

USING COMPUTER MODELING AS A TOOL TO SOLVE NONLINEAR EQUATIONS OF MOTION IN DYNAMICS TEACHING

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Abstract. While teaching Dynamics, one of the most important disciplines in a Mechanical Engineering course, one of the handicaps is to obtain solutions for nonlinear equations of motion which are present for the majority of situations and to study the behavior of such systems under diverse initial conditions. This paper discusses the use of computer environment for symbolic, numerical and graphical computation applied to the teaching of classical mechanics, with special emphasis on Dynamics. The physical modeling leads to videos, representing the whole system motion. Some of the main advantages of this approach are: to train the student in computer modeling; to reinforce its insight about the true behavior of nonlinear mechanical systems; and to let the scholar to test by himself new configurations. Some examples are discussed, including: the parameters setting; the initial conditions choice; the equations of motion programming; its numerical integration; plots of the mechanical system coordinates versus time; and the system animation showing the resulting movement. It will also be shown how some relatively simple situations may lead to a surprising behavior, which strengthens the importance and the advantage of the use of computer modeling in Dynamics teaching.

Keywords: Computer Modeling, Teaching Dynamics, Animations, Nonlinear Equations of Motion, Numerical Integration.

1. INTRODUCTION

Computer modeling permeates the engineering realm with a growing importance. It is natural, therefore, to apply the numeric available techniques also as a teaching support, particularly in mechanical engineering.

As it is well known, most of the situations involving dynamical analysis of mechanical systems results in nonlinear differential equations (Tenenbaum, 1997) — even for, say, the elementary problem of the one-degree of freedom simple pendulum.

Classical textbooks on Dynamics for undergraduate students usually present the analysis of forces and torques applied to a mechanical system, study its kinematics — with the calculus of velocities, accelerations, and angular velocities and accelerations, in spite of treating carelessly the angular velocity concept (Tenenbaum, 1993) —, and establish the corresponding equations of motion (Beer and Johnson, 1972, Meriam, 1990, Williams, Jr, 1996, Hibeller, 2005).

Of course, since more than 90% of the situations result in nonlinear differential equations, these texts stop the analysis at this point, since no simple analytical solution can be provided for the differential equations. This is not the case, however, of few books intended to be used by graduate students on Dynamics (see, for instance, Kane and Levinson, 1985), where coordinates evolution plots, obtained via numerical integration, are presented and discussed for some examples. Another noticeable exception is the nice book *Dinâmica de Sistemas Mecânicos*, by Ilmar F. Santos (2001). There is a CD-ROM that goes with the book showing interesting animations. However, the files do not permit the student to interact with the mechanical systems, by changing parameters or initial conditions.

In this author's opinion, showing coordinates plots or even video files is not enough for the sake of deep understanding of the dynamical principles by the actual undergraduate student of engineering and correlated sciences. Once being presented to a computer program capable to deal with algebraic manipulation and numerical integration of differential equations and, more than that, using by him or herself such a program, increases in an astonishing way the student's insight about the applied dynamical principles and the system behavior.

In the next sections, it will be shown some examples of animations for dynamic systems, including the setting of system parameters and initial conditions, the numerical integration of equations of motion, the plot of coordinates versus time, the computer modeling of the physical system itself and the animation of the system's motion, which furnishes to the student a reliable way to verify the correctness of the model. The computer models were programmed with one of the available commercial programs for algebraic manipulation and numerical integration (Maple™). The mechanical systems shown here were extracted from the files associated with the book *Fundamentals of Applied Dynamics* (Tenenbaum, 2005) or the ones found in the CD-ROM that goes with the book *Dinâmica Aplicada* (Tenenbaum, 2006), which contains the computer modeling with animations for almost all examples considered in the text.

