# NUMERICAL INTEGRATION BY GAUSS-LEGENDRE QUADRATURE OVER TRIANGULAR DOMAINS

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***Abstract**. The finite element method has been used as a powerful tool to obtain approximate solution to several phenomena common in engineering field. When using the finite element method to discretize a given problem some integrals may arise and its computation must be done numerically. For two dimensional domains as in this work it will be two dimensional integrals. For triangular finite elements there will be two dimensional integrals over triangular domains. For such domains there are not so much methods to accomplish that integration with accuracy. In this work it is proposed another algorithm to do numerical integration over two dimensional domains shaped as triangles. To accomplish that a first linear transform is done to change the general triangle into a right triangle having unitary sides. Afterwards the standard triangle is transformed nonlinearly into a standard rectangle suitable for the two dimensional Gauss-Legendre integration. The well known Gauss-Legendre quadrature points and weights for the two dimensional integration over a standard rectangle are then mapped back to the standard triangle and then to the general triangle. By using a supposed new transformation it was possible to obtain a set of Gauss-Legendre symmetric points and weights.*

***Keywords:** numerical integration, Gauss-Legendre, finite element, triangular element.*

## 1. INTRODUCTION

The Finite Element Method (FEM) is a main numerical tool used to obtain approximate solution for mathematical problems that arise from physical modeling, for example, those ones associated to Fluid Mechanics and Heat Transfer. This method, computationally developed and coded, shows good results when applied to solve problems under steady state and unsteady state flow regimes, for linear and non-linear equations, and to one, two and three-dimensional domains.

An issue, among others, that need attention in the context of FEM is about doing the integrals that appear during the discretization process of a given problem. Such integrals must be done for each kind of element being used. Effective numerical methods to perform that job are not easily found in literature. The most used types of elements described in papers are the triangular and quadrilateral.

Hammer (1956), till we know, was the first to accomplish a solution to the problem of doing two-dimensional numerical integration of a function defined over a general triangular domain. In the same year Turner *et al.* (1956) presented the first work about the Finite Element Method, followed by Clough (1960) and Argyris (1963). Afterwards, many others researchers such as Cowper (1973), Lannoy (1977), Laurie (1977), Reddy and Shippy (1981) were elaborating and improving the integration formulae.

The purpose of this work is to obtain a method or algorithm to do numerical integration over general triangle shapes using and adapting the ideas of the Gauss – Legendre quadrature.

## 2. NUMERICAL METHOD FORMULATION TO DO INTEGRATION OVER A TRIANGLE

Initially, to find a suitable integration method to be used with FEM it is necessary to transform a general triangle into an intermediate standard right triangle with the following corner coordinates (0,0); (1,0) and (0,1). To map a general triangle into the standard one it is necessary to do:

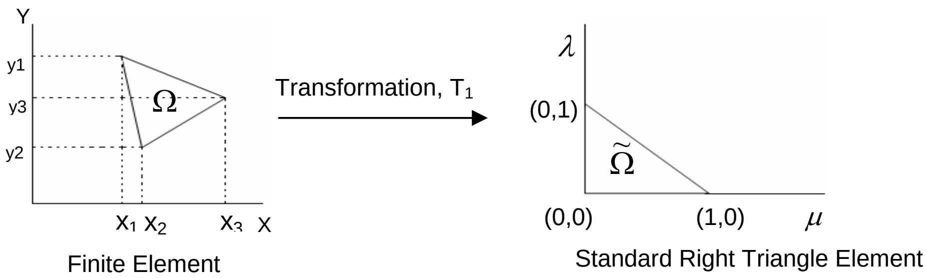


Figure 1. Coordinate transformation from a general triangular element into a right angle triangular element.

Such linear geometric transformation can be written as follows,

$$\begin{bmatrix} \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} - \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} \tag{1a,b}$$

or

$$\mu = a_{11}x + a_{12}y + \tilde{a} \quad and \quad \lambda = a_{21}x + a_{22}y + \tilde{b} \ . \tag{2}$$

It is enough to find the coefficients in equation (1), and for that is used the following constraints:

$$\begin{aligned} 0 &= a_{11}x_1 + a_{12}y_1 + \tilde{a} \\ 0 &= a_{21}x_1 + a_{22}y_1 + \tilde{b} \\ 1 &= a_{11}x_2 + a_{12}y_2 + \tilde{a} \\ 0 &= a_{21}x_2 + a_{22}y_2 + \tilde{b} \\ 0 &= a_{11}x_3 + a_{12}y_3 + \tilde{a} \\ 1 &= a_{21}x_3 + a_{22}y_3 + \tilde{b} \end{aligned} \tag{3}$$

Solving the linear system (3), we get explicit equations for $a_{11}$ , $a_{12}$ , $a_{21}$ and $a_{22}$ :

$$a_{11} = \frac{y_3 - y_1}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)} , \tag{4a}$$

$$a_{12} = \frac{x_3 - x_1}{(y_2 - y_1)(x_3 - x_1) - (y_3 - y_1)(x_2 - x_1)} , \tag{4b}$$

$$a_{21} = \frac{y_2 - y_1}{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)} , \tag{4c}$$

$$a_{22} = \frac{x_2 - x_1}{(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)} . \tag{4d}$$

It is known from analytical geometry that the area, *A*, of a general triangle can be obtained from the coordinates of its corners. Such equation is well-known and is expressed by

$$A = \frac{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}{2} \tag{5}$$

Using this equation into equations (4a-d) it is possible to rewrite them as:

$$a_{11} = \frac{y_3 - y_1}{2A} , \quad a_{12} = \frac{x_3 - x_1}{-2A} , \quad a_{21} = \frac{y_2 - y_1}{-2A} \quad \text{and} \quad a_{22} = \frac{x_2 - x_1}{2A} . \tag{6a-d}$$

Using the definition of the Jacobean for the transformation $T_1$

$$J_{T_1} \equiv \frac{\partial(x,y)}{\partial(\mu,\lambda)} \equiv \begin{bmatrix} \frac{\partial x}{\partial \mu} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial \mu} & \frac{\partial y}{\partial \lambda} \end{bmatrix} \tag{7}$$

we obtain from equations (1a-b) and (6a-d) the following result

$$J_{T_1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{y_3 - y_1}{2A} & \dfrac{x_3 - x_1}{-2A} \\ \dfrac{y_2 - y_1}{-2A} & \dfrac{x_2 - x_1}{2A} \end{bmatrix}^{-1} = \begin{bmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{bmatrix} \Rightarrow det(J_{T_1}) = 2A \tag{8}$$

Then, let us consider the integration over a general triangular domain, Ω,

$$I_\Omega \equiv \int_\Omega f(x,y)dxdy = \int_{\tilde{\Omega}} f(\mu,\lambda)det[J(\mu,\lambda)]d\mu d\lambda = \int_{\tilde{\Omega}} f(\mu,\lambda)2Ad\mu d\mu = 2A\int_{\tilde{\Omega}} f(\mu,\lambda)d\mu d\lambda \tag{9}$$

With the previous transformation was possible to change the integration over a general triangle, Ω, into an integration over a standard right triangle, $\tilde{\Omega}$, however, it is not yet useful to apply directly the Gauss-Legendre quadrature, so it is necessary to do another transformation in order to change the standard right triangle domain into a standard square domain. To do such transformation, $T_2$, we propose the following formulae

$$\mu = (1-\frac{v}{2})u; \quad \lambda = (1-\frac{u}{2})v. \tag{10}$$

This transformation is an extension of that one proposed by Rathod et al. (2004). The difference is that this one is symmetric, while Rathod's isn't. In Figure 2 it is shown the transformation from the triangular domain, $\tilde{\Omega}$, to the new transformed square domain, $\hat{\Omega}$.

