$$\begin{aligned} L_h\left((q, \mathbf{v}, \tau)\right) \;\; = \;\; & \Delta t^2\left(\mathbf{f}, \nabla q - \nabla \cdot \tau\right) + \Delta t < \mathbf{g}, (\tau - Iq)\vec{\nu} >_{1/2, \partial\Omega} + \Delta t\left(\mathbf{f}, \mathbf{v}\right) + \\ & \left(\mathbf{u}_h^{n-1}, \mathbf{v}\right) + \Delta t\left(\mathbf{u}_h^{n-1}, \nabla q - \nabla \cdot \tau\right) + \frac{\lambda}{2\eta}\left(\sigma_h^{n-1}, \tau\right) - \left(\mathbf{u}_h^0, \mathbf{v}\right). \end{aligned}$$

we set the following prblem to approximate (5) for every $n, n = 0, 1, 2, \ldots$

$$\left\{ \begin{array}{l} \text{Find } (p_h^n, \mathbf{w}_h^n, \sigma_h^n) \in Q_h \times \mathbf{V}_h^0 \times \Sigma_h \text{ such that} \\ a\left((p_h^n, \mathbf{w}_h^n, \sigma_h^n), (q, \mathbf{v}, \tau)\right) = L\left((q, \mathbf{v}, \tau)\right) \;\; \forall (q, \mathbf{v}, \tau) \in Q_h \times \mathbf{V}_h^0 \times \Sigma_h \end{array} \right. \tag{13}$$

For problem (13) the following result holds.

Proposition 41 *Problem (13) has a unique solution for every Δt and every n (cf. Brasil Jr, Carneiro de Araujo and Ruas (To appear)).*

Now we give the following convergence result (cf. Ruas and Brasil Jr (To appear)).

Theorem 41 *For every Δt and λ the solution of (13) converges to the solution of (13) in the norm of $L^2(\Omega) \times L^2(\Omega)^N \times L^2(\Omega)^{N \times N}$ as h goes to 0, provided for every n the solution of (13) is such that $p \in H^1(\Omega)$.*

5. STATIONARY CASE

Now we consider the approximation of stationary system (6) by means of the stationary counterpart of the finite element discretized problem (13), namely:

$$\left\{ \begin{array}{l} \text{Find } (\bar{p}_h, \bar{\mathbf{w}}_h, \bar{\sigma}_h) \in Q_h \times \mathbf{V}_{\underline{h}}^0 \times \Sigma_h \text{ such that} \\ \bar{a}\left((\bar{p}_h, \bar{\mathbf{w}}_h, \bar{\sigma}_h), (q, \mathbf{v}, \tau)\right) = \bar{L}_h\left((q, \mathbf{v}, \tau)\right) \;\; \forall (q, \mathbf{v}, \tau) \in Q_h \times \mathbf{V}_h^0 \times \Sigma_h \end{array} \right. \tag{14}$$

where $\bar{\mathbf{w}}_h = \bar{\mathbf{u}}_h - \mathbf{u}_h^0$, $\bar{\mathbf{u}}_h$ being the approximation of $\bar{\mathbf{u}}$ in $\mathbf{V}_h^g$, and for every $(p, \mathbf{u}, \sigma) \in Q \times \mathbf{V} \times \Sigma$ and $(q, \mathbf{v}, \tau) \in Q \times \mathbf{V} \times \Sigma$ we set

$$\begin{aligned} \bar{a}\left((p, \mathbf{u}, \sigma), (q, \mathbf{v}, \tau)\right) := & \Delta t^2\left(\nabla p - \nabla \cdot \sigma, \nabla q\right) + \Delta t\left(\mathbf{u}, \nabla \cdot \tau - \nabla q\right) + \\ & \Delta t\left(\nabla p - \nabla \cdot \sigma, \mathbf{v}\right) + \frac{\Delta t}{2\eta}\left(\sigma, \tau\right) + \Delta t^2\left(\nabla \cdot \sigma - \nabla p, \nabla \cdot \tau\right) \end{aligned} \tag{15}$$