2. INTEGRATING NONLINEAR EQUATIONS OF MOTION

Let us start with a classical example of dynamics of particles, a one-degree of freedom system, as depicted in Fig. 1. This example shows how a small ball (taken as a particle) behaves while moving inside a rotating curved pipe that moves with a prescribed angular velocity. The simulation involves different values for the angular velocity ω , corresponding

to distinct behaviors of the motion under study. The nonlinear differential equation that rules the coordinate $\theta(t)$ is (Tenenbaum, 1997)

$$\ddot{\theta}(t) + \left(\omega^2 \sin \theta - \frac{g}{r} \right) \cos \theta = 0, \quad (1)$$

which, stated in Maple language (see Redfern, 1996, for the syntax), becomes

$$eq := D(D(\theta))(t) + \omega^2 * \cos(\theta(t)) * \sin(\theta(t)) = g/r * \cos(\theta(t)); \quad (2)$$

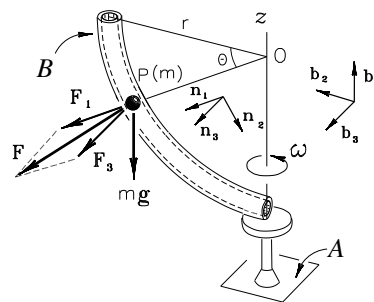


Figure 1. Example of dynamics of a particle P that slides without friction inside a rotating curved pipe B with a prescribed angular velocity ω .

The system parameters (radius r and angular velocity ω) are given as (in SI units)

$$r := 5; \omega := 1; \quad (3)$$

The initial conditions ($\theta(0) = \dot{\theta}(0) = 0$) are established as

$$ci := \theta(0) = 0, D(\theta)(0) = 0; \quad (4)$$

Then, the numerical integration is performed with the commands

$$\begin{aligned} Solution &:= dsolve(eq, ci, \theta(t), type = numeric, method = classical[rk4], corrections = 4, \\ &maxfun = -1, output = listprocedure); \end{aligned} \quad (5)$$

The graph of the solution for $\theta(t)$ and $\dot{\theta}(t)$ are obtained with

$$odeplot(Solution, [t, \theta(t)], 0..60, numpoints = 1000); \quad (6)$$

$$odeplot(Solution, [t, diff(\theta(t), t)], 0..60, numpoints = 1000); \quad (7)$$

Of course, for this quite simple example, the most important student gain is the fact that he or she begin to master the computer programming for algebraic manipulation and numerical integration of equations of motion.

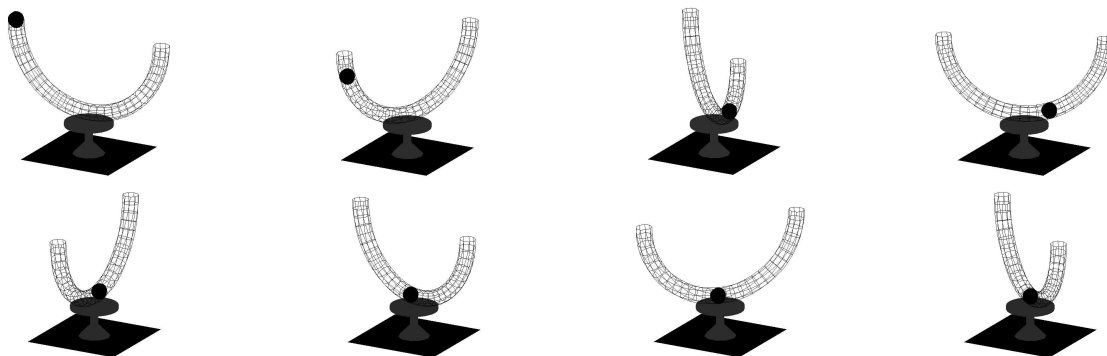


Figure 2. Animation of the system depicted in Fig. 1, presented here as a sequence of stills at equal time intervals. Observe that the ball speed in B reduces from the initial position and stops at the bottom of the pipe (for $\omega = \sqrt{2g/r}$).

