A DESCRIPTION OF A BOUNDARY INTEGRAL METHOD FOR FREE SURFACE INVOLVING MAGNETIC FLUIDS

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Abstract. *The present work is concerned with a new general three-dimensional hydrodynamic-magnetic boundary integral formulation to be used in numerical simulations for describing the deformation of a ferrofluid drop undergoing a magnetic field and a shear flow at low Reynolds numbers. Most of works have been concerned with axisymmetric or two-dimensional boundary integral formulation for axisymmetric magnetic drops in electric or magnetic field, which only require treatment of line integrals. The present work, however, will consider the more difficult case of a three-dimensional integral formulation for a hydrodynamic-magnetic surface distortion. The formulation is based on an extension of the Lorentz reciprocal theorem for the incompressible flow of a magnetic fluid. Combining the reciprocal theorem and the fundamental solution of a creeping flow we obtain the integral representation of the flow in terms of hydrodynamic and magnetic potentials. According to this formulation, the magnetic and hydrodynamic quantities which are necessary for determination of the dynamics of a magnetic liquid are established by means of appropriate integral equations at the boundary of the region occupied by the magnetic liquid. The method can be applied to compute by boundary integral numerical simulations the distortion and orientation of a three-dimensional ferrofluid droplet under the action of shearing motions and magnetic fields.*

Keywords: *creeping flow, boundary integral, reciprocal theorem, magnetic drops.*

1. INTRODUCTION

In recent years the deformation of fluid interfaces under an applied field have been considered as subject of numerous investigations. Applications include the breakup of rain drops in thunderstorms, electrohydrodynamic atomization, the behavior of jets, drops in ink-jets plotters and optimization of high-voltage car spraying tools. Magnetic drop deformation has been first studied by Arkhipenko, Barkov and Bashtovoi (1978), and by Drozdova, Skrobotova and Chekanov (1979). Experiments carried out by Bacri and Salin (1982) and Bacri and Salin (1983) have been shown that when the magnetic field is increased and subsequently reduced, hysteresis in the deformation of the drop is observed. In addition, Bacri, Salin and Massart (1982) shows that the drop shape jumps from a slightly elongated to a slender one, when the magnetic field intensity is risen. Recently, Lavrova et al. (2006) used a coupled system of Maxwell equations and incompressible Navier-Stokes equation together with the Young-Laplace equation in order to obtain equilibrium shapes of ferrofluid drop. A related problem of a colapsing bubble in a magnetic fluid was studied by Cunha, Souza and Morais (2002).

The present work is concerned with a general three-dimensional boundary integral formulation to be used in numerical simulations for describing the deformation of a three-dimensional ferrofluid drop undergoing a magnetic field and a shear flow at low Reynolds numbers. The boundary integral formulation for a Stokes flow regime was first described, in a theoretical way, by Ladyzhenskaya (1969) within the framework of hydrodynamic potentials. Boundary integral methods have been successfully used for simulations of potential flow around three-dimensional bodies (Alvarenga and Cunha, 2006), nonmagnetic drop deformation and breakup (Cristini, Blawzdziewicz and Loewenberg, 1998), drop-to-drop interaction (Loewenberg and Hinch, 1996; Guido and Simeone, 1998; Cunha, Almeida and Loewenberg, 2003; Cunha and Loewenberg, 2003), characterization of nonmagnetic emulsion rheology (Loewenberg and Hinch, 1997; Zinchenko and Davis, 2002) and emulsion expansion and foam-drop dynamics (Cunha, Souza and Loewenberg, 2003; Cunha and Loewenberg, 2003; Kraynik, Reinelt and Princen, 1991).

Although most work has been concerned with axisymmetric or two-dimensional boundary integral formulation for axisymmetric magnetic drops in electric or magnetic field, which only require numerical treatment of line integrals (e.g. Sherwood, 1988; Bacri et al., 1995; Bacri et al., 1996), the present work will consider the more difficult case of a three-dimensional integral formulation for a hydrodynamic-magnetic surface distortion. The problem falls naturally into two parts: that of finding the magnetic potential, and that of determining the fluid motion. In this way, the motion of a free surface with arbitrary magnetic properties and with the viscosity of the magnetic liquid and the surrounding fluid not equal may be explored with the present formulation. Two relevant physical parameter are revealed in the present hydrodynamic-magnetic boundary integral formulation; the ratio of the magnetic permeability and the magnetic Bond number. We have combined the hydrodynamic and the magnetic problem by means of a boundary integral technique. The next sections are devoted to the mathematical development of a general three-dimensional boundary integral formulation for a ferrofluid deformable interface undergoing a magnetic field and a shear flow.

2. GOVERNING EQUATIONS

The theoretical formulation developed in this article may be applied for modeling the motion of drops of viscosity η', magnetic permeability μ' and undisturbed radius a immersed in a second immiscible fluid of viscosity η, magnetic permeability μ with externally imposed velocity $\boldsymbol{u}^{\infty}$ and magnetic $\boldsymbol{H}^{\infty}$ fields, as described in Fig. (1)

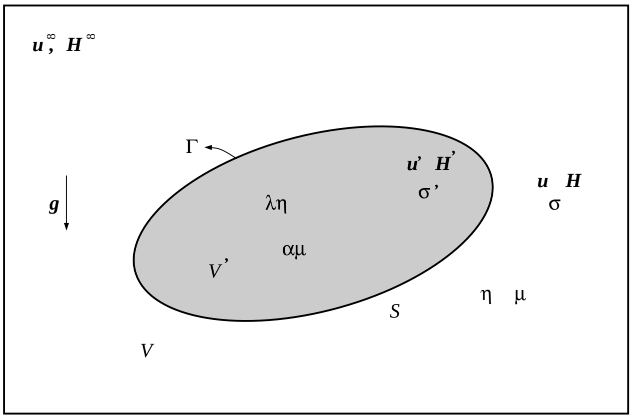

Figure 1. Sketch of a magnetic fluid drop in an emulsion under an externally imposed velocity and magnetic fields.

Hereupon, $\lambda = \eta'/\eta$ and $\alpha = \mu'/\mu$ denotes, respectively, the viscosity ratio and the magnetic permeability ratio between internal and external fluids. This problem is non-linear and gives rise to non-Newtonian effects assigned to the magnetic stresses and the coupling between magnetism and hydrodynamics.