and for given $\mathbf{f}, \mathbf{g}$, we set for every $(q, \mathbf{v}, \tau) \in Q \times \mathbf{V} \times \Sigma$

$$\bar{L}\left((q, \mathbf{v}, \tau)\right) = \Delta t^2 \left(\mathbf{f}, \nabla q - \nabla \cdot \tau\right) + \Delta t < \mathbf{g}, (\tau - Iq)\vec{\nu} >_{1/2, \partial\Omega} + \Delta t \left(\mathbf{f}, \mathbf{v}\right).$$

Proposition 5.1 *Problem (14) has a unique solution.*

Proof. First we note that problem (14) is equivalent to a linear system of algebric equations with an equal number of unknowns and equations. Therefore it has a unique solution if and only if it admits only the trivial solution, once its right hand side is set to zero.

Let us then assume that the triple $(\bar{p}_h, \bar{\mathbf{w}}_h, \bar{\sigma}_h)$ satisfies

$$\bar{a}\left((\bar{p}_h, \bar{\mathbf{w}}_h, \bar{\sigma}_h), (q, \mathbf{v}, \tau)\right) = 0 \quad \forall (q, \mathbf{v}, \tau) \in Q_h \times \mathbf{V}_h^0 \times \Sigma_h$$

Taking $(q, \mathbf{v}, \tau) = (\bar{p}_h, \bar{\mathbf{w}}_h, \bar{\sigma}_h)$ we readily obtain

$$\Delta t^2 \parallel \nabla \cdot \bar{\sigma}_h - \nabla \bar{p}_h \parallel^2 + \frac{\Delta t}{2\eta} \parallel \bar{\sigma}_h \parallel = 0,$$

which implies that $\bar{\sigma}_h = 0$ and $\bar{p}_h = 0$.

This trivially implies that

$$(D(\bar{\mathbf{w}}_h), \tau - qI) = 0 \quad \forall (q, \tau) \in Q_h \times \Sigma_h \tag{16}$$

Next we endeavour to establish that relation (16) implies that $\bar{\mathbf{w}}_h = \mathbf{0}$. For this purpose we will take systematically $q = 0$. Now for every node P of T_h not belonging to $\partial\Omega$ we will choose N orthonormal frames B_P^i, $1 \leq i \leq N$, in such a way that one of the axes of B_P^i, say e_P^i, is the edge of an element of T_h having P as vertex. Now assume that P is the vertex of an element T_P such that $\bar{\mathbf{w}}_h$ vanishes at all the other N vertices of T_P. This is for instance the case of elements having an edge for $N = 2$ or a face for $N = 3$, contained in $\partial\Omega$. The axes e_P^i of B_P^i will be chosen in such a way that they are oriented from P to the vertices of T_P, say S_P^i for $1 \leq i \leq N$, respectively. Now we number the unit vectors of B_P^i in such a way that e_P^i is the first one, and we take $\tau = \tau_P^i$ where τ_P^i is the tensor whose representation in terms of the frame B_P^i writes

$$\tau_P^i = \begin{pmatrix} f_P^i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where f_P^i is the function of S_h whose value equals one at S_P^i and zero at every other node of T_h. Finally defining T_P^i to be the subset of T_h consisting of those elements having PS_P^i as a common edge, by straightforward calculations we derive

$$\left(D(\bar{\mathbf{w}}_h), \tau_P^i\right) = \Sigma_{T \in T_P^i} \frac{area(T)}{N+1} \left(\frac{-\bar{\mathbf{w}}_h(P) \cdot e_P^i}{l_P^i}\right)$$

where $l_P^i = \overline{PS_P^i}$. Letting i vary from one to N we immediately conclude from (16) that $\bar{\mathbf{w}}_h(P) = \mathbf{0}$.

