OPTIMIZATION OF A NONLINEAR DYNAMIC VIBRATION ABSORBER

Romes Antonio Borges, rborges@mecanica.ufu.br
Valder Steffen, Jr, vsteffen@mecanica.ufu.br
Federal University of Uberlândia, School of Mechanical Engineering
P. O. Box 593 – ZIP 38400-902 – Uberlândia – MG – Brazil

Abstract. *Dynamic vibration absorbers (DVAs) are mechanical devices used normally to attenuate the vibration level in different types of structures and machines. They were developed in the beginning of the last century and have been used in a number of applications in engineering, such as in ships, in power lines, in aeronautic structures, in civil engineering constructions subjected to seismic induced excitations, etc. In this work, a damped nonlinear dynamic vibration absorber will be studied. The nonlinear effect is introduced in the system by nonlinear springs. Then, the main propose is to verify the nonlinear effects, intended to increase the efficiency of the DVA into the frequency band of interest. The first part presents the equation of motion of the nonlinear DVA. Next, the response functions of the system are obtained and the optimal operation conditions of the nonlinear DVA are calculated by using a global heuristic optimization procedure (genetic algorithms). Finally, some numerical examples are presented to evaluate the performance of the optimal nonlinear DVA.*

Keywords*: dynamic vibration absorber, optimization, suppression band, nonlinear spring, nonlinear vibrations.*

1. INTRODUCTION

Dynamic vibrations absorbers (DVAs) have been used to attenuate vibrations in several types of structures and machines. They were first developed in the beginning of last century by Frahm (1911) and have been used extensively in many applications in the mechanical, civil and aeronautical engineering. Practical applications of these devices can be seen in ships, power lines, aircrafts and helicopters, buildings and towers, etc. A comprehensive study on the theory and practice of DVA's is given by Koronev & Reznikov (1993). More recently, Cunha Jr (1999) studied some complex configurations of DVAs such as those dedicated to multi-degree-of-freedom and distributed parameter systems. Rade & Steffen (1999) studied the optimization of DVA's parameter over a frequency band using a substructure coupling technique.

Pai & Schulz (1998) performed a theoretical study on how to use saturations phenomena to design nonlinear vibration absorbers and how to improve their stability and effective frequency bandwidth, leading to a refined nonlinear vibration absorber. Rice & McCraith (1987) used optimization techniques for designing a nonlinear DVA with an asymmetric nonlinear Duffing-type element incorporated for narrow-band absorption applications.

Today, new engineering applications demand the structures to be lighter and more flexible. Besides, design constraints make the vibration control problem an important design issue.

In this work a study regarding a damped nonlinear dynamic vibration absorber will be performed, for which the springs of the DVA have nonlinear characteristics. With this aim, the contribution of the nonlinearity to the improvement of the efficiency of vibration attenuation provided by the absorber will be analyzed.

An optimization problem is solved by using genetic algorithms (GAs) (Goldberg, 1989) to determine the optimal DVA's parameters such as the band of frequency for which the DVA is the most effective in absorbing vibrations.

In the remainder, the equations of motion are first presented, followed by the definition of the performance indexes to be optimized. Then, numerical examples are shown to illustrate the main features of the proposed methodology.

2. DYNAMIC MODEL

Consider the vibratory system represented by the two degree-of-freedom model shown in Fig. 1. This device consists of a damped primary system attached to the ground by a suspension that includes either a linear or nonlinear spring and a damped secondary mass coupled to the primary system by a spring with nonlinear characteristics (Nissen *et al.*,1985), (Natsiavas, 1992).

The oscillations are imposed to the primary system through a harmonic disturbance given by eq. (1). The coordinate x_1 represents de displacement of the primary system with respect to the ground and the coordinate x_2 is the displacement of the DVA's mass with respect to the primary system.

$$F_1(t) = p\cos(\omega t) \qquad (1)$$

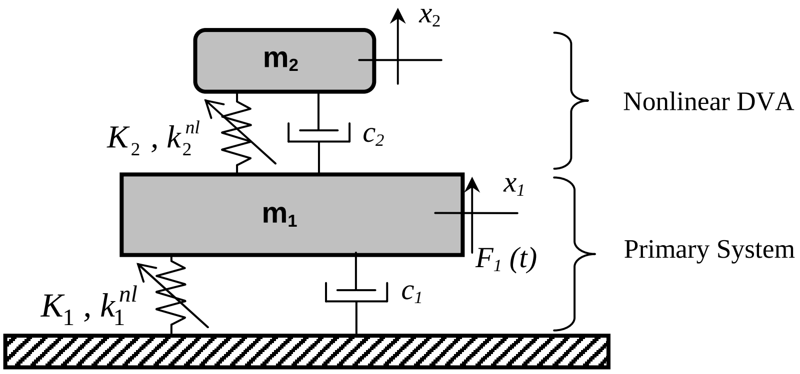

Figure 1. Two degree-of-freedom mechanical system (the primary system and the nonlinear DVA