But even in this simple example, some interesting insight can be obtained, for instance, with the study of the conditions required so that the ball arrives to the pipe bottom with zero velocity. It can be shown (Tenenbaum, 2005) that this condition is given by $\omega = \sqrt{2g/r}$ and the motion animation becomes the one shown in Fig 2.

3. OBSERVING COORDINATES EVOLUTION

One of the main features of computer modeling while teaching dynamics is the possibility to observe the coordinates time evolution for strong nonlinear systems at different initial conditions. As an example, let us consider the classical two-degrees of freedom double pendulum moving on a vertical plane, as depicted on Fig. 3.

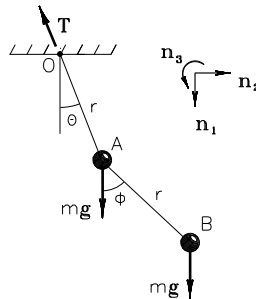


Figure 3. Example of the double pendulum moving on a vertical plane.

The differential equations for the coordinates $\theta(t)$ and $\phi(t)$ are, in Maple language:

$$\begin{aligned}
 eq1 := & D(D(phi))(t) * (sin(phi(t)) * tan(theta(t)) + cos(phi(t))) \\
 & + (D(phi)(t))^2 * (cos(phi(t)) * tan(theta(t)) \\
 & - sin(phi(t))) + 2 * D(D(theta))(t) * sec(theta(t)) = -2 * g/r * tan(theta(t)); \quad (8)
 \end{aligned}$$

$$\begin{aligned}
 eq2 := & (2 + cos(phi(t) - theta(t))) * D(D(theta))(t) + (1 + cos(phi(t) - theta(t))) * D(D(phi))(t) \\
 & - sin(phi(t) - theta(t)) * ((D(phi)(t))^2 - (D(theta)(t))^2) \\
 & = -g/r * (2 * sin(theta(t)) + sin(phi(t))); \quad (9)
 \end{aligned}$$

In this example we have one of the most remarkable and simple situations of chaotic motion. Here, it is of the utmost importance to test various initial conditions to observe the strong dependence on these conditions in the final behavior. For instance, given the initial conditions as $\theta(0) = \phi(0) = \pi/2$ and $\dot{\theta}(0) = \dot{\phi}(0) = 0$, the coordinates evolution is as depicted in Fig. 4, for the first 20 seconds.

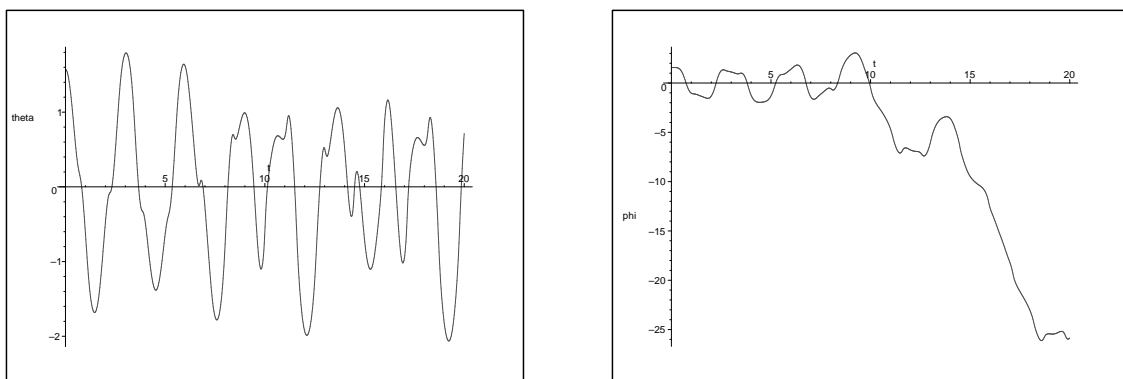


Figure 4. Evolution of $\theta(t)$ (left) and $\phi(t)$ (right), both in radians, for the first 20 seconds. Note the nonlinear oscillation behavior of $\theta(t)$. Observe the complete turn of $\phi(t)$ after three oscillation cycles of $\theta(t)$ and the three complete turns of $\phi(t)$ after one more cycle of $\theta(t)$.