Now the question is: is it possible to find a path linking all the nodes of T_h, starting from a node P having N neighboring nodes on $\partial\Omega$, in such a way that every new node of the path has N neighboring nodes at which it was previously established that $\bar{\mathbf{w}}_h$ vanishes. The answer is yes according to the following argument.

Once we eliminate from the mesh the set Γ_h^1 of all the elements of T_h having at least N vertices on $\partial\Omega$, in which $\bar{\mathbf{w}}_h$ vanishes identically according to the above argument, we come up with a new domain $\Omega_h^1 \subset \Omega - \Gamma_h^1$ namely the set of elements of T_h in which $\bar{\mathbf{w}}_h$ possibly does not vanish identically. If Ω_h^1 is empty the proof is complete. Otherwise $\bar{\mathbf{w}}_h$ vanishes on the boundary of Ω_h^1, and this domain necessarily contains at least one element having exactly one vertex that does not belong to its boundary at which possibly $\bar{\mathbf{w}}_h \neq \mathbf{0}$. More precisely such element has a common face with an element of $\Omega - \Omega_h^1$ and a vertex in the interior of Ω_h^1. Let Γ_h^2 be the union of all such elements. Then we apply the same construction for the elements of Γ_h^1, to those of Γ_h^2, thereby establishing that $\bar{\mathbf{w}}_h$ vanishes identically in Γ_h^2 too. Again we come up with a sub domain $\Omega_h^2 \not\subseteq \Omega_h^1 \not\subseteq \Omega$, namely the union of all elements of Ω_h^1 in which possibly $\bar{\mathbf{w}}_h$ does not vanish identically. If Ω_h^2 is empty the proof is complete. Otherwise the procedure continues in the same way until we reach a domain $\Omega_h^r \not\subseteq \Omega_h^{r-1} \cdots \not\subseteq \Omega_h^1 \not\subseteq \Omega$ for a certain integer r, which contains no element having more than one vertex that does not belong to its boundary at which possibly $\bar{\mathbf{w}}_h \neq \mathbf{0}$. Finally treating all the elements of Ω_h^r in the same manner as those of Γ_h^1 we establish that $\bar{\mathbf{w}}_h = \mathbf{0}$ everywhere in Ω. ■

Next we have

Theorem 5.1 *For every $\Delta t > 0$ the solution $(p_h^n, \mathbf{u}_h^n, \sigma_h^n)$ of (13) converges to $(\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$ as n goes to infinity.*

Proof. First we set $\bar{p}_h^n = p_h^n - \bar{p}_h$, $\bar{\mathbf{u}}_h^n = \mathbf{u}_h^n - \bar{\mathbf{u}}_h$, $\bar{\sigma}_h^n = \sigma_h^n - \bar{\sigma}_h$ and take $q = \bar{p}_h^n$, $\mathbf{v} = \bar{\mathbf{u}}_h^n$ and $\tau = \bar{\sigma}_h^n$ in both (13) and (14), thereby obtaining, after combining the resulting relations:

$$\beta \parallel \bar{\mathbf{u}}_h^n \parallel^2 + \parallel \bar{\sigma}_h^n \parallel^2 - \beta\Delta t\,(\nabla\cdot\bar{\sigma}_h^n, \bar{\mathbf{u}}_h^n) + \beta\Delta t\,(\nabla\bar{p}_h^n, \bar{\mathbf{u}}_h^n) + \beta\Delta t^2 \parallel \nabla\cdot\bar{\sigma}_h^n - \nabla\bar{p}_h^n \parallel^2 = \beta\left(\bar{\mathbf{u}}_h^{n-1}, \bar{\mathbf{u}}_h^n\right) + \alpha\left(\bar{\sigma}_h^{n-1}, \bar{\sigma}_h^n\right) + \beta\Delta t\left(\bar{\mathbf{u}}_h^{n-1}, \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n\right)$$