In the model above, the dampers are linear but the springs have nonlinear characteristics. The constitutive forces of the springs are given by:

$$r_i(x_i) = k_i x_i + k_i^{nl} x_i^3, \qquad i = 1,2. \tag{2}$$

The displacements are normalized with respect to the length of vector x_c , then:

$$y_i = \frac{x_i}{x_c} \tag{3}$$

The time is given by:

$$\tau = \omega t \tag{4}$$

Now, the following parameters will be introduced:

$$\overline{\omega}_i^2 = \frac{k_i}{m_i}, \quad \omega_i = \frac{\omega_i}{\omega}, \quad \zeta_i = \frac{c_i}{2\sqrt{k_i m_i}}, \quad \delta_i = 2\zeta_i \omega_i, \quad \mu = \frac{m_2}{m_1}, \quad \eta_i = \omega_i^2,$$
$$\varepsilon_i = \frac{k_i^{nl} x_c^2}{m_i \omega^2}, \quad \rho = \frac{\omega_2}{\omega_1}, \quad P = \frac{p}{m_1 \overline{\omega}^2 x_c}, \quad \beta = \frac{P}{\eta_1}, \ \Omega = \frac{\omega}{\overline{\omega}_1}. \tag{5}$$

Using the second Newton's law and after some algebraic manipulations, the following matrix equations of motion are obtained for the system:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f} \tag{6}$$

where **M**, **C** and **K** are the mass, damping and stiffness matrices, respectively.

$$\mathbf{M} = \begin{bmatrix} 1+\mu & \mu \\ \mu & \mu \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \mu\delta_2 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} \eta_1 & 0 \\ 0 & \mu\eta_2 \end{bmatrix} \tag{7}$$

The displacement and force vectors are given by:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}; \qquad f = \begin{bmatrix} P\cos\tau - \varepsilon_1 y_1^3 \\ -\mu\varepsilon_2 y_2^3 \end{bmatrix} \tag{8}$$

2.1. Steady-State Response of the System

A number of perturbation methods are based on averaging. For this aim the unknown functions of the problem are now considered dependent variables, by making a shift of variables from the original dependent variable (Thonsem, 2003). These methods encompass techniques such as the following: Krylov-Bogoliubov method, Krylov-Bogoliubov-Mitropolsky method, the method of the generalized average (Nayfeh, 2000).

In the present work, the Krylov-Bogoliubov method will be used to integrate the equation of motion, namely Eq. (6). This method leads to an approximate solution of nonlinear differential equations. The procedure can be described as follows:

$$\mathbf{y}(\tau) = \mathbf{u}(\tau)\cos\tau + \mathbf{v}(\tau)\mathrm{sen}\,\tau \tag{9}$$

where the time dependence of $\mathbf{u} = (\mathrm{u}_1\ \mathrm{u}_2)^{\mathrm{T}}$ and $\mathbf{v} = (\mathrm{v}_1\ \mathrm{v}_2)^{\mathrm{T}}$ is assumed to be small for high order terms, such as the vectors **u** and **v**.

When the transformation of variables is made, an additional and independent equation, Eq. (9), will be necessary to guarantee that the transformation is unique. For this additional equation the velocity is similar to that of the linear case and is written as:

$$\dot{\mathbf{y}}(\tau) = -\mathbf{u}(\tau)\mathrm{sen}\,\tau + \mathbf{v}(\tau)\cos\tau \tag{10}$$

The transformation of variables given by Eq. (9) and Eq. (10) is known as the *Van der Pol Transformation* (Thomsem, 2003) and (Hagedorn, 1977). The differentiation of Eq. (9) was performed with respect to τ .

$$\dot{\mathbf{y}}(\tau) = -\dot{u}(\tau)\cos\tau - u(\tau)\mathrm{sen}\,\tau + \dot{v}(\tau)\mathrm{sen}\,\tau + v(\tau)\cos\tau \tag{11}$$

Substituting Eq. (10) into Eq. (11) results:

$$\dot{u}(\tau)\cos\tau + \dot{v}(\tau)\mathrm{sen}\,\tau = 0 \tag{12}$$

By differentiating Eq. (10) with respect to τ gives:

$$\ddot{\mathbf{y}}(\tau) = -\dot{u}(\tau)\mathrm{sen}\,\tau - u(\tau)\cos\tau + \dot{v}(\tau)\cos\tau + v(\tau)\mathrm{sen}\,\tau \tag{13}$$