4. COMPUTER MODELING OF PHYSICAL SYSTEMS

Let us consider now a mechanical system that consists of two rigid and homogeneous bars, articulated at the common point O, moving on a frictionless horizontal plane as depicted on Fig. 5. The given initial conditions are bar B_1 with angular velocity ω_0 and bar B_2 with no angular velocity, as shown.

Since there is no net force applied, it is natural to conclude that the mass center velocity, $\mathbf{v}^* = \frac{1}{4}r\omega_0 \mathbf{n}_1$ remains invariant during the system's motion. However, it is not trivial to perceive the rotating motion of the bars. To calculate this motion, the physical system must be modeled and the resultant equations of motion should be numerically integrated. The animation shown in Fig. 6 presents the unsuspected system behavior.

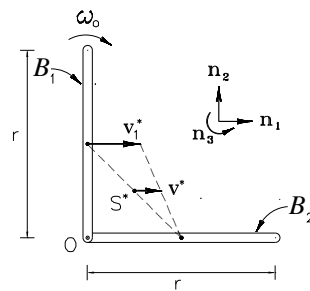


Figure 5. Mechanical system consisting of two homogeneous and rigid bars, articulated at the common point O, sliding on a frictionless horizontal plane.

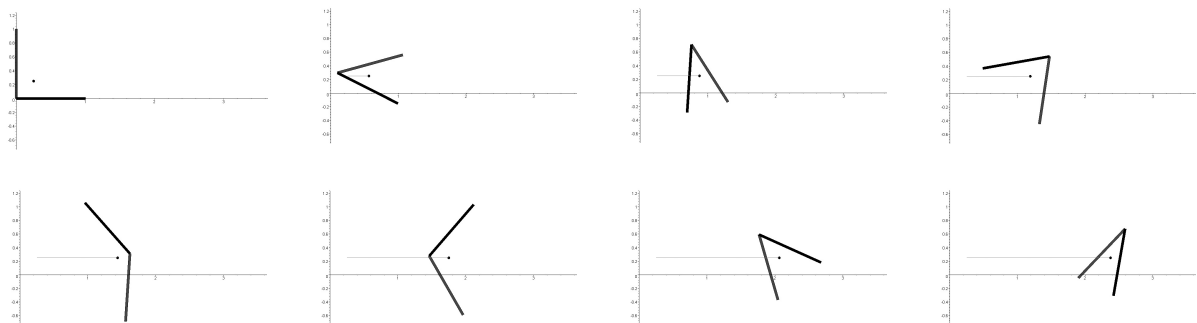


Figure 6. Animation of the system depicted in Fig. 5, presented as a sequence of stills at equal time intervals. Observe the rotating motion of the two bars around the system's mass center (indicated as a dot), which moves in a straight line.

5. SOME "SURPRISING" RESULTS

Sometimes, the mechanical system behavior strikes the student's common sense and the animation is the better way — in the absence of a well controlled experimental setup — to "see" what actually happens with it. Let us consider the mechanical system depicted in Fig. 7, which represents a sphere that rolls inside a cylindrical vertical pipe.

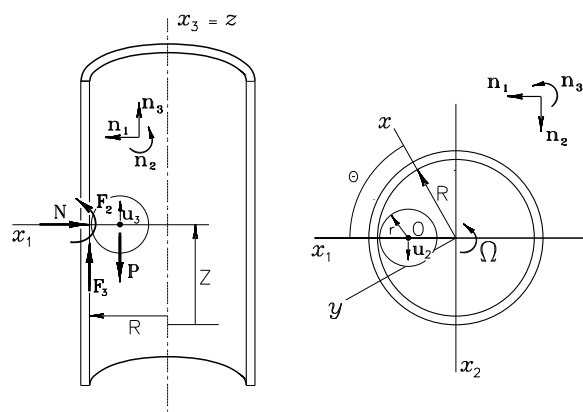


Figure 7. A sphere that rolls inside a rough cylindrical pipe with vertical axis .

