THE GENERALIZED FINITE ELEMENT METHOD APPLIED TO FREE VIBRATION OF UNIFORM STRAIGHT BARS

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Abstract. *This work deals with the Generalized Finite Element Method (GFEM) applied to free vibration of uniform straight bars. The variational problem of free vibration is formulated and the main aspects of GFEM are discussed. The Composite Element Method (CEM) and the GFEM are presented. The CEM is developed by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from closed form solutions of classical theory. This approach results in a hierarchical refinement called c-version. The GFEM is developed by enrichment of the conventional FEM solution space with characteristics of the solution known a priori applying the partition of unity method (PUM). The application of GFEM in vibration analysis of straight bars is investigated. The eigenvalues obtained by GFEM are compared with those obtained by analytical solution, by CEM and by h-version of FEM. The numerical results show that GFEM presented in this work have higher rates of convergence than those obtained by the h-version of FEM and obtained by CEM.*

Keywords*: generalized finite element method, free vibration, vibration analysis, partition of unity method*

1. INTRODUCTION

Engineers involved in the analysis and design of structures subjected to dynamic loading need a precise understanding of the dynamical behavior of these structures. Nowadays many researchers have developed vibration analysis methods.

The Finite Element Method (FEM) is commonly used in vibration analysis. In the FEM, the approximated solution can be improved using two refinement techniques: *h* and *p*-versions. The *h*-version consists of the refinement of element mesh. Recent works (Ribeiro, 2001) (Campion and Jarvis, 1996) define *p*-version as the increase in the number of form functions in the element without change in the mesh. The conventional *p*-version of FEM consists of increasing the polynomial degree in the solution. Some researchers (Ganesan and Engels, 1991) (Ribeiro, 2001) have used non-polynomial form functions to refine FEM solutions.

Zeng (1998a, b) developed the Composite Element Method (CEM). The CEM is obtained by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from analytical solutions of simple vibration problems. This approach results in a hierarchical refinement called *c*-version. The CEM have showed rates of convergence greater than those obtained by FEM in free vibration problems.

Another method named Generalized Finite Element Method (GFEM) was developed based on the ideas of the partition of unity method (PUM) (Babuska, Banerjee and Osborn, 2004). In the GFEM, the local solution spaces are formed for functions, not necessarily polynomial, that reflect the available information on the unknown solution. This procedure ensures good local approximation. The local spaces are grouped in the approximated solution space by the partition of unity method that ensures good global approximation.

This work presents the variational form to the problem of free axial vibration of straight bars and analyses the application of GFEM in the solution of these problems.

2. VARIATIONAL FORM OF THE FREE VIBRATION OF BARS

The bar consists of a straight rod with axial strain (Fig. 1). The basic hypotheses are (Craig, 1981): (a) The cross sections which are straight and normal to the axis of the bar before deformation remain straight and normal after deformation; and (b) The material is elastic, linear and homogeneous.

The vibration of the bar is a time dependent problem. The movement equation that governs this problem is a partial differential equation. The problem consists of find the axial displacement $\bar{u} = \bar{u}(x,t)$ which satisfies

$$\rho A \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial \bar{u}}{\partial x} \right) = p(x,t) \tag{1}$$

where A is the cross section area, E is the Young modulus, ρ is the specific mass, p is the externally applied axial force per unit length and t is the time. The solution $\overline{u} = \overline{u}(x,t)$ must satisfy boundary and initial conditions defined in the problem.

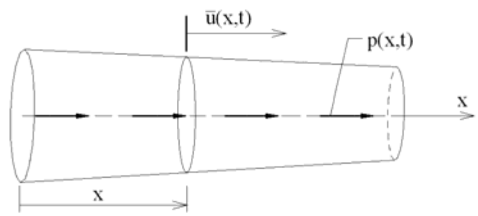

Figure 1. Straight bar.

According to Carey and Oden (1984), the most popular form to obtain the variational form of a time dependent problem is consider the time t like a real parameter and develop a family of variationals problems in t. This consists in select test functions $v = v(x)$, independent of t, and apply the weighted-residual method. If the finite element method is used to represent the spatial behavior of the solution, one obtains a system of ordinary differential equations of the degrees of freedom as functions of the parameter t. This approach is called semidiscrete formulation of the problem.