Substituting equations (9), (11) and (12) into the equation of motion, Eq. (6), the following equation is obtained:

$$(\mathbf{M\dot{v}} - \mathbf{Mu} + \mathbf{Cv} + \mathbf{Ku})\cos\tau - (\mathbf{M\dot{u}} - \mathbf{Mv} + \mathbf{Cu} + \mathbf{Kv})\mathrm{sen}\,\tau = \mathbf{f(u, v, \tau)} \tag{14}$$

Next, Eq. (12) is multiplied by $(\mathbf{M}\cos\tau)$ and Eq. (14) is multiplied by $(-\mathbf{M}\mathrm{sen}\tau)$. Then, these equations must be added up. The resulting equation is then integrated over the period (0 to 2π). It is worth mentioning that u and v are taken as constants in this period (it represents a very short time interval) (Natsiavas, 1992). After some algebraic manipulations one obtains:

$$\mathbf{M}\dot{u} = \frac{1}{2}(\mathbf{K} - \mathbf{M})\mathbf{v} - \begin{pmatrix} \frac{1}{2}\delta_1 u_1 - \frac{3}{8}\left(v_1^3\varepsilon_1 - u_1^2 v_1\varepsilon_1\right) \\ \frac{1}{2}\mu u_2\delta_2 - \frac{3}{8}\left(\mu v_2^3\varepsilon_2 - \mu u_2^2 v_2\varepsilon_2\right) \end{pmatrix} \tag{15}$$

Similarly, equations (12) and (14) must be multiplied by $(\mathbf{M}\mathrm{sen}\tau)$ and $(\mathbf{M}\cos\tau)$, respectively, and then integrated over the period (0 to 2π). Then:

$$\mathbf{M}\dot{u} = \frac{1}{2}(\mathbf{K} - \mathbf{M})\mathbf{v} - \begin{pmatrix} -\frac{1}{2}P + \frac{1}{2}\delta_1 v_1 - \frac{3}{8}\left(u_1^3\varepsilon_1 - v_1^2 u_1\varepsilon_1\right) \\ \frac{1}{2}\mu v_2\delta_2 - \frac{3}{8}\left(\mu u_2^3\varepsilon_2 - \mu v_2^2 u_2\varepsilon_2\right) \end{pmatrix} \tag{16}$$

Equations (15) and (16) represent a first order ordinary differential equation system with four variables. The solution originated from the averaging method corresponds to motions of period 2π for the original system given by Eq. (6). In the case of steady-state periodic vibrations, the following condition can be used:

$$\dot{\mathbf{u}} = \dot{\mathbf{v}} = \mathbf{0} \tag{17}$$

By substituting Eq. (17) into equations (15) and (16), a nonlinear algebraic system with four equations and four variables u_1, u_2, v_1, v_2 is obtained:

$$\left(1+\mu-\omega_1^2\right)u_1+\mu u_2-2\zeta_1\omega_1 v_1-\frac{3\varepsilon_1\left(u_1^2+v_1^2\right)u_1}{4}+\beta\omega_1^2=0$$
$$\mu u_1+\left(\mu-\mu\rho^2\omega_1^2\right)u_2-\mu\left(2\zeta_2\rho\omega_1^2 v_2+\frac{3\varepsilon_2\left(u_2^2+v_2^2\right)u_2}{4}\right)=0$$
$$\left(\omega_1^2-1-\mu\right)v_1-\mu v_2-2\zeta_1\omega_1 u_1+\frac{3\varepsilon_1\left(u_1^2+v_1^2\right)v_1}{4}=0$$
$$-\mu v_1+\left(\mu\rho^2\omega_1^2-\mu\right)v_2-\mu\left(2\zeta_2\rho\omega_1^2 u_2-\frac{3\varepsilon_2\left(u_2^2+v_2^2\right)v_2}{4}\right)=0 \tag{18}$$

2.2 Simulation Results

The system represented by Eq. (18) is numerically solved by using the function "fsolve" from MATLAB® toolbox (Coleman, 1996). Then, the values of u_1,u_2,v_1,v_2 can be calculated and the vibration amplitude of the primary and secondary masses of the nonlinear DVA obtained. The amplitude values are given by r_1 and r_2 in Eq. (19), respectively:

$$r_i=\sqrt{u_i^2+v_i^2},\quad i=1,2. \tag{19}$$

With the intention of graphically observing and analyzing the results previously obtained some case-studies involving different situations of interest will be studied, particularly with respect to parameters that introduce some type of nonlinearity to the system.