2.1 Magnetostatics

In the absence of an electrical field and, if magnetic fields do not vary with time, Maxwell's equations (Grant and Philips, 1990) reduce to the magnetostatic limit that is described by the following equations

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \nabla \times \boldsymbol{H} = \boldsymbol{0}, \tag{1}$$

where ∇ denotes the partial differential operator, $\boldsymbol{B}$ is the magnetic induction and $\boldsymbol{H}$ the magnetic intensity vector. In addition, the magnetic relation

$$\boldsymbol{B} = \mu_0(\boldsymbol{M} + \boldsymbol{H}) \tag{2}$$

is valid at every point of the material. Here, $\boldsymbol{M}$ is the local magnetization that reveals the intrinsic polarization state of the continuum material promoted by the magnetic field and $\mu_0 = 4\pi \times 10^{-7} H/m$ is the permeability of free space. The magnetizable liquid is assumed to obey a linear relation $\boldsymbol{M} = \chi \boldsymbol{H}$, with χ being the magnetic susceptibility. We focus therefore on dilute soft magnetic material, i.e., superparamagnetic fluid, that has a very short memory, resulting in an instantaneous alignment of the particles with $\boldsymbol{H}$. Under this condition Eq. (2) reduces to $\boldsymbol{B} = (K-1)\boldsymbol{H} = \mu \boldsymbol{H}$, where $\mu = \mu_0(1+\chi)$ denotes the permeability of the magnetic liquid and $K = \mu/\mu_0 = 1 + \chi$ is the relative permeability of the magnetic liquid. It should be important note that, in our formulation, we consider the magnetic permeability as being a magnetic material constant. Therefore, $\nabla \cdot \boldsymbol{H} = 0$. Remembering that $\boldsymbol{H}$ is a irrotational field, then $\boldsymbol{H} = \nabla \phi$, where ϕ is the magnetic potential field, and the problem is governed by the Laplace equation $\nabla^2 \phi = 0$.

2.2 Hydrodynamics

As mentioned in §2. a hydrodynamic-magnetic coupled problem is studied here. Therefore, besides the magnetic equations, we must describe the hydrodynamic balance equations. In this sense, neglecting fluid inertia and compressibility the hydrodynamic balance equations reduces to the Stokes flow regime described by Happel and Brenner (1965)

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0}, \tag{3}$$

where $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ represents the Eulerian velocity field and the stress tensor of the fluid. In a ferrohydrodynamic problem, as the one explored herein, the coupling between magnetism and hydrodynamics is given by the stress tensor $\boldsymbol{\sigma}$ that considers magnetic effects on the flow, namely

$$\boldsymbol{\sigma} = -P\boldsymbol{I} + 2\eta \boldsymbol{D} + \boldsymbol{B}\boldsymbol{H} \tag{4}$$

where the notation $\boldsymbol{BH}$ corresponds to the dyadic or tensorial product between the $\boldsymbol{B}$ and $\boldsymbol{H}$ usually written as $\boldsymbol{B} \otimes \boldsymbol{H}$, η denotes the fluid shear viscosity, $\boldsymbol{I}$ is the identity tensor, P is the total pressure and $\boldsymbol{D} = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ represents the rate of strain tensor. In addition, $\nabla \boldsymbol{u}^T$ denotes the transpose tensor of $\nabla \boldsymbol{u}$. Herein, P is defined as $P = p_h + p_m$, where p_h is the static pressure and $p_m = (1/2)\mu_0(\boldsymbol{H} \cdot \boldsymbol{H})$ is the magnetic pressure.

3. BOUNDARY CONDITIONS

On a drop interface S with surface tension Γ, the boundary conditions requires a continuous velocity across the interface and a balance between the net surface traction and surface forces that express the discontinuity in the interfacial surface forces. Moreover, the magnetostatic regime states that the normal component of $\boldsymbol{B}$ and the tangential component of $\boldsymbol{H}$ to an interface between different media must be continuous (at all points across the interface). Mathematically, these conditions are expressed as

$$\boldsymbol{u} \to \boldsymbol{u}^\infty \quad |\boldsymbol{x}| \to \infty; \quad \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}'(\boldsymbol{x}); \quad \boldsymbol{n} \cdot \boldsymbol{B}'(\boldsymbol{x}) = \boldsymbol{n} \cdot \boldsymbol{B}(\boldsymbol{x}); \quad \boldsymbol{n} \times \boldsymbol{H}'(\boldsymbol{x}) = \boldsymbol{n} \times \boldsymbol{H}(\boldsymbol{x}), \quad \forall \quad \boldsymbol{x} \in S, \tag{5}$$

where $\boldsymbol{u}'$, $\boldsymbol{H}'$ and $\boldsymbol{B}'$ denotes, respectively, the flow, the magnetic field and the magnetic induction inside the drop and $\boldsymbol{n}$ is the unit normal vector to S. Considering an interface free of surface viscosity, surface elasticity and surface module of bending and dilatation, the constitutive equation for the traction jump $\Delta \boldsymbol{t} = [[\boldsymbol{n} \cdot \boldsymbol{\sigma}]]$ is written as Pozrikidis (1992)

$$\Delta \boldsymbol{t} = [[\boldsymbol{n} \cdot \boldsymbol{\sigma}]] = \Gamma \nabla^s \cdot \boldsymbol{nn} - (\boldsymbol{I} - \boldsymbol{nn}) \cdot \nabla \Gamma \tag{6}$$

The notation $[[\quad]]$ denotes a jump in flow quantities, $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ is the surface traction, $\nabla^s = (\boldsymbol{I} - \boldsymbol{nn})$ denotes the gradient operation tangent to the interface, consequently $\nabla^s \cdot \boldsymbol{n}$ denotes twice the mean curvature of the interface κ. The normal component includes the effect of the surface tension Γ while the tangential component is that due to interfacial tension gradients, associated with the presence of surfactants in the fluid, named Marangoni effects. Furthermore, using a Lagrangian representation for the interface evolution of a drop, one gets a kinematic constraint relating changes in the interface position to the local velocity

$$\frac{D\boldsymbol{x}}{Dt} = \boldsymbol{u}(\boldsymbol{x}), \quad \forall \quad \boldsymbol{x} \in S, \tag{7}$$

where D/Dt denotes the material derivative.

4. MAGNETIC BOUNDARY INTEGRAL FORMULATION

In this section we discuss a three-dimensional boundary integral method to solve Laplace equation, resulting from the magnetostatic conditions given in Eq. (1), in terms of singularities at the interface between two magnetic fluids.

4.1 Reciprocal theorem for a magnetic potencial field

Consider a closed region of fluid V bounded by a surface S. Following this assumption, consider two distinct magnetic potencial fields ϕ and ϕ' acting, respectively, over two different magnetic fluids with permeabilities μ and μ'. According to the Green's second identity (Jaswon and Symm, 1977), we have

$$\int_S (\phi \nabla \phi' - \phi' \nabla \phi) \cdot \boldsymbol{n} dS = \int_V (\phi \nabla^2 \phi' - \phi' \nabla^2 \phi) dV, \tag{8}$$

where ϕ and ϕ' are two scalar functions of position. Here, $\boldsymbol{n}_i$ and $\boldsymbol{n}$ means, respectively, the inwardly and outwardly directed unit vector normal to the surface S, then $\boldsymbol{n} = -\boldsymbol{n}_i$. Being ϕ and ϕ' harmonic functions, like as the magnetic potential field, the RHS (right hand side) of (8) vanishes, carrying out the reciprocal theorem for harmonic functions

$$\int_S \phi \nabla \phi' \cdot \boldsymbol{n} dS = \int_S \phi' \nabla \phi \cdot \boldsymbol{n} dS. \tag{9}$$

The Eq. (9) means that if a solution for the magnetic potencial field ϕ' is known any field of interest ϕ can be determined.