Using the weighted-residual method, according to Carey and Oden (1984), to develop an integral statement of Eq.(1), the solution $\overline{u} = \overline{u}(x,t)$ must satisfy

$$\int_0^L \rho A \frac{\partial^2 \overline{u}}{\partial t^2}.v.dx - \int_0^L \frac{\partial}{\partial x}\left(EA \frac{\partial \overline{u}}{\partial x} \right).v.dx = \int_0^L p(x,t).v.dx \quad (2)$$

for admissible test functions $v = v(x)$ at any time $t \in (0,T]$.

Integrating Eq. (2) by parts, one obtains:

$$\int_0^L \rho.A \frac{\partial^2 \overline{u}}{\partial t^2}.v.dx - \left[v.EA \frac{\partial \overline{u}}{\partial x} \right]_0^L + \int_0^L \left(EA \frac{\partial \overline{u}}{\partial x} \right).\frac{\partial v}{\partial x}.dx = \int_0^L p(x,t).v.dx \quad (3)$$

It is necessary to introduce the boundary and initial conditions to complete the problem. The admissible test functions $v = v(x)$ must satisfy the same boundary conditions of the solution $\overline{u}(x,t)$. To common combination of boundary conditions in boundaries $x = 0$ and $x = L$, the second term in Eq. (3) vanishes and it becomes:

$$\int_0^L \rho.A \frac{\partial^2 \overline{u}}{\partial t^2}.v.dx + \int_0^L \left(EA \frac{\partial \overline{u}}{\partial x} \right).\frac{\partial v}{\partial x}.dx = \int_0^L p(x,t).v.dx \quad (4)$$

Particularizing the problem to the case of free vibration of an uniform straight bar, where E, A and ρ are constants and $p(x,t) = 0$, the Eq. (4) becomes:

$$\rho.A \int_0^L \frac{\partial^2 \overline{u}}{\partial t^2}.v.dx + EA \int_0^L \frac{\partial \overline{u}}{\partial x}.\frac{\partial v}{\partial x}.dx = 0 \quad (5)$$

According to Carey and Oden (1984), in vibration problems, one assumes periodic solutions $\overline{u}(x,t) = e^{i\omega.t}u(x)$, where ω is the vibration frequency. The free vibration of an uniform bar becomes in an eigenvalue problem with variational statement: find a pair (λ, u), with $u \in H^1(0,L)$ and $\lambda \in \mathbf{R}$, so that

$$EA\int_0^L \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}.dx - \rho A\lambda\int_0^L uvdx = 0 \tag{6}$$

for admissible test functions $v \in H^1(0,L)$, where $\lambda = \omega^2$.The variational statement of this eigenvalue problem can also be written as: find (λ, u), with $u \in H^1(0,L)$ and $\lambda \in \mathbf{R}$, so that

$$B(u,v) = \lambda.(u,v) \tag{7}$$

for all admissible test functions $v \in H^1(0,L)$, where $B(u,v)$ is the bilinear form and (u,v) is the scalar product in L^2 space, obtained from

$$B(u,v) = \frac{E}{\rho}\int_0^L \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}.dx \tag{8}$$

$$(u,v) = \int_0^L uvdx\,. \tag{9}$$

3. FINITE ELEMENT METHOD

The approximation by finite elements of the free vibration problem of a uniform straight bar consists to rewrite the variational form of the problem (Eq. (7)) in an approximated subspace $H^h \subset H^1(0,L)$. The approximated eigenvalue problem in Eq. (7) becomes: find $\lambda_h \in \mathbf{R}$ and $u_h \in H^h(0,L)$ so that

$$B(u_h, v_h) = \lambda_h.(u_h, v_h) \qquad \forall v_h \in H^h \tag{10}$$

The approximated solution $u_h(x)$ from finite elements can be written, to discretization in *N* nodes, in the following form:

$$u_h(x) = \sum_{j=1}^{N} u_j \phi_j(x) \tag{11}$$

where $\{\phi_j\}$ are the global base functions of the subspace H^h and $\{u_j\}$ are the degrees of freedom. Replacing the Eq. (11) in Eq. (10) and taking $v_h = \phi_i,\ i = 1,2,\ldots,N$, one obtains:

$$\sum_{j=1}^{N}\left(EA\int_0^L \frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}dx\right).u_j = \lambda_h \sum_{j=1}^{N}\left(\rho A\int_0^L \phi_i\phi_j dx\right).u_j\ , \qquad i = 1,2,\ldots,N \tag{12}$$

or, in matrix form,

$$\mathbf{Ku} = \lambda_h \mathbf{Mu} \tag{13}$$

where **K** is the stiffness matrix and **M** is the mass matrix, and they are defined in Eq. (12). The Equation (12) corresponds to a generalized eigenvalue problem where λ_h are the eigenvalues related to the vibration frequencies ω and the vectors **u** are the eigenvectors related to the vibration shape modes of the bar.

After the discratization of the problem domain Ω by sub domains Ω_e, called elements, it is necessary to determinate the contribution of each element to the elementary stiffness matrix coefficients (k_{ij}^e) and elementary mass matrix coefficients (m_{ij}^e), obtained by:

$$k_{ij}^e = EA\int_{\Omega_e} \frac{\partial \psi_i^e}{\partial x}\frac{\partial \psi_j^e}{\partial x}dx \tag{14}$$

$$m_{ij}^e = \rho A\int_{\Omega_e} \psi_i^e \psi_j^e dx \tag{15}$$

where the local form function ψ_i^e is the restriction of the base function ϕ_i in element Ω_e.

The adequated choice of the local form functions ψ_i^e determinates different solution methods with distinct features and rates of convergence. In general, according to Reddy (1986), the form functions are developed to master elements and then they are mapped to the real elements obtained from the finite element mesh. In this work the master element domain is Ω_e (0,1).

The conventional FEM uses polynomials form functions. Taking the uniform bar element (Fig. 2) with one degree of freedom per each node, the approximated solution in the element domain can be defined as:

$$u_h^e(\xi) = \psi_1^e(\xi).u_1 + \psi_2^e(\xi).u_2 \tag{16}$$

or in matrix form: $u_h^e(\xi) = \mathbf{N}^T\mathbf{q}$ (17)

where $\xi = \dfrac{x}{L}$, $\mathbf{N}^T = [\psi_1^e \quad \psi_2^e]$ and $\mathbf{q}^T = [u_1 \quad u_2]$, L is the element length, and, u_1 and u_2 are the nodal displacements.

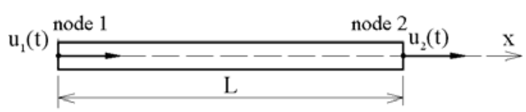

Figure 2. Bar element.

Using Hermitian polynomials as local form functions, one obtains:

$$\mathbf{N}^T = [1-\xi \quad \xi] \tag{18}$$

The next topics present form functions sets that generate the Composite Element Method (CEM) and the Generalized Finite Element Method (GFEM).

4. COMPOSITE ELEMENT METHOD (CEM)

A new method can be obtained using a FEM conventional element with local form functions enriched by adding non-polynomial functions related to the closed form solutions from classical theory.

Weaver Junior and Loh (1985) used analytical solution as form functions to lateral displacements in the analysis of local vibration mode shapes of trusses. After, this approach was applied by Ganesan and Engels (1991) to obtain a hierarchical model of finite elements of Euler-Bernoulli beams. Zeng (1998b) developed elements of trusses, Euler-Bernoulli beams and frames using this approach to vibration analysis. In the work of Zeng (1998a, b) this technique was called Composite Element Method (CEM).

According to CEM, the approximated solution in the element domain of a bar is obtained by:

$$u_h^e = \mathbf{N}^T\mathbf{q} + \mathbf{Ø}^T.\mathbf{c} \tag{19}$$

where $\mathbf{q}^T = [u_1 \quad u_2]$ is the same of Eq. (17), the vector **N** contains the FEM form functions obtained in Eq. (18) and the vectors $\boldsymbol{\emptyset}$ and **c** are obtained by:

$$\boldsymbol{\emptyset}^{\mathrm{T}}(\xi) = [F_1 \quad F_2 \quad \ldots \quad F_r \quad \ldots \quad F_n] \tag{20}$$

$$\mathbf{c}^T = [c_1 \quad c_2 \quad \cdots \quad c_n] \tag{21}$$

$$F_r = \sin(r.\pi.\xi) \tag{22}$$

where c_i are the coefficients that multiply the analytical solutions F_r obtained from the solution of the free vibration problem of a rod with all end displacements constrained, for $r = 1,2,\ldots$.

The new degrees of freedom related to the enriched form functions do not have direct physical meaning and they were called *c* degrees of freedom by Zeng (1998b). The enrichment proposed by CEM produce hierarchical models and better results than those obtained from *h*-version of FEM (Arndt, Machado and Hecke, 2002 and 2003). The hierarchical refinement produced when the number of analytical functions in the approximated solution is increased was called *c* refinement by Zeng (1998a, b).

5. GENERALIZED FINITE ELEMENT METHOD (GFEM)

The Generalized Finite Element Method (GFEM) is a Galerkin method which main goal is the construction of an approximation using local knowledge of the solution that ensures good local and global results. The GFEM was initially named Partition of Unity Finite Element Method (PUFEM) by Melenk and Babuska (1996). The local enrichment in the approximated space is incorporated by the partition of unity approach. The classical FEM is a special case of the GFEM (Babuska, Banerjee and Osborn, 2004).

The approximated solution $u_h(x)$ from GFEM can be written, with *N* nodes, in the following form:

$$u_h(x) = \sum_{i=1}^{N} \eta_i(x).u_i + \sum_{i=1}^{N^e} \eta_i(x) \left(\sum_{j=1}^{n_f} \gamma_j .a_{ij} \right) \tag{23}$$

where $\{\eta_i\}$ are the partition of unity functions, $\{\gamma_j\}$ are the functions of enrichment, $\{u_i\}$ and $\{a_{ij}\}$ are the degrees of freedom, N^e is the number of enriched degrees of freedom and n_f is the number of functions of enrichment.

The partition of unity functions η_i defined on the supports w_i, when $\Omega = \{w_i\}_{i=1}^{N}$, present the following properties (Duarte et al, 2001):

$$\eta_i \in C_0^S(w_i), \quad S \geq 0 \quad , \quad 1 \leq i \leq N \tag{24}$$

$$\sum_{i=1}^{N} \eta_i(x) = 1, \qquad \forall x \in \Omega \tag{25}$$

The first property implies that the functions η_i, $i = 1,\ldots,N$ are non-zero only over the supports w_i. It is easy to verify that the form functions of conventional FEM in Eq. (18) represent a partition of unity.

In this work, the approximated solution by GFEM in the element domain is:

$$u_h^e(\xi) = \sum_{i=1}^{2} \psi_i^e(\xi).u_i + \sum_{i=1}^{2} \psi_i^e(\xi). \left[\sum_{j=1}^{n_e} F_j .a_{ij} \right] \tag{26}$$

where ψ_i^e are the FEM form functions in Eq. (18), F_j are the enrichment functions used by CEM in Eq. (22), n_e is the number of functions of enrichment, u_i are the nodal displacements and a_{ij} are the degrees of freedom related to the functions of enrichment. Figure 3 shows the form functions of GFEM in Eq. (26). This approach allows a hierarchical *p* refinement of GFEM.

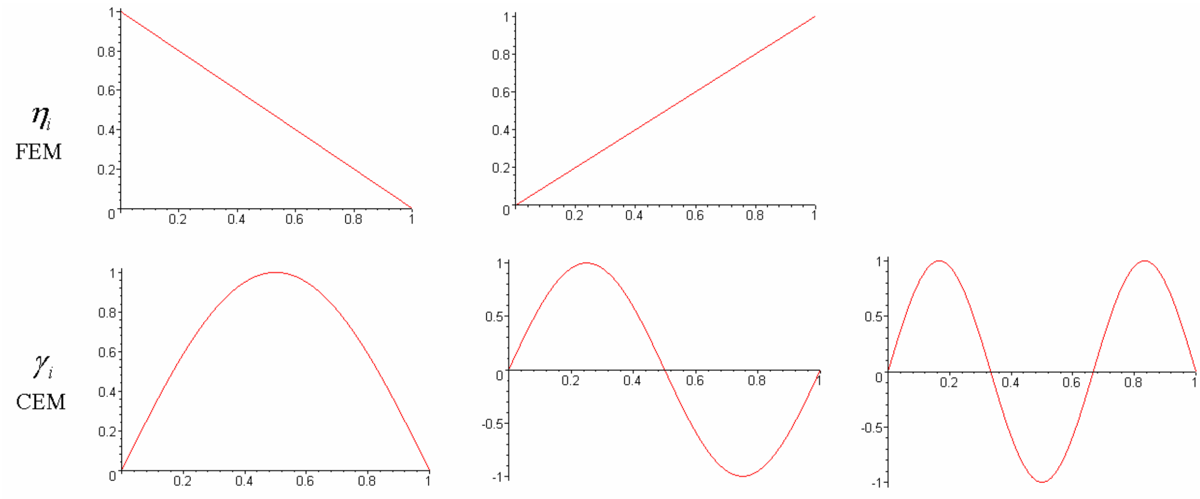

Figure 3. GFEM form functions.

6. APPLICATION

For the numerical verification of the presented methods, the free vibration of a uniform fixed-free bar in axial motion (Fig. 4), with length *L*, elasticity modulus *E*, mass density ρ and cross section area *A*, is analyzed.

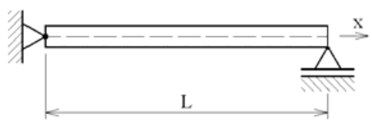

Figure 4. Uniform fixed-free bar.

The exact natural frequencies (ω_r) of the uniform fixed-free bar in axial motion are (Craig, 1981):

$$\omega_r = \frac{(2r-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \quad , \; r = 1,2,\ldots \tag{27}$$

A parameter β is used to compare the exact solution with the approximated solutions. The parameter is given by:

$$\beta = \frac{\rho L^2 \omega^2}{E} \tag{28}$$

The results were also compared to those obtained from the *h*-version of FEM taking a regular subdivision of the mesh. In the analyses by CEM and GFEM, the bar was described geometrically by just one element and the successive refinements were obtained increasing the number of enriched form functions.

Figures 5, 6 and 7 present the evolution of relative error for the six earliest eigenvalues of the proposed problem as function of the total number of degrees of freedom used in each method. The relative error is presented in logarithmic scale and calculated by:

$$error = 100\frac{\beta_h - \beta_e}{\beta_e} \tag{29}$$

where β_h is the approximated eigenvalue (parameter β) and β_e is the exact eigenvalue obtained by Eq. (27) and Eq.(28).

Analyzing the results obtained for the fixed-free bar, one observes that the proposed GFEM have convergence rates greater than those obtained by CEM and *h* refinement of FEM.

To obtain higher eigenvalues with good precision by GFEM, it is necessary a refinement with total number of degrees of freedom greater than those necessary for the first eigenvalues with same precision. But for all eigenvalues the error decreases very quickly when new enrichment functions are incorporated in GFEM.

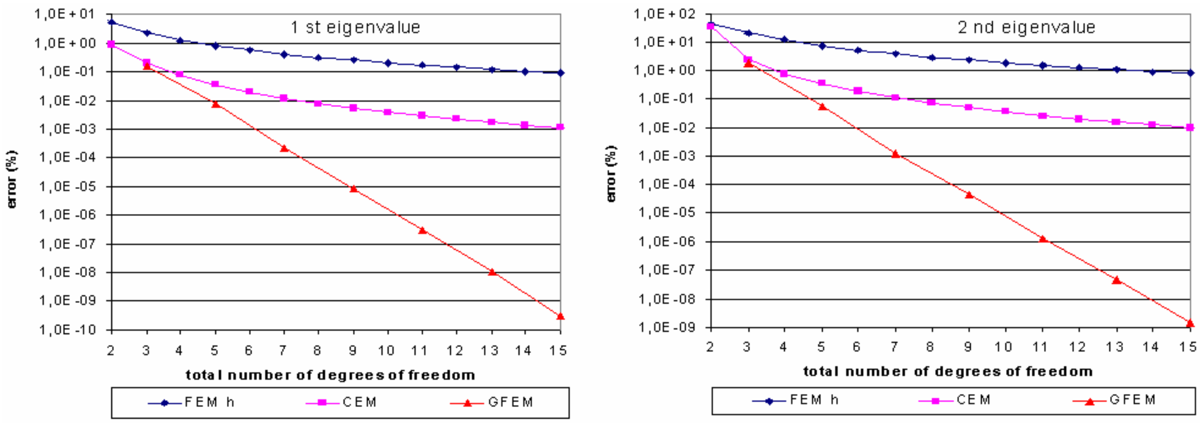

Figure 5. Relative error (%) for the 1st and 2nd bar eigenvalues.

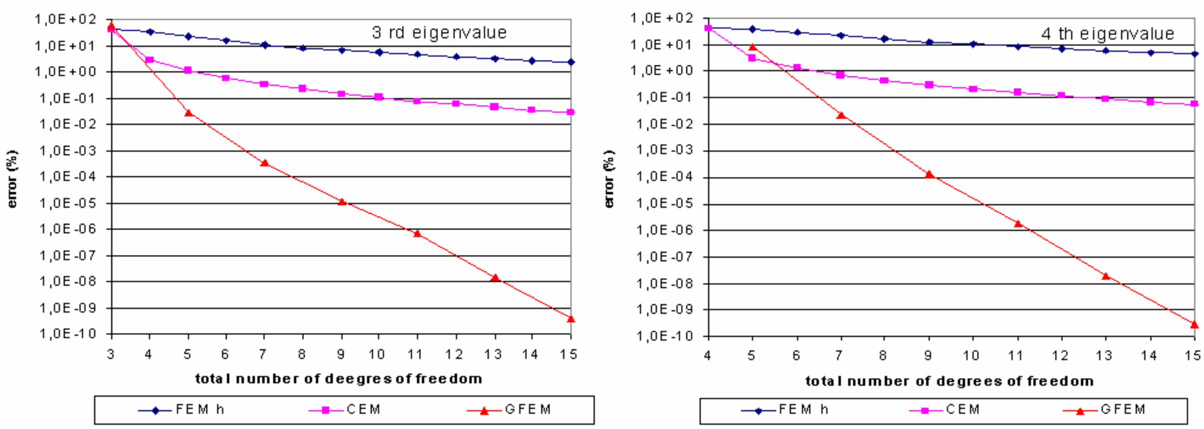

Figure 6. Relative error (%) for the 3rd and 4th bar eigenvalues.

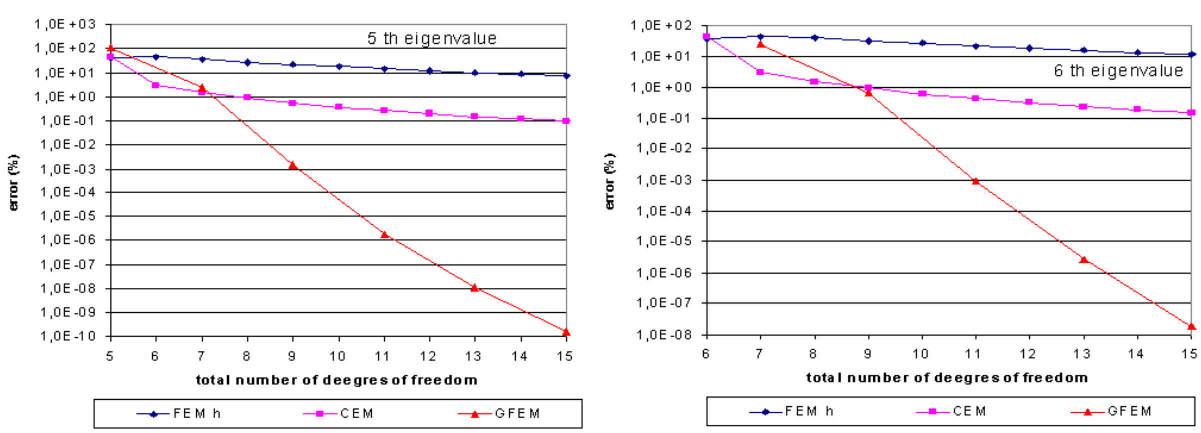

Figure 7. Relative error (%) for the 5th and 6th bar eigenvalues.

7. CONCLUSION

This work presents the variational form to the free vibration problem of the uniform straight bar with common boundary conditions. The presented procedure allows obtaining the variational form for the free vibration of non uniform bars and for the forced vibration of rods.

Two different methods of enrichment of the FEM were presented.

In the Composite Element Method (CEM), the local form functions of a FEM conventional element are enriched by adding non-polynomial functions obtained from closed form solutions (exact) of the classical theory. This approach produces a hierarchical refinement called *c* refinement.

In the proposed Generalized Finite Element Method (GFEM), the same enrichment functions of the CEM are added to the FEM form functions by the partition of unity approach. This technique produces a hierarchical *p* refinement.

To compare these methods, some eigenvalues of free vibration of a fixed-free bar were calculated. The analytical solution of this problem is well-known. The CEM and GFEM were also compared to *h* refinement of FEM.

The results have shown that the GFEM presents convergence rates greater than those obtained from CEM and *h* refinement of FEM. However the latest eigenvalues obtained by GFEM in each analysis have poor precision, worst than those obtained by CEM. This disadvantage of the proposed GFEM is compensated by the raised rate of convergence of the hierarchical *p* refinement developed.

The application of GFEM in vibration analysis of trusses and beams and the use of others enriched functions will be investigated in future works.

8. REFERENCES

Arndt, M., Machado, R. D. and Hecke, M. B., 2002, “An introduction to the composite element method applied to the vibration analysis of trusses”, Shock and Vibration, Vol. 9, pp.155-164.

Arndt, M., Machado, R. D. and Hecke, M. B., 2003, “The composite element method applied to free vibration analysis of trusses and beams”, Applied Numerical Mathematics, Vol. 47, pp.59-73.

Babuska, I., Banerjee, U. and Osborn, J. E., 2004. “Generalized finite element methods: main ideas, results, and perspective”, Technical Report 04-08, TICAM, University of Texas at Austin.

Campion, S. D. and Jarvis, J. L., 1996, “An investigation of the implementation of the p-version finite element method”, Finite Elements in Analysis and Design, Vol. 23, pp.1-21.

Carey G. F. and Oden J. T., 1984, “Finite elements: computational aspects”, Vol. 3, Ed. Prentice Hall, New Jersey, USA.

Craig R. R., 1981, “Structural dynamics: an introduction to computer methods”, Ed. Jonh Wiley, New York, USA.

Duarte, C. A., Hamzeh, O. N., Liszka, T. J. and Tworzydlo, W. W., 2001, “A generalized finite element method for the simulation of three-dimensional dynamic crack propagation”, Computer Methods in Applied Mechanics and Engineering, Vol. 190, pp.2227-2262.

Ganesan, N. and Egels, C., 1991, “Hierarchical Bernoulli-Euller beam finite elements”, Computers & Structures, Vol. 43, No. 2, pp.297-304.

Melenk, J. M. and Babuska, I., 1996, “The partition of unity finite element method: basic theory and applications”, Computer Methods in Applied Mechanics and Engineering, Vol. 139, No. 1-4, pp.289-314.

Reddy, B. D., 1986, “Functional analysis and boundary-value problems: and introductory treatment”, Ed. Longman, New York, USA.

Ribeiro, P., 2001, “Hierarchical finite element analyses of geometrically non-linear vibration of beams and plane frames”, Journal of Sound and Vibration, Vol. 246, No. 2, pp.225-244.

Weaver Junior W. and Loh C. L., 1985, “Dynamics of trusses by component-mode method”, Journal of Structural Engineering, Vol. 111, No. 12, pp.2565-2575.

Zeng P., 1998a, “Introduction to composite element method for structural analysis in engineering”, Key Engineering Materials, Vol. 145-149, pp.185-190.

Zeng P., 1998b, “Composite element method for vibration analysis of structure, part I: principle and C^0 element (bar)”, Journal of Sound and Vibration, Vol. 218, No. 4, pp.619-658.

9. RESPONSIBILITY NOTICE

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