2.2.1. Nonlinear DVA

Initially, for illustrating the effect of the non-linear coefficient associated to the stiffness of the absorber, the frequency response of the system for different values of the nonlinear coefficient $(\varepsilon_1 \text{ and } \varepsilon_2)$ is shown in Fig. 2.

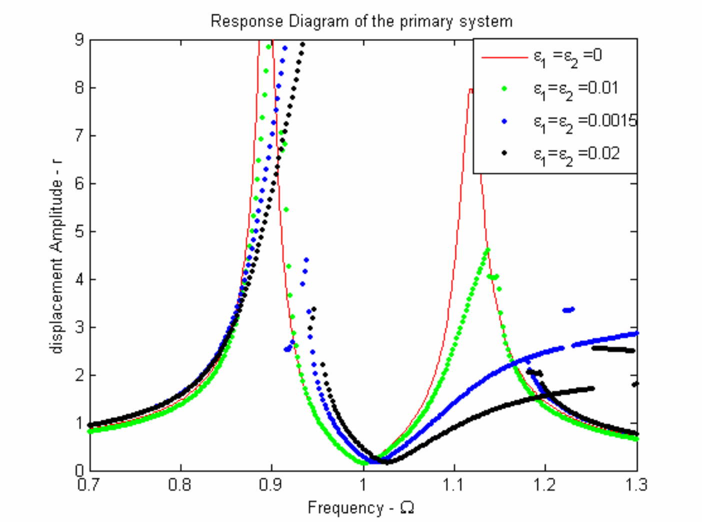

Figure 2 – Effect of ε_1 and ε_2 on the system response (other parameters $\beta=0.1;\ \zeta_1=\zeta_2=0.01;\ \mu=0.05;\ \rho=1.$)

In the following cases, the primary mass and the absorber mass are assembled through *hardening springs*. In Fig. 3, the steady-state frequency response diagram of the main mass amplitude versus the normalized frequency Ω is shown. It is considered in this case that the spring that connects the main mass to the absorber mass (k_2) has non-linear characteristics. For comparison purposes, Fig. 3 presents the case for which spring (k_1) exhibits linear behavior. The absorber was then assembled in conjunction with a spring with non-linear characteristics $(\varepsilon_2\neq 0)$ and the other parameters are given by $\varepsilon_1=0;\ \varepsilon_2=0.01;\ \beta=0.1;\ \zeta_1=\zeta_2=0.01;\ \mu=0.05;\ \rho=1$. In the same figure, the case for which both springs have linear characteristics (classical linear two degree-of-freedom system, $\varepsilon_1=\varepsilon_2=0$) is also presented (the same system parameters of the non-linear case were used). It can be observed that the nonlinear absorber case leads to a significant change in the response for $\Omega>1$.

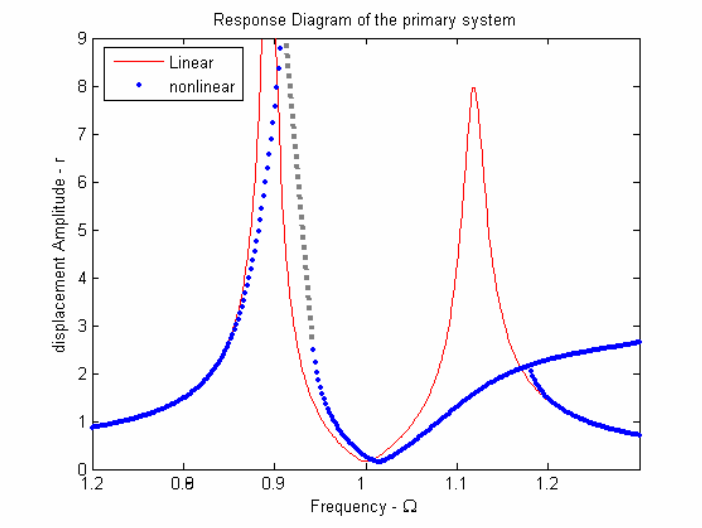

Figure 3: Response diagram of the main mass displacement for
$\varepsilon_1 = 0;\ \varepsilon_2 = 0.01;\ \ \beta = 0.1;\ \ \zeta_1 = \zeta_2 = 0.01;\ \ \mu = 0.05;\ \ \rho = 1.$

Figure 4 shows the response diagram of the main mass displacement using the same parameters as those of the previous case. However, different values for ε_2 , namely $\varepsilon_2 = 0.01$, $\varepsilon_2 = 0.01$ and $\varepsilon_2 = 0.02$ were considered.