4.2 Integral representation for a magnetic potencial field

Now, let's consider the particular case of interest with magnetic potencial field ϕ. Here, the known magnetic potencial field ϕ' is that given by the fundamental solution of $\nabla^2 \phi' = h\delta(r)$

$$\phi'(r) = \frac{h}{4\pi\mu_0 r} = \frac{h}{4\pi\mu_0}\, \mathcal{C}(r), \quad \text{with} \quad \nabla \phi'(r) = -h\frac{\boldsymbol{r}}{4\pi\mu_0\, r^3} = \frac{h}{4\pi\mu_0} \nabla \mathcal{C}(r) \tag{10}$$

where $\mathcal{C}(r) = 1/r$ is the free space Green's function corresponding to a source point and $\nabla\mathcal{C}(r) = -\boldsymbol{r}/r^3$ denotes a potential dipole. The solution in Eq. (10) corresponds to the magnetic potencial field due to a point force with strength h. Here, $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$ with $\boldsymbol{x}$ being an arbitrary point of the domain V and $\boldsymbol{x}_0$ the location of the pole and $r = |\boldsymbol{r}|$. Thus we are interested to apply the solution in (10) to the reciprocal theorem given by Eq. (9). Here, ϕ is the unknown potential in the domain V and ϕ' is a potential of a source point that is singular as $r \to 0$. Next, we consider two situations with respect to the location of the singularity in the domain.

Singularity outside V: In the case $\delta(r) = 0$ inside V, the reciprocal theorem in Eq. (9), after discarding the arbitrary constant $h \neq 0$, takes the form

$$\int_S [\phi(\boldsymbol{x})\nabla\mathcal{C}(r) - \mathcal{C}(r)\nabla\phi(\boldsymbol{x})] \cdot \boldsymbol{n} dS = 0 \tag{11}$$

because the potencial ϕ' is not singular inside V, if $\boldsymbol{x}_0$ is outside V.

Singularity inside V: When exists a singularity located at $\boldsymbol{x}_0$ into V, it is needed to be excluded from the region of integration. To overcome this problem, we place a small bounded sphere of radius ε and volume V_ε centered at $\boldsymbol{x}_0$ involving this singularity, as described in Fig. (2).

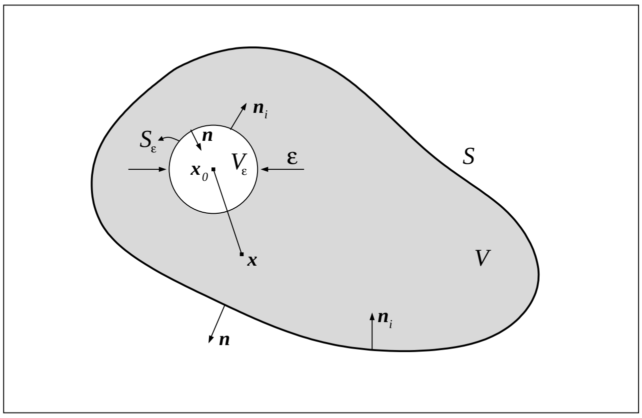

Figure 2. Fluid domain V bounded by a surface S broken down into V_ε and $V - V_\varepsilon$.

Then, outside the small sphere, throughout the remaining volume $V - V_\varepsilon$ the functions within the square brackets in Eq. (9) are continuous. In this way, again discarding the arbitrary constant h, the reciprocal theorem (9), applied to the surface $S - S_\varepsilon$ that bounds the volume $V - V_\varepsilon$, becomes

$$\int_S [\phi(\boldsymbol{x})\nabla\mathcal{C}(r) - \mathcal{C}(r)\nabla\phi(\boldsymbol{x})] \cdot \boldsymbol{n}_i dS + \int_{S_\varepsilon} [\phi(\boldsymbol{x})\nabla\mathcal{C}(r) - \mathcal{C}(r)\nabla\phi(\boldsymbol{x})] \cdot \boldsymbol{n}_i dS = 0 \tag{12}$$

Now, considering the integral over S_ε containing the singularity $\boldsymbol{x}_0$, with $dS_\varepsilon = \varepsilon^2 d\Omega$; $d\Omega$ denotes the infinitesimal solid angle. Based on the fundamental solution given in (10), the expressions for the potential monopole $\mathcal{C}(r)$ and the potential dipole $\nabla\mathcal{C}(r)$ inside S_ε are given by

$$\mathcal{C}(r) \approx \frac{1}{\varepsilon} \quad \text{and} \quad \nabla\mathcal{C}(r) \approx -\frac{\boldsymbol{r}}{\varepsilon^3} \tag{13}$$

with the inwardly directed unit normal vector being $\boldsymbol{n}_i = \boldsymbol{r}/\varepsilon$. Therefore, for the limit $\varepsilon \to 0$, one obtains

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} \mathcal{C}(r)\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{n}_i dS = \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{1}{\varepsilon}\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{n}_i \varepsilon^2 d\Omega = \mathcal{O}(\varepsilon) \to 0 \quad \text{and} \tag{14}$$

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} \phi(\boldsymbol{x})\nabla\mathcal{C}(r) \cdot \boldsymbol{n}_i dS = -\lim_{\varepsilon\to 0} \int_{S_\varepsilon} \phi(\boldsymbol{x})\frac{1}{\varepsilon^2}\varepsilon^2 d\Omega = -\phi(\boldsymbol{x}_0). \tag{15}$$

With the results (14) and (15), the Eq. (12) reduces to

$$\phi(\boldsymbol{x}_0) = -\int_S [\phi(\boldsymbol{x})\nabla\mathcal{C}(r) - \mathcal{C}(r)\nabla\phi(\boldsymbol{x})] \cdot \boldsymbol{n} dS. \tag{16}$$

By analogy with corresponding results in the theory of eletrostatics (Pozrikidis, 1992) and elastostatics (Jaswon and Symm, 1977), the two integrals on the RHS of Eq. (16) are termed the single-layer and double-layer potentials. They represent, respectively, a boundary distribution of the Green's functions $\mathcal{C}(r)$ and $\nabla\mathcal{C}(r)$, amounting to boundary distributions of magnetic point sources and magnetic point dipoles.

4.3 Integral representation in terms of jump conditions

At this point, we present two situations related with the position of the singularity when we have two different fluids separated by an interface S. Here, $\boldsymbol{n}$ is the unit outwardly directed normal to the drop surface and, again, $\boldsymbol{n} = -\boldsymbol{n}_i$.