where $\alpha = \frac{\lambda}{\lambda+\Delta t}$ and $\beta = \frac{2\eta}{\lambda+\Delta t}$. Then,

$$(1-\alpha) \parallel \bar{\sigma}_h^n \parallel^2 + \alpha \parallel \bar{\sigma}_h^n \parallel^2 + \beta\left[\parallel \bar{\mathbf{u}}_h^n \parallel^2 + \Delta t\,(\bar{\mathbf{u}}_h^n, \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n] + \Delta t^2 \parallel \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n \parallel^2\right] \leq \beta\left(\bar{\mathbf{u}}_h^{n-1}, \bar{\mathbf{u}}_h^n + \Delta t\,(\nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n)\right) + \alpha\left(\bar{\sigma}_h^{n-1}, \bar{\sigma}_h^n\right)$$

from which after simple calculations we derive: for all n,

$$(1-\alpha) \parallel \bar{\sigma}_h^n \parallel^2 + \frac{\alpha}{2} \parallel \bar{\sigma}_h^n \parallel^2 + \frac{\beta}{2}\left[\parallel \bar{\mathbf{u}}_h^n \parallel^2 + \Delta t^2 \parallel \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n \parallel^2\right] \leq \frac{\beta}{2} \parallel \bar{\mathbf{u}}_h^{n-1} \parallel^2 + \frac{\alpha}{2} \parallel \bar{\sigma}_h^{n-1} \parallel^2$$

This implies that $[\beta \parallel \bar{\mathbf{u}}_h^n \parallel^2 + \alpha \parallel \bar{\sigma}_h^n \parallel^2]/2$ is a decreasing sequence of positive numbers and hence a converging one. Therefore since $0 < \alpha < 1$ and $\beta > 0$ we have $\parallel \bar{\sigma}_h^n \parallel \to 0$ and $\parallel \nabla\bar{p}_h^n - \nabla\cdot\bar{\sigma}_h^n \parallel \to 0$ which implies that $\parallel \bar{p}_h^n \parallel \to 0$, since $\bar{p}_h^n \in L_0^2(\Omega)$ for every n.

As for the convergence of $\bar{\mathbf{u}}_h^n$ to zero, we employ an argument similar to the one of Proposition 5.1; indeed from the convergence to zero of $\bar{\sigma}_h^n$ and $\bar{p}_h^n$, we readily infer from (14) and (13) that

$$(\bar{\mathbf{u}}_h^n, \nabla\cdot\tau - \nabla q) \to 0 \quad \forall (q,\tau) \in (Q_h, \Sigma_h).$$

Then choosing $q = 0$ and $\tau = \tau_P^i$ (cf. Proposition 5.1), and sweeping the mesh in the way indicated in the proof of that result, we derive $\bar{\mathbf{u}}_h^n(P) \to 0$ for every vertex P of the mesh, and the result follows. ■

As a consequence we establish the following convergence result:

Theorem 5.2 *Assume that the solution of (5) is such that $p^n \in H^1(\Omega)$ for every n. Then as h goes to zero the solution $(\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$ of (14) converges to the solution $(\bar{p}, \bar{\mathbf{u}}, \bar{\sigma})$ of (6) in $L^2(\Omega) \times L^2(\Omega)^N \times L^2(\Omega)^{N\times N}$.*

Proof. Let $W = (\bar{p}, \bar{\mathbf{u}}, \bar{\sigma})$ and $W_h = (\bar{p}_h, \bar{\mathbf{u}}_h, \bar{\sigma}_h)$. We further set $W^n = (p^n, \mathbf{u}^n, \sigma^n)$ and $W_h^n = (p_h^n, \mathbf{u}_h^n, \sigma_h^n)$. Still denoting by $\parallel \cdot \parallel$ the norm of $L^2(\Omega) \times L^2(\Omega)^N \times L^2(\Omega)^{N\times N}$, we have:

$$\parallel W - W_h \parallel \leq \parallel W - W^n \parallel + \parallel W^n - W_h^n \parallel + \parallel W_h^n - W_h \parallel$$