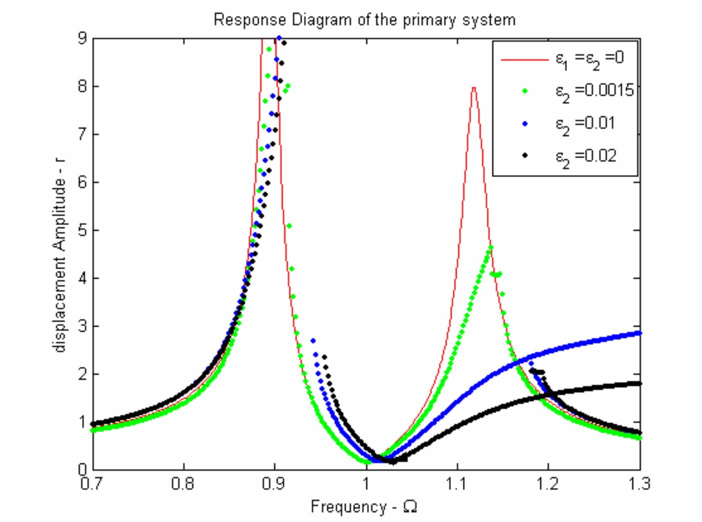

Figure 4. Response diagram of the main mass displacement for various values of ε_2 and
$\beta = 0.1;\ \ \zeta_1 = \zeta_2 = 0.01;\ \ \mu = 0.05;\ \ \rho = 1..$

It can be noticed that by increasing the value of ε_2 , the advantages in using a nonlinear DVA also increases because the amplitude response in the region that corresponds to Ω>1 decreases significantly. However, if the nonlinearities increase to critical levels unstable responses may occur. For example, in the case where ε_2 = 0.02, a new instable solution appears in the resonance area (near Ω=1). In many cases, the instabilities can be avoided without losing the advantages gained by the presence of non-linearities. In order to achieve this, the system parameters have to be changed. Optimization techniques should be used to obtain the best possible results. Next, the case for which the absorber is assembled with springs of the *softening* type will be addressed. In Fig. 5 the same parameters used to obtain the results shown in Fig.3 were used, except for ε_2 = -0.01.

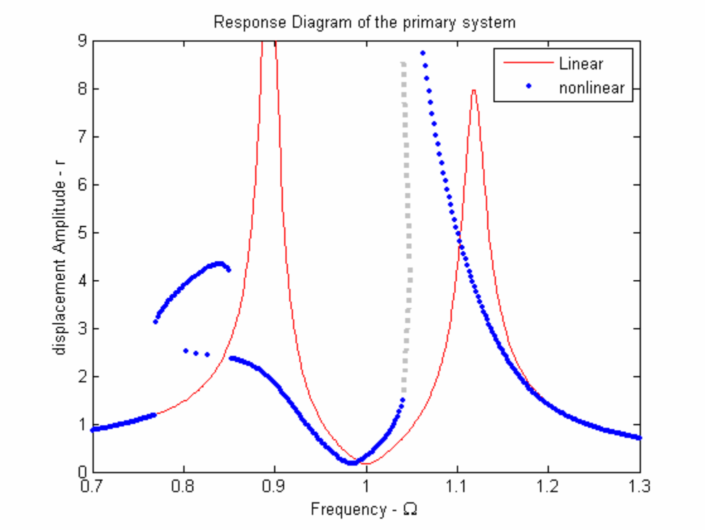

Figure 5 – Response diagram of the main mass displacement for
$\varepsilon_1 = 0;\ \varepsilon_2 = -0.01;\ \beta = 0.1;\ \zeta_1 = 0.01;\ \zeta_2 = 0.01;\ \mu = 0.05;\ \rho = 1.$

It can be noticed, by analyzing the case presented above as compared to the linear absorber, that the non-linear case becomes more efficient (from the vibration absorbing view point) for frequencies that correspond to Ω<1.

2.2.2. DVA mass assembled using a linear spring

Now, the case where only the main mass is connected to the ground through a nonlinear spring, i.e. $\varepsilon_2=0$, is addressed. Figure 6(a) illustrates the frequency response diagram for $\varepsilon_1 = 0.01$. Figure 6(b) shows the same diagram for $\varepsilon_1= -0.01$.