Singularity inside the external fluid domain V: According to the reciprocal theorem in (11) for the internal fluid ϕ' (inside the particle) with the point $\boldsymbol{x}_0$ (singularity) exterior to the particle, we obtain

$$\int_S [\phi'(\boldsymbol{x})\nabla\mathcal{C}(r) - \mathcal{C}(r)\nabla\phi'(\boldsymbol{x})] \cdot \boldsymbol{n} dS = 0. \tag{17}$$

Now, applying Eq. (16) for the external fluid under an externally imposed $\phi^\infty(\boldsymbol{x}_0)$ and subtracting the Eq. (17) of it, one obtains in terms of the jump condition $\phi - \phi'$ and $\nabla(\phi - \phi')$ that

$$\phi(\boldsymbol{x}_0) = \phi^\infty(\boldsymbol{x}_0) - \int_S \mathcal{C}(r)\nabla[\phi(\boldsymbol{x}) - \phi'(\boldsymbol{x})] \cdot \boldsymbol{n} dS + \int_S [\phi(\boldsymbol{x}) - \phi'(\boldsymbol{x})]\nabla\mathcal{C}(r) \cdot \boldsymbol{n} dS. \tag{18}$$

Singularity inside the internal fluid domain V': By the same procedure used to obtain (18), we determine the integral representation for the internal fluid applying Eq. (16) as being

$$\phi'(\boldsymbol{x}_0) = -\int_S [\phi'(\boldsymbol{x})\nabla\mathcal{C}(r) - \mathcal{C}(r)\nabla\phi'(\boldsymbol{x})] \cdot \boldsymbol{n} dS. \tag{19}$$

In addition, using the reciprocal identity (11) for the external fluid ϕ (outside the particle) with a point $\boldsymbol{x}_0$ that is located in the interior of the particle and adding the result to Eq. (19) one obtains

$$\phi'(\boldsymbol{x}_0) = \phi^\infty(\boldsymbol{x}_0) - \int_S \mathcal{C}(r)\nabla[\phi(\boldsymbol{x}) - \phi'(\boldsymbol{x})] \cdot \boldsymbol{n} dS + \int_S [\phi(\boldsymbol{x}) - \phi'(\boldsymbol{x})]\nabla\mathcal{C}(r) \cdot \boldsymbol{n} dS. \tag{20}$$

4.4 Integral representation for the interface

Now, we are interested in the solution of the magnetic potencial field at the interface, that may be found by the application of the jump condition $(1/2)[\phi(\boldsymbol{x}_0) + \phi'(\boldsymbol{x}_0)]$ to the Eqs. (18) and (20). Limiting $\boldsymbol{x}_0$ to the interface, $\phi(\boldsymbol{x}_0) = \phi'(\boldsymbol{x}_0)$, $\phi'(\boldsymbol{x}) = \phi(\boldsymbol{x})$ and $\mu\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{n} = \mu'\nabla\phi'(\boldsymbol{x}) \cdot \boldsymbol{n}$. Therefore, the integral representation for the interface S between two magnetic materials is given by

$$\phi(\boldsymbol{x}_0) = \phi^\infty(\boldsymbol{x}_0) + \left(\frac{1-\alpha}{\alpha}\right) \int_S \mathcal{C}(r)\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{n} dS. \tag{21}$$

As before $\alpha = \mu'/\mu$. The Eq. (20) represents the solution for each point $\boldsymbol{x}_0$ at the interface as the summation of all disturbance flows induced by the other points over the surface, located at point $\boldsymbol{x}$.

5. HYDRODYNAMIC BOUNDARY INTEGRAL FORMULATION

In this section, a boundary integral formulation for computing the Stokes flow of a magnetic drop is derived by solving integral equations for functions that are evaluated over the boundaries. This formulation couples the integral equations for the velocity and magnetic potential fields.

5.1 Reciprocal theorem for the flow of a magnetic fluid

Consider a closed region of fluid V bounded by a surface S. Then consider two unrelated incompressible flows of two different magnetic fluids with densities ρ and ρ', viscosities η and η', magnetic permeabilities μ and μ' and stress fields $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$, respectively.

Flow 1: $\boldsymbol{u}$, $\boldsymbol{H}$, $\boldsymbol{\sigma}$ (ρ, η, μ). The balance equations for mass and momentum and the constitutive equation for a magnetic fluid are respectively

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\sigma} = -P\boldsymbol{I} + 2\eta\boldsymbol{D} + \mu\boldsymbol{H}\boldsymbol{H} \tag{22}$$

Here, locally, $\boldsymbol{u}$ is the Eulerian velocity, $\boldsymbol{\sigma}$ is the stress field and $\boldsymbol{H}$ is the magnetic field. $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{D} = (1/2)[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]$ is the rate of strain tensor. Again, $P = p_h + p_m$, where p_h is the hidrodinamic pressure and $p_m = (1/2)\mu_0 H^2$ is the magnetic pressure.

Flow 2: $\boldsymbol{u}'$, $\boldsymbol{H}'$, $\boldsymbol{\sigma}'$ (ρ', η', μ'). In the same sense shown in Eq. (22), the balance equations for mass and momentum and the constitutive equation for this flow are, respectively,

$$\nabla\cdot\boldsymbol{u}' = 0, \quad \nabla\cdot\boldsymbol{\sigma}' = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\sigma} = -P'\boldsymbol{I} + 2\eta'\boldsymbol{D}' + \mu'\boldsymbol{H}'\boldsymbol{H}' \tag{23}$$

where $\boldsymbol{D}' = (1/2)[\nabla\boldsymbol{u}' + (\nabla\boldsymbol{u}')^T]$ and P' are the rate of strain tensor and the pressure field, respectively. Furthermore, remind that the following tensorial operation for a incompressible fluid is valid, $\boldsymbol{I}:\boldsymbol{D} = \nabla\cdot\boldsymbol{u} = 0$ and $\boldsymbol{I}:\boldsymbol{D}' = \nabla\cdot\boldsymbol{u}' = 0$. Therefore, one may obtains

$$\boldsymbol{\sigma}:\boldsymbol{D}' = 2\eta\boldsymbol{D}:\boldsymbol{D}' + \mu\boldsymbol{H}\boldsymbol{H}:\boldsymbol{D}' \tag{24}$$

and, similarly,

$$\boldsymbol{\sigma}':\boldsymbol{D} = 2\eta'\boldsymbol{D}':\boldsymbol{D} + \mu'\boldsymbol{H}'\boldsymbol{H}':\boldsymbol{D}. \tag{25}$$