The given initial conditions are the two mass center velocity components, u_2 and u_3 , in the shown directions (see Fig. 7). The sphere is modeled as a rigid body, that means, with rotational inertia. The first student guess is that the sphere mass center circumferential speed, u_2 , remains constant (which is correct), indicating that $\dot{\theta}(t)$ is also constant, and that the vertical velocity component, u_3 , suffers a continuous rate of change, due to the gravitational acceleration.

However, it can be shown (see Tenenbaum, 2006) that the mass center vertical coordinate, $z(t)$, must satisfy the

differential equation

$$\ddot{z} + \frac{2}{7}\Omega^2 z + \frac{5}{7}g = 0, \tag{10}$$

where $\Omega = -\dot{\theta} = u_2/(R - r)$ and g , as usual, stands for the gravitational acceleration. The general solution of Eq. (10) is, then

$$z(t) = \frac{r}{\omega_0} (A \sin \omega_0 t + B(1 - \cos \omega_0 t)), \tag{11}$$

where $\omega_0 = \sqrt{\frac{2}{7}}\Omega$, $A = u_3/r$, and $B = -(5g)/(7r\omega_0)$. The behavior is, of course, periodic (due to the realistic consideration of the rotational inertia of the sphere), unlike the “common sense” suggests. In Fig. 8 the animation of the system is presented.

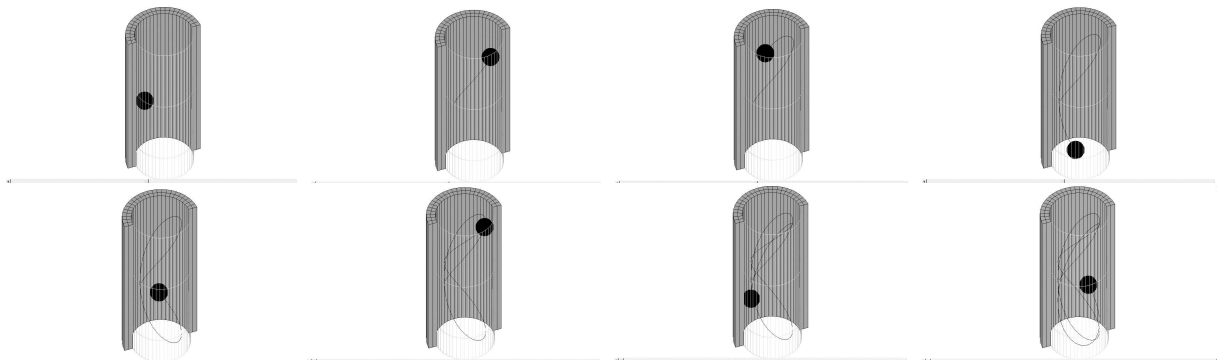


Figure 8. Animation of the periodic motion of a sphere rolling inside a vertical cylindrical pipe, showed as a sequence of stills with equal time intervals. The thin line describes the sphere mass center trajectory. Observe the changes in the direction of the vertical velocity component, u_3 .

6. GYROSCOPIC MOTION

Another situation where the use of computer modeling is fundamental to reinforce the insight in dynamics is in the study of the gyroscopic motion. We will consider then the gyroscopic motion with an applied torque (resulting from weight) with regard to fixed point O. This is the paradigm for the motion of a top. The geometry is shown in Fig. 9.

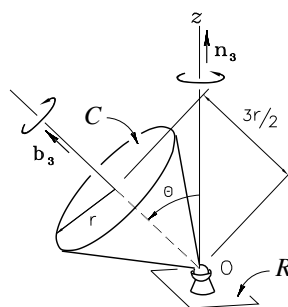


Figure 9. Geometry for the example of a gyroscopic motion with an applied torque. Note the spin direction, \mathbf{b}_3 , and the inclination angle, θ .