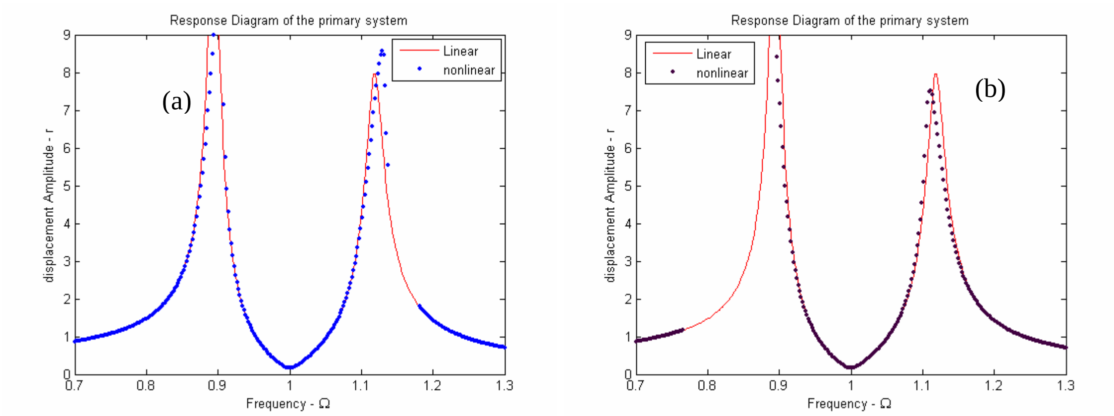

Figure 6: Frequency response diagram of the main mass displacement:
$\varepsilon_2 = 0;\ \beta = 0.1;\ \zeta_1 = 0.01;\ \zeta_2 = 0.01;\ \mu = 0.05;\ \rho = 1.$.

For both cases shown by fig. 6(a) and (b) it is possible to see that the effect of the nonlinear spring is not relevant, i.e., no vibration reduction is achieved. This is due to the small vibration amplitude that the main mass is submitted with respect to the absorber amplitude leading to a small deformation of the non-linear spring. As a consequence, the difference between the response of the linear and nonlinear systems in the neighborhood of Ω=1 is small. This conclusion remains true for different values of the force parameter β.

2.2.3. DVA mounted on springs with non-linear characteristics

First of all, in Fig. 7 (a), it is admitted that both the main mass and the absorber mass are mounted on a "hard spring" type of suspension. The non-linearity coefficients for both springs are equal to 0.01 ($\varepsilon_1 = \varepsilon_2 = 0.01$). The other parameters are the same as those given in Fig.5 previously presented. In Fig. 7 (b) a "soft spring" replaced the hard spring that connects the absorber mass to the primary mass and it was observed that better results are obtained for Ω<1.

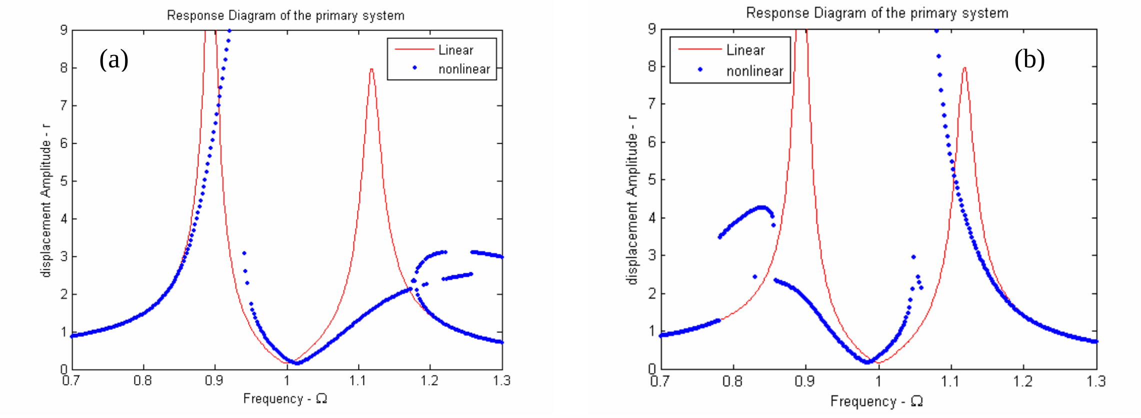

Figure 7 – (a) Response diagram of the main mass displacement for $\varepsilon_1 = 0.01;\ \varepsilon_2 = 0.01;\ \beta = 0.1;\ \zeta_1 = 0.01;\ \zeta_2 = 0.01;\ \mu = 0.05;\ \rho = 1$. (b) Response diagram of the main mass displacement for $\varepsilon_1 = 0.01;\ \varepsilon_2 = -0.01;\ \beta = 0.1;\ \zeta_1 = 0.01;\ \zeta_2 = 0.01;\ \mu = 0.05;\ \rho = 1$.