Now for a given $\varepsilon > 0$ we choose n in such a way that $\parallel W - W^n \parallel < \varepsilon/3$ and $\parallel W_h^n - W_h \parallel < \varepsilon/3$, which is possible according to theorems 3.1 and 5.1. Next for such n we choose h in such a way that $\parallel W^n - W_h^n \parallel < \varepsilon/3$, which is possible according to our regularity assumptions and to Theorem 4.1. This means that for every $\varepsilon > 0$ we may choose h in such a way that $\parallel W - W_h \parallel < \varepsilon$, and the result follows. ■

6. NUMERICAL RESULTS

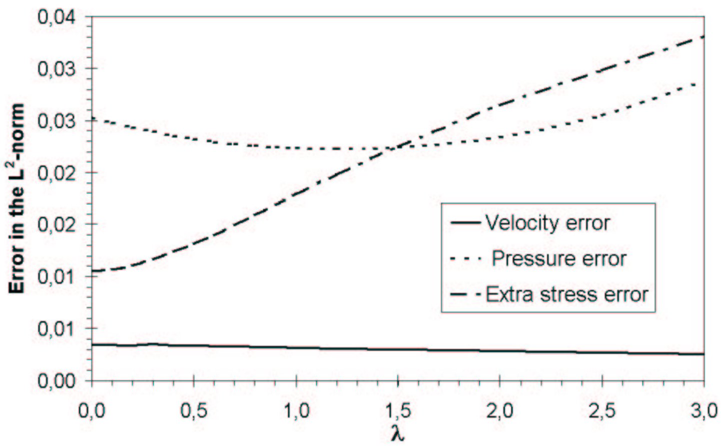

Figure 1.The evolution with λ of the L^2-errors of velocity, pressure and extra stress.

We present below numerical results for a test-problem with known analytic solution, in order to certify the adequacy of our numerical approach. In the test the initial values of both velocity and extra stress tensor are zero.

Owing to the dimension of the flow domain and the prescribed values of velocity, the characteristic parameter λ coincides with the Weissenberg number for viscolastic flows (cf. Marchal and Crochet (1987)). We consider a constitutive law of the upper convected Maxwell type. The non linear terms including the acceleration were computed explicitely at every iteration.

More specifically we solved a three-dimensional problem whose analytical solution is given by:

$$\begin{array}{c} u_1 = u_2 = 0\,; u_3 = x_1 x_2 (1 - x_2)\,;\;\; p = \eta(0.5 - 2x_2 x_3);\; \sigma_{11} = \sigma_{12} = \sigma_{22} = 0; \\ \sigma_{33} = 2\lambda\eta[(x_2 - x_2^2)^2 + (x_1 - 2x_1 x_2)^2];\; \sigma_{13} = \eta(x_2 - x_2^2);\; \sigma_{23} = \eta(x_1 - 2x_1 x_2). \end{array} \tag{17}$$

The computations were performed with a mesh constructed upon a $10 \times 10 \times 10$ uniform partition of a unit cube, thereby generating 6000 tetrahedrons. The body force term is given by $-\nabla \cdot \sigma + \nabla p$. The pressure is prescribed at the node $x_1 = x_2 = x_3 = 0.5$. In this example the value of Δt is 0.0002, and the values of λ range from $0.$ up to 3.0, with increments of 0.5.

In Figure 1 we display the evolution with λ of the L^2-errors of velocity, pressure and extra stress. The errors correspond to the values of the computed approximations of the unknown flow fields, once convergence is attained for a tolerance of 10^{-5} in the maximum norm.

7. ACKNOWLEDGEMENTS

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8. Responsibility notice