The simmetry of both $\boldsymbol{D}$ and $\boldsymbol{D}'$ requires that $\boldsymbol{D}:\boldsymbol{D}' = \boldsymbol{D}':\boldsymbol{D}$. Using this argument, the Eq. (24) becomes

$$\boldsymbol{D}:\boldsymbol{D}' = \boldsymbol{D}':\boldsymbol{D} = \frac{1}{2\eta}(\boldsymbol{\sigma}:\boldsymbol{D}' - \mu\boldsymbol{H}\boldsymbol{H}:\boldsymbol{D}'), \tag{26}$$

and, substituting the result (26) into Eq. (25), we obtain

$$\boldsymbol{\sigma}':\boldsymbol{D} = \frac{\eta'}{\eta}\boldsymbol{\sigma}:\boldsymbol{D}' - \mu\left(\frac{\eta'}{\eta}\boldsymbol{H}\boldsymbol{H}:\boldsymbol{D}' - \frac{\mu'}{\mu}\boldsymbol{H}'\boldsymbol{H}':\boldsymbol{D}\right) \tag{27}$$

It should be important to note that for superparamagnetic materials $\mu_0\boldsymbol{M}\times\boldsymbol{H} = \boldsymbol{0}$ (the magnetic torque). In this case, the stress tensor is symmetric. In this way, using Cauchy's equation given in Eq. (22), we may write

$$\boldsymbol{\sigma}:\boldsymbol{D}' = \boldsymbol{\sigma}:\nabla\boldsymbol{u}' = \nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{\sigma}) - \boldsymbol{u}'\cdot(\nabla\cdot\boldsymbol{\sigma}) = \nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{\sigma}). \tag{28}$$

Similarly, one may obtain that

$$\boldsymbol{\sigma}':\boldsymbol{D} = \nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{\sigma}'). \tag{29}$$

Thereafter, we can evaluate the term $\boldsymbol{H}\boldsymbol{H}:\boldsymbol{D}'$. Note that $\boldsymbol{H}\boldsymbol{H}$ is a second rank symmetric tensor. Accordingly

$$\boldsymbol{H}\boldsymbol{H}:\boldsymbol{D}' = \boldsymbol{H}\boldsymbol{H}:\nabla\boldsymbol{u}' = \nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{H}\boldsymbol{H}) - \boldsymbol{u}'\cdot\nabla\cdot(\boldsymbol{H}\boldsymbol{H}) \tag{30}$$

but, using a vectorial identity, the magnetostatic regime balance equations $\nabla\times\boldsymbol{H} = \boldsymbol{0}$ and $\nabla\cdot\boldsymbol{B} = 0$ and the assumption of a constant magnetic susceptibility $\nabla\cdot\boldsymbol{H} = 0$. In this way, one obtains that

$$\nabla\cdot(\boldsymbol{H}\boldsymbol{H}) = \boldsymbol{H}\cdot\nabla\boldsymbol{H} + \boldsymbol{H}(\nabla\cdot\boldsymbol{H}) = \nabla\left(\frac{H^2}{2}\right) + \boldsymbol{H}\times(\nabla\times\boldsymbol{H}) + \boldsymbol{H}(\nabla\cdot\boldsymbol{H}) = \nabla\left(\frac{H^2}{2}\right). \tag{31}$$

Then, substituting (31) into (30), one may obtain

$$\boldsymbol{H}\boldsymbol{H}:\boldsymbol{D}' = \nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{H}\boldsymbol{H}) - \boldsymbol{u}'\cdot\nabla\left(\frac{H^2}{2}\right). \tag{32}$$

If the same steps are applied to the term $\boldsymbol{H}'\boldsymbol{H}':\boldsymbol{D}$, it must reduces in an analogous fashion to

$$\boldsymbol{H}'\boldsymbol{H}':\boldsymbol{D} = \nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{H}'\boldsymbol{H}') - \boldsymbol{u}\cdot\nabla\left(\frac{H'^2}{2}\right). \tag{33}$$

Now, substituting the results (28), (29), (32) and (33) into Eq. (27) and, after we determine

$$\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{\sigma}') = \frac{\eta'}{\eta}\nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{\sigma}) - \mu\left\{\frac{\eta'}{\eta}\left[\nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{H}\boldsymbol{H}) - \boldsymbol{u}'\cdot\nabla\left(\frac{H^2}{2}\right)\right] - \frac{\mu'}{\mu}\left[\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{H}'\boldsymbol{H}') - \boldsymbol{u}\cdot\nabla\left(\frac{H'^2}{2}\right)\right]\right\}. \tag{34}$$

Finally, after making few algebraic manipulations, we obtain the expression for the generalized Lorentz reciprocal theorem for a Stokes flow of a magnetic fluid

$$\eta\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{\sigma}') - \eta'\nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{\sigma}) = \mu'\eta\left[\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{H}'\boldsymbol{H}') - \boldsymbol{u}\cdot\nabla\left(\frac{H'^2}{2}\right)\right] - \mu\eta'\left[\nabla\cdot(\boldsymbol{u}'\cdot\boldsymbol{H}\boldsymbol{H}) - \boldsymbol{u}'\cdot\nabla\left(\frac{H^2}{2}\right)\right]. \tag{35}$$

5.2 Integral representation for a Stokes flow of a magnetic fluid

Consider the particular flow of interest with velocity $\boldsymbol{u}$, magnetic field $\boldsymbol{H}$ and stress tensor $\boldsymbol{\sigma}$. The known flow is the one due to a point force with strength $\boldsymbol{f}$, and located at a point $\boldsymbol{x}_0$. Suppose that the inertia of both fluids has a negligible influence on the motion of the fluid elements, and by convenience takes $\eta = \eta'$, $\rho = \rho'$ and $\mu' = 0$. Flow 1 and flow 2 for this particular situation are described as following.

Flow 1: $\boldsymbol{u}$, $\boldsymbol{H}$, $\boldsymbol{\sigma}$. The equations for conservation of mass and momentum for the flow 1 and for the constitutive equation are respectively

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0}, \quad \text{and} \quad \boldsymbol{\sigma} = -P\boldsymbol{I} + 2\eta\boldsymbol{D} + \mu\boldsymbol{H}\boldsymbol{H}. \tag{36}$$