It can be shown (Beer and Johnson, 1972) that, for special initial conditions, the cone starts moving with an almost vertical spin (with a very small deviation) and then starts a more general movement, with precession and nutation, but with the particular characteristic of being periodic. These initial conditions are given with the following commands (in Maple language):

$$\begin{aligned} ci := theta(0) = theta_0, D(theta)(0) = der_t theta, psi(0) = 0, D(psi)(0) = 0, phi(0) = 0, \\ D(phi)(0) = omega[0]; \end{aligned} \tag{12}$$

The equations of motion for the three Euler angles are stated as:

$$eq1 := In * (D(D(theta))(t) - D(psi)(t)^2 * sin(theta(t)) * cos(theta(t))) + J * (D(psi)(t) * cos(theta(t)) + D(phi)(t) * D(psi)(t) * sin(theta(t))) = 3/4 * m * g * a * sin(theta(t)); \quad (13)$$

$$eq2 := In * (D(D(psi))(t) * sin(theta(t)) + 2 * D(psi)(t) * D(theta)(t) * cos(theta(t))) - J * (D(psi)(t) * cos(theta(t)) + D(phi)(t) * D(theta)(t)) = 0; \quad (14)$$

$$eq3 := J * (D(D(psi))(t) * cos(theta(t)) - D(psi)(t) * D(theta)(t) * sin(theta(t)) + D(D(phi))(t)) = 0; \quad (15)$$

The numerical integration of the equations of motion is performed then with

$$sol := dsolve(eq1, eq2, eq3, ci, theta(t), phi(t), psi(t), type = numeric, method = rosenbrock, maxfun = -1, output = listprocedure); \quad (16)$$

The graph showing the spin evolution with time, $\dot{\phi}(t)$, is obtained (see Fig. 10) with

$$odeplot(sol, [t, diff(phi(t), t)], 0..5, numpoints = 1000); \quad (17)$$

Another interesting plot is the spin evolution as a function of the angle θ (see Fig. 10), which is given by

$$odeplot(sol, [theta(t), diff(phi(t), t)], 0..5, numpoints = 1000); \quad (18)$$

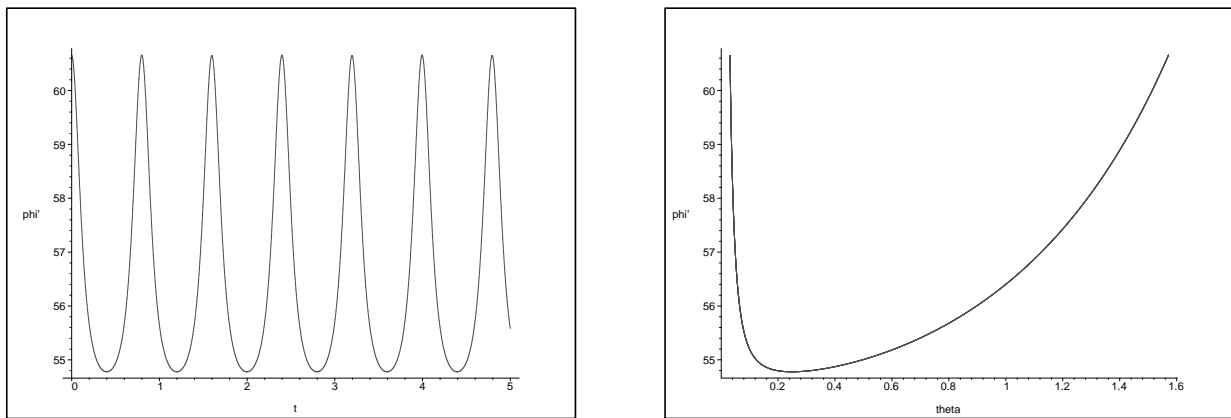


Figure 10. Evolution of $\dot{\phi}(t)$ (left) and $\dot{\phi}(\theta)$ (right), both in rad/s. One can observe the periodic behavior of the spin and the singular dependence of it with the inclination angle of the conic top.