In Fig. 7(a), by comparing with the linear case, it is seen that the results obtained for Ω>1 are very satisfactory since the amplitude values decrease significantly. The response shown in Fig. 7(b) was obtained for the non-linear coefficients $\varepsilon_1 = 0.01$ and $\varepsilon_2 = -0.01$. However, if the main mass is mounted on a "softening" type of a spring, the results obtained are similar to those obtained in the previous cases as illustrated in Fig. 8 for $\varepsilon_1 = -0.01$ and $\varepsilon_2 = -0.01$.

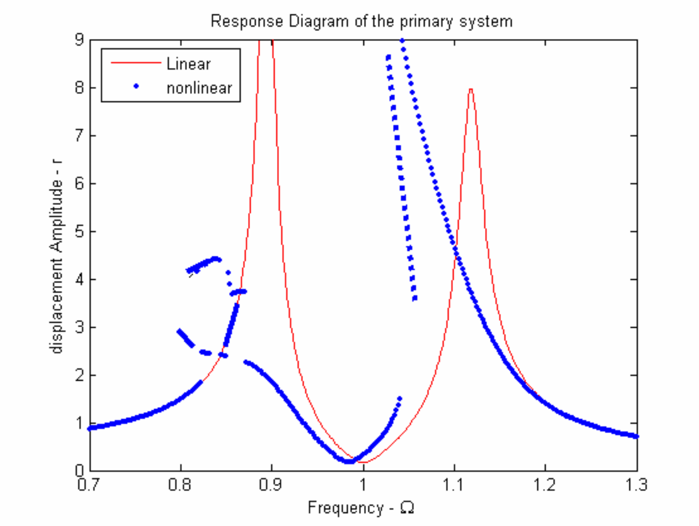

Figure 8 – Response diagram of the main mass displacement for $\varepsilon_1 = -0.01;\ \varepsilon_2 = -0.01;\ \beta = 0.1;\ \zeta_1 = 0.01;\ \zeta_2 = 0.01;\ \mu = 0.05;\ \rho = 1$.

Figure 8, provides a good example regarding a difficulty situation that appears when using a DVA with non-linear characteristics. This is represented by the bending of the resonance branch (the second resonance peak) above the stable solution branch characterized by small amplitudes. This results in the coexistence of stable solutions with low and high amplitudes that depend on the initial conditions. The risk just mentioned does not occur in the system corresponding to the Fig. 3, from which it can be seen that the resonance peaks tilt outside of the interest area. If the non-linearities are increased, they can result in unwanted instabilities.

2.2.4. Optimal design for the damped nonlinear DVA

The optimal design for the damped nonlinear DVA is obtained by using Genetic Algorithms (Holland, 1975). The goal is to obtain a larger "suppression band", namely, the frequency range over which the ratio of main mass displacement amplitude to the amplitude of the forcing function is less than unity (Hunt and Nissen 1982), (Rice, 1986) as depicted in Fig. 9.

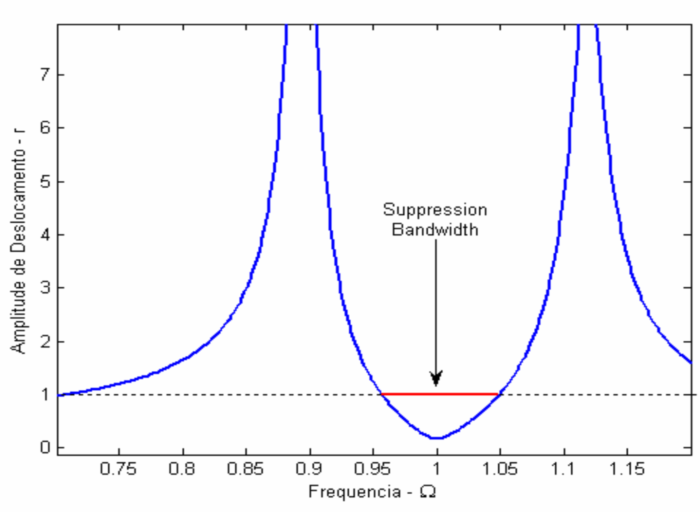

Figure 9. Suppression bandwidth characterization

We can define the problem of nonlinear optimal design as the determination of the values of the design variables x_i, ($i=1,\ldots,n$) such that the objective function attains an extreme value while simultaneously all constraints are satisfied (Steffen and Inman, 2002). The problem above is formulated as

$$Min\{f(x)/h(x)=0, g(x)\leq 0\}, x\in R^n \tag{20}$$

where R^n *is* n-dimensional set of real numbers, x is the vector containing the n design variables, $f(x)$ is objective function, $g(x)$ is vector of p inequality constraints and $h(x)$ is the vector of q equality constraints. The corresponding feasible domain is defined as

$$X=\{x\in R^n / h(x)=0,\ g(x)\leq 0\} \tag{21}$$

Side constraints were imposed for delimitation of the design space (Borges and Steffen, 2003).