Flow 2: $\boldsymbol{u}'$, $\boldsymbol{\sigma}'$. The fundamental solution for Stokes equations correspond to the velocity and stress fields at a point $\boldsymbol{x}$ produced by a point force $\boldsymbol{f}$ located at $\boldsymbol{x}_0$

$$\nabla \cdot \boldsymbol{u}' = 0, \quad \nabla \cdot \boldsymbol{\sigma}' = -\boldsymbol{f}\delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{37}$$

with $|\boldsymbol{u}'| \to 0$ and $|\boldsymbol{\sigma}'| \to \infty$ as $|\boldsymbol{x}| \to \infty$. The solution of such equations may be derived using Fourier transforms

$$\boldsymbol{u}'(\boldsymbol{x}) = \frac{\boldsymbol{f}}{8\pi\eta} \cdot \boldsymbol{\mathcal{G}}(\boldsymbol{r}); \quad \boldsymbol{\sigma}'(\boldsymbol{x}) = -\frac{3\boldsymbol{f}}{4\pi} \cdot \boldsymbol{\mathcal{T}}(\boldsymbol{r}), \quad \text{where} \quad \boldsymbol{\mathcal{G}}(\boldsymbol{r}) = \frac{\boldsymbol{I}}{r} + \frac{\boldsymbol{r}\boldsymbol{r}}{r^3} \quad \text{and} \quad \boldsymbol{\mathcal{T}}(\boldsymbol{r}) = \frac{\boldsymbol{r}\boldsymbol{r}\boldsymbol{r}}{r^5} \tag{38}$$

are the stokeslet $\boldsymbol{\mathcal{G}}$ and the stresslet $\boldsymbol{\mathcal{T}}$. The above functions are the kernels or the free-space Green's functions that maps the force $\boldsymbol{f}$ at $\boldsymbol{x}_0$ to the fields at $\boldsymbol{x}$ in an unbounded three-dimensional domain. Here $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$, and $r = |\boldsymbol{r}|$. Physically, $\boldsymbol{u} = \boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \boldsymbol{f}$ expresses the velocity field due to a concentrated point force $\boldsymbol{f}\delta(\boldsymbol{r})$ placed at the point $\boldsymbol{x}_0$, and may be seen as the flow produced by the slow settling motion of a small particle. $\mathcal{T}_{ijk}$ is the stress tensor associated with the Green's function G_{ij} and $\sigma_{ik}(\boldsymbol{x}) = \mathcal{T}_{ijk} f_j$ is a fundamental solution of the Stokes produced by the hydrodynamic dipole $\boldsymbol{D} \cdot \nabla\delta(\boldsymbol{r})$. In addition, $\mathcal{T}_{ijk} = \mathcal{T}_{kji}$ as required by symmetry of the stress tensor $\boldsymbol{\sigma}$. Finally, substituting the expressions of the point-force solution (38) into (35) and discarding the arbitrary constant $\boldsymbol{f}$ ones obtain

$$-\frac{3}{4\pi}\nabla \cdot [\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\mathcal{T}}(\boldsymbol{r})] - \frac{1}{8\pi\eta}\nabla \cdot [\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \boldsymbol{\sigma}(\boldsymbol{x})] = -\frac{\mu}{8\pi\eta}\left\{\nabla \cdot [\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \boldsymbol{H}\boldsymbol{H}(\boldsymbol{x})] - \boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \nabla\left(\frac{H^2(\boldsymbol{x})}{2}\right)\right\}. \tag{39}$$

Now using for the second term on the RHS of Eq. (39) the incompressibility condition of the singular solution $\nabla \cdot \boldsymbol{\mathcal{G}} = \boldsymbol{0}$ and the symmetry of $\boldsymbol{\mathcal{G}}$ tensor, so that $\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \nabla(H^2/2) = \nabla \cdot [\boldsymbol{\mathcal{G}}(\boldsymbol{r})(H^2/2)]$, Eq. (39) becomes

$$-\frac{3}{4\pi}\nabla \cdot [\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\mathcal{T}}(\boldsymbol{r})] - \frac{1}{8\pi\eta}\nabla \cdot [\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \boldsymbol{\sigma}(\boldsymbol{x})] = -\frac{\mu}{8\pi\eta}\nabla \cdot \left\{\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\right\}. \tag{40}$$

The above equation is valid everywhere except at the singular point $\boldsymbol{x}_0$. Now, consider a material volume of fluid V bounded by a singly or multiply connected surface S in order to evaluate the integration of Eq. (40). The surface S may be composed of fluid surfaces, fluid interfaces or solid surfaces. There are two situations to be considered next.

Singularity outside V: In this case, we select a point $\boldsymbol{x}_0$ outside V. Then, all terms of the reciprocal theorem are regular throughout V, and thus after integration of Eq. (40) the integral representation of the reciprocal theorem takes the form

$$\begin{aligned}-\frac{3}{4\pi}\int_V \nabla \cdot [\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\mathcal{T}}(\boldsymbol{r})]dV - \frac{1}{8\pi\eta}\int_V \nabla \cdot [\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \boldsymbol{\sigma}(\boldsymbol{x})]dV = \\ -\frac{\mu}{8\pi\eta}\int_V \nabla \cdot \left\{\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\right\}dV.\end{aligned} \tag{41}$$

Besides, the volume integrals in Eq. (41) are converted to the surface integrals over S, by using the divergence theorem

$$\begin{aligned}-\frac{1}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})dS - \frac{3}{4\pi}\int_S \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\mathcal{T}}(\boldsymbol{r}) \cdot \boldsymbol{n}(\boldsymbol{x})dS + \\ \frac{\mu}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r}) \cdot \left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right] \cdot \boldsymbol{n}(\boldsymbol{x})dS = \boldsymbol{0},\end{aligned} \tag{42}$$

where $\boldsymbol{n}$ is the unit outward normal to the surface S. Eq. (42) is the integral representation of the flow if the singularity is outside V. It will be shown that the integral equation (42) is a useful identity for developing new integral equations in terms of jump conditions on an interface.

Singularity inside V: Similar to the analysis developed in Sec. §4.2 , if exists a singularity located at $\boldsymbol{x}_0$ into V, it is needed to be excluded of our integration step. In order to make this integration, we define a small spherical volume V_ε of radius ε centered at $\boldsymbol{x}_0$, as shown in Fig. (2). In addition, the functions into Eq. (9) are regular throughout the reduced volume $V - V_\varepsilon$. Then, integrating the Eq. (9) over $V - V_\varepsilon$ and converting the volume integral into a surface integral using the divergence theorem, gives

$$
\begin{aligned}
&-\frac{1}{8\pi\eta}\int_{S,S_\varepsilon} \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\boldsymbol{\sigma}(\boldsymbol{x})\cdot\boldsymbol{n}_i(\boldsymbol{x})dS - \frac{3}{4\pi}\int_{S,S_\varepsilon} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}_i(\boldsymbol{x})dS + \\
&\qquad \frac{\mu}{8\pi\eta}\int_{S,S_\varepsilon} \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\cdot\boldsymbol{n}_i(\boldsymbol{x})dS = \boldsymbol{0},
\end{aligned}
\tag{43}
$$

where S_ε is the spherical surface enclosing V_ε, as indicated in Fig. (2). Letting the radius ε tends to zero we obtain the following expressions for the leading order terms in ε for the tensors $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{T}}$, namely

$$
\boldsymbol{\mathcal{G}}(\boldsymbol{r}) \approx \frac{\boldsymbol{I}}{\varepsilon} + \frac{\boldsymbol{r}\boldsymbol{r}}{\varepsilon^3}; \quad \boldsymbol{\mathcal{T}}(\boldsymbol{r}) \approx \frac{\boldsymbol{r}\boldsymbol{r}\boldsymbol{r}}{\varepsilon^5}. \tag{44}
$$