Lastly, we have the solution of the equations for precession, $\dot{\psi}(t)$, which also behaves periodically, as shown in Fig. 11.

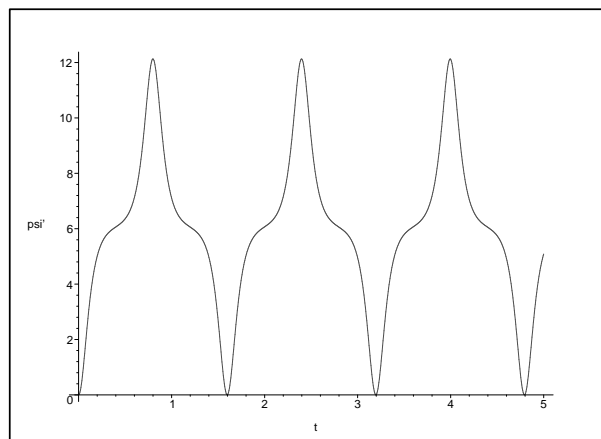


Figure 11. Time evolution of the precession, $\dot{\psi}$, showing a periodic behavior.

Note that the precession returns to zero whenever the angle to the vertical is also null. The possibilities to explore the system dynamics in order to increase the student's understanding are almost infinite. For instance, in this same example,

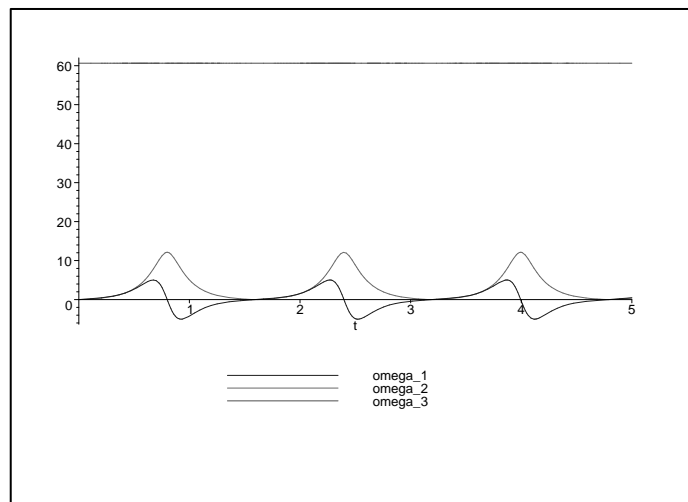


Figure 12. Time evolution of the three components of the angular velocity.

one can plot the time evolution of the three scalar components of the angular velocity vector, ω_1 , ω_2 , and ω_3 . As can be seen in Fig. 12, two of these components are periodic while the third one is constant, as previewed in the theory.

One of the main advantages of applying computer modeling to mechanical systems, integrating its equations of motion, plotting the desired coordinates, and animating its movement is the possibility to interact virtually with the system, by changing parameters values and initial conditions without the need of re-programming the model, and, by this way, to test as many situations as desired, improving ones feeling about the subject.

7. CONCLUSIONS

As discussed above, computer modeling applied to dynamics of mechanical systems shows to be a powerful and flexible tool to help teaching Dynamics. It permits to set system's parameters and initial conditions, to numerically integrate nonlinear differential equations, to analyze coordinates time evolutions, to physically model the system itself, and to observe the system's motion in a virtual animation, that can be saved also as a video file.

Engineering students do not offer resistance to learn one of the available software for mathematics manipulation. Unlike, they are fully interested and they actually master the software in a couple of weeks. More than that, to master computing modeling at undergraduate level is of great utility for his future professional life.

The gain in deep understanding of mechanical systems and the increase in the insight about the discussed examples are unbeatable.

We encourage every teacher to make large use of this powerful tool while teaching.

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