The optimization problem was written such as the design variables are the following: the mass ratio (μ), the nonlinearity coefficient of the spring that fixes the main mass to the absorber (ε_2) and the damping factor related to primary mass suspension (ζ_2). The goal is to reduce the vibration amplitude and to increase the suppression bandwidth simultaneously. In the present contribution only the suppression band was used in the objective function.

As an initial configuration of the system, the case illustrated in Fig. 3 is taken into account ($\varepsilon_1=0$ and $\varepsilon_2=0.01$). For this initial configuration the design variables correspond to:

$$\varepsilon_2=0.01;\quad \zeta_2=0.01;\quad \mu=0.05 \tag{23}$$

The other parameters of the system remain fixed. After the optimization run with Genetic Algorithms, the following optimal values were obtained:

$$\varepsilon_2^*=0.0261;\quad \zeta_2^*=0.0931;\quad \mu^*=0.1456 \tag{24}$$

The dynamic responses of the system for both system configurations (initial and optimal cases) are shown in Fig. 10.

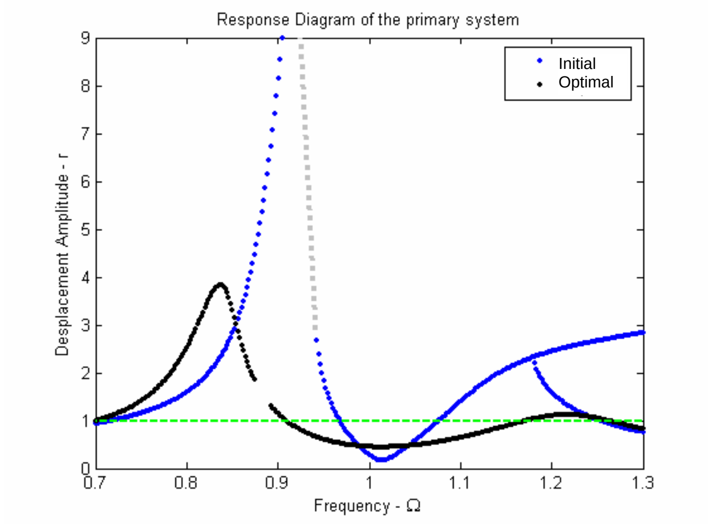

Figure. 10 – Optimal configuration for the nonlinear dynamic vibration absorber

It can be observed that the optimal configuration provides an important improvement in the behavior of the system. The objective function (the suppression band) was increased with respect to the initial configuration of the system. Besides, it was possible to obtain smaller vibration amplitudes as a side benefit. Obviously, it would be interesting to build a multi-objective problem dedicated to simultaneously increase the suppression bandwidth and to decrease the vibrations.

2.3. Conclusion

In this paper, a DVA that may exhibit nonlinear characteristics in the spring that connects the main mass to the base and in the spring that connects the absorber mass to the absorber mass was presented. The equations of motion of the nonlinear two degree-of-freedom system were integrated by using the so-called "method of the average" that provides an approximate solution to the problem. From this point, the nonlinear algebraic equation system was numerically solved by using the *"fsolve"* function of MATLAB® toolbox and the roots of the equations were determined. Several configurations were taken into account aiming at demonstrating the dynamic behavior of the system. It is worth mentioning that the magnitude of the non-linearity of the system stiffness can lead to conflicting situations: a) the vibration amplitude of the system is reduced, and b) dynamic instabilities may appear. Consequently, the necessity of determining the optimal non-linearity coefficient that guarantees the best solution for a given system was described. Finally, optimal design for the system was determined. The most important objective in the present contribution was to increase the suppression bandwidth. However, in performing optimization good results were obtained in the sense that both the "suppression band" was increased and the vibration amplitude was decreased satisfactorily. The optimal design was obtained by Genetic Algorithms, since classical optimization techniques failed due to various local minima found in the design space. Further research work will be dedicated to writing a multi-objective function in such a way that vibration attenuation and suppression bandwidth increase are both contemplated in the optimization scheme.

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