Over S_ε, $\boldsymbol{n}_i = \boldsymbol{r}/\varepsilon$ and $dS = \varepsilon^2 d\Omega$, where, as defined before, Ω is the differential solid angle. Substituting these expressions along with Eq. (43) and taking the limit $\varepsilon \to 0$ we obtain

$$
\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \boldsymbol{\mathcal{G}}(\varepsilon)\cdot\boldsymbol{\sigma}(\boldsymbol{x})\cdot\boldsymbol{n}_i(\boldsymbol{x})dS = \lim_{\varepsilon\to 0}\int_{S_\varepsilon}\left(\frac{\boldsymbol{I}}{\varepsilon} + \frac{\boldsymbol{r}\boldsymbol{r}}{\varepsilon^3}\right)\cdot\boldsymbol{\sigma}(\boldsymbol{x})\cdot\boldsymbol{n}_i(\boldsymbol{x})\varepsilon^2 d\Omega = \boldsymbol{\mathcal{O}}(\varepsilon) \to 0. \tag{45}
$$

As $\varepsilon \to 0$, the values of $\boldsymbol{u}$, $\boldsymbol{H}$ and $\boldsymbol{\sigma}$ tend to their corresponding values at the center of V_ε, i.e. to $\boldsymbol{u}(\boldsymbol{x}_0)$, $\boldsymbol{H}(\boldsymbol{x}_0)$ and $\boldsymbol{\sigma}(\boldsymbol{x}_0)$, respectively. By analogy the following term tends to zero in the limit $\varepsilon \to 0$ decreasing linearly in ε

$$
\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\cdot\boldsymbol{n}_i(\boldsymbol{x})\varepsilon^2 d\Omega = \boldsymbol{\mathcal{O}}(\varepsilon) \to 0. \tag{46}
$$

Also, the contribution of the velocity can be evaluated. Thus, in the limit $\varepsilon \to 0$, we have

$$
\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}_i(\boldsymbol{x})dS = \lim_{\varepsilon\to 0}\int_{S_\varepsilon} \boldsymbol{u}(\boldsymbol{x})\cdot\left(\frac{\boldsymbol{r}\boldsymbol{r}\boldsymbol{r}}{\varepsilon^5}\right)\cdot\boldsymbol{n}_i(\boldsymbol{x})dS = \frac{\boldsymbol{u}(\boldsymbol{x}_0)}{\varepsilon^4}\cdot\int_{S_\varepsilon}\boldsymbol{r}\boldsymbol{r}dS = \frac{4\pi}{3}\boldsymbol{u}(\boldsymbol{x}_0). \tag{47}
$$

Then, substituting the results (45-47) into Eq. (43), it gives

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}_0) = &\frac{1}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\boldsymbol{\sigma}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})dS + \frac{3}{4\pi}\int_S \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS - \\
&\frac{\mu}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\cdot\boldsymbol{n}(\boldsymbol{x})dS.
\end{aligned}
\tag{48}
$$

Eq. (48) is the integral representation for the Stokes flow of a magnetic fluid in terms of boundary distributions involving the Green's functions $\boldsymbol{\mathcal{G}}$ and the stresslet $\boldsymbol{\mathcal{T}}$. The first distribution on the RHS of (48) is termed the single-layer potential, the second distribution is termed the double-layer potential. Both integrals have already appeared in three-dimensional boundary integral formulations of non-magnetic fluids. The last integral however represents an extra single-layer potential contribution by the fact that the fluid is polar.

5.3 Integral representation in terms of the traction jump

Singularity inside the external fluid domain V: Using the reciprocal identity (42) for the internal flow $\boldsymbol{u}'$ (inside the particle) with the point $\boldsymbol{x}_0$ located exterior to the particle, one obtain

$$
\begin{aligned}
&-\frac{1}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\boldsymbol{\sigma}'(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})dS - \frac{3\lambda}{4\pi}\int_S \boldsymbol{u}'(\boldsymbol{x})\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS + \\
&\qquad \frac{\alpha\mu}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left[\boldsymbol{H}'\boldsymbol{H}'(\boldsymbol{x}) - \left(\frac{H'^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\cdot\boldsymbol{n}(\boldsymbol{x})dS = \boldsymbol{0},
\end{aligned}
\tag{49}
$$

where, as defined before, $\lambda = \eta'/\eta$ and $\alpha = \mu'/\mu$. Now, applying Eq. (48) for the external flow subject to an ambient flow $\boldsymbol{u}^\infty(\boldsymbol{x}_0)$, and combining the result with Eq. (49), the integral representation is obtained as a function of the traction jump $\Delta\boldsymbol{t}(\boldsymbol{x}) = [\boldsymbol{\sigma}(\boldsymbol{x}) - \boldsymbol{\sigma}'(\boldsymbol{x})]\cdot\boldsymbol{n}(\boldsymbol{x})$,

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}_0) \;=\; &\boldsymbol{u}^\infty(\boldsymbol{x}_0) - \frac{1}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\Delta\boldsymbol{t}(\boldsymbol{x})dS - \frac{3}{4\pi}\int_S [\boldsymbol{u}(\boldsymbol{x}) - \lambda\boldsymbol{u}'(\boldsymbol{x})]\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS + \\
&\frac{\mu}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left\{\left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right] - \alpha\left[\boldsymbol{H}'\boldsymbol{H}'(\boldsymbol{x}) - \left(\frac{H'^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\right\}\cdot\boldsymbol{n}(\boldsymbol{x})dS.
\end{aligned}
\tag{50}
$$

Singularity inside the internal fluid domain V': We repeat the above procedure for the internal flow. Hence, the integral representation of the internal flow is obtained when Eq. (48) is applied,

$$\boldsymbol{u}'(\boldsymbol{x}_0) = \frac{1}{8\pi\lambda\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\boldsymbol{\sigma}'(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})dS + \frac{3}{4\pi}\int_S \boldsymbol{u}'(\boldsymbol{x})\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS - \frac{\alpha\mu}{8\pi\lambda\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left[\boldsymbol{H}'\boldsymbol{H}'(\boldsymbol{x}) - \left(\frac{H'^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\cdot\boldsymbol{n}(\boldsymbol{x})dS. \quad (51)$$

Again, using the reciprocal identity (40) for the external flow $\boldsymbol{u}$ (outside the particle) with the singularity $\boldsymbol{x}_0$ located in the interior of the particle and combining its result with (51) results

$$\lambda\boldsymbol{u}'(\boldsymbol{x}_0) = \boldsymbol{u}^\infty(\boldsymbol{x}_0) - \frac{1}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\Delta\boldsymbol{t}(\boldsymbol{x})dS - \frac{3}{4\pi}\int_S [\boldsymbol{u}(\boldsymbol{x}) - \lambda\boldsymbol{u}'(\boldsymbol{x})]\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS + \frac{\mu}{8\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left\{\left[\boldsymbol{H}\boldsymbol{H}(\boldsymbol{x}) - \left(\frac{H^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right] - \alpha\left[\boldsymbol{H}'\boldsymbol{H}'(\boldsymbol{x}) - \left(\frac{H'^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\right\}\cdot\boldsymbol{n}(\boldsymbol{x})dS. \quad (52)$$

5.4 Integral representation for the interface

The integral representation for the flow solution at the interface is found by applying the jump condition $(1/2)[\boldsymbol{u}(\boldsymbol{x}_0) + \lambda\boldsymbol{u}'(\boldsymbol{x}_0)]$ to the Eqs. (50) and (52). For the limit of $\boldsymbol{x}_0$ going to the interface, $\boldsymbol{u}(\boldsymbol{x}_0) = \boldsymbol{u}'(\boldsymbol{x}_0)$ (continuity of velocity), $\boldsymbol{H}_t = \boldsymbol{H}'_t$ (continuity of tangential component of magnetic field), $\mu\boldsymbol{H}_n = \mu'\boldsymbol{H}'_n$ (continuity of normal components of magnetic induction) and the traction discontinuity $\Delta\boldsymbol{t}$ is given by the Eq. (6). Under these conditions only the integral representation for the fluid-fluid interface S need to be considered, hence

$$(1+\lambda)\boldsymbol{u}'(\boldsymbol{x}_0) = 2\boldsymbol{u}^\infty(\boldsymbol{x}_0) - \frac{1}{4\pi\eta}\int_S \Gamma(\nabla^s\cdot\boldsymbol{n})\boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS - \frac{3}{2\pi}(1-\lambda)\int_S \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{\mathcal{T}}(\boldsymbol{r})\cdot\boldsymbol{n}(\boldsymbol{x})dS + \frac{\mu(1-\alpha)}{4\pi\eta}\int_S \boldsymbol{\mathcal{G}}(\boldsymbol{r})\cdot\left\{\left[\boldsymbol{H}_t\boldsymbol{H}_t(\boldsymbol{x}) - \left(\frac{H_t^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right] - \frac{1}{\alpha}\left[\boldsymbol{H}_n\boldsymbol{H}_n(\boldsymbol{x}) - \left(\frac{H_n^2(\boldsymbol{x})}{2}\right)\boldsymbol{I}\right]\right\}\cdot\boldsymbol{n}(\boldsymbol{x})dS. \quad (53)$$

where the vector field $\boldsymbol{H}_n = (\boldsymbol{H}\cdot\boldsymbol{n})\boldsymbol{n}$ and $\boldsymbol{H}_t = \boldsymbol{H}\cdot(\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n})$.

5.5 Dimensionless integral representation

All quantities above are made dimensionless using the undisturbed drop size a, the relaxation rate $\Gamma/\mu a$ and a characteristic magnetic field H_0. The following dimensionless quantities $\tilde{\boldsymbol{\mathcal{G}}}(\tilde{\boldsymbol{r}}) = a\boldsymbol{\mathcal{G}}(\boldsymbol{r})$, $\tilde{\boldsymbol{u}} = (\eta/\Gamma)\boldsymbol{u}$, $\tilde{\boldsymbol{\mathcal{T}}}(\tilde{\boldsymbol{r}}) = a^2\boldsymbol{\mathcal{T}}(\boldsymbol{r})$ and $\tilde{\boldsymbol{H}} = (\boldsymbol{H}/H_0)$. In this manner, we can make dimensionless the Eq. (53). Then, one may obtain

$$\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}_0) = \frac{2\tilde{\boldsymbol{u}}^\infty(\tilde{\boldsymbol{x}}_0)}{(1+\lambda)} - \frac{1}{4\pi(1+\lambda)}\int_S (\tilde{\nabla}^s\cdot\boldsymbol{n})\tilde{\boldsymbol{\mathcal{G}}}(\tilde{\boldsymbol{r}})\cdot\boldsymbol{n}(\tilde{\boldsymbol{x}})d\tilde{S} - \frac{3}{2\pi}\frac{(1-\lambda)}{(1+\lambda)}\int_S \tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}})\cdot\tilde{\boldsymbol{\mathcal{T}}}(\tilde{\boldsymbol{r}})\cdot\boldsymbol{n}(\tilde{\boldsymbol{x}})d\tilde{S} + \quad (54)$$

$$\frac{Ca_m(1-\alpha)}{4\pi(1+\lambda)}\int_S \tilde{\boldsymbol{\mathcal{G}}}(\tilde{\boldsymbol{r}})\cdot\left\{\left[\tilde{\boldsymbol{H}}_t\tilde{\boldsymbol{H}}_t(\tilde{\boldsymbol{x}}) - \left(\frac{\tilde{H}_t^2(\tilde{\boldsymbol{x}})}{2}\right)\boldsymbol{I}\right] - \frac{1}{\alpha}\left[\tilde{\boldsymbol{H}}_n\tilde{\boldsymbol{H}}_n(\tilde{\boldsymbol{x}}) - \left(\frac{\tilde{H}_n^2(\tilde{\boldsymbol{x}})}{2}\right)\boldsymbol{I}\right]\right\}\cdot\boldsymbol{n}(\tilde{\boldsymbol{x}})d\tilde{S}. \quad (55)$$

For linear shearing motions, $\tilde{\boldsymbol{u}}^\infty(\tilde{\boldsymbol{x}}_0) = Ca(\tilde{\boldsymbol{E}}^\infty + \tilde{\boldsymbol{W}}^\infty)\cdot\tilde{\boldsymbol{x}}$, with $Ca = \dot{\gamma}\eta a/\Gamma$ being the capillary number, that represents the ratio of viscous to surface tension stress. In addition, $Ca_m = \mu H_0^2 a/\Gamma$ is the magnetic capillary number, that represents the ratio of magnetic stresses to surface tension stress. Note that our magnetic capillary number is defined as a function of the magnetic permeability of the drop fluid.

6. Final Remarks

Equations (21) and (55) are considered the key results of the coupled magnetic-hydrodynamic boundary integral formulation presented here. The analysis described in this paper will be used in a future work to investigate by numerical simulation the full time-dependent low Reynolds number problem for three-dimensional ferrofluid droplet deformation under the action of shearing motion and magnetic fields, and thereby infer some key properties of flowing magnetic emulsions, when the viscosity ratio and the magnetic permeability ratio of the two phases are not necessarily $O(1)$. The hydrodynamic integral representation coupled with the magnetic potential integral will determine the drop shape evolution. The mathematical formulation developed here may be extended in a straightforward manner to the problem with multiple polydisperse drops, for the case of general shear flows in the presence of magnetic field where no experimental studies of drop shape evolution are at present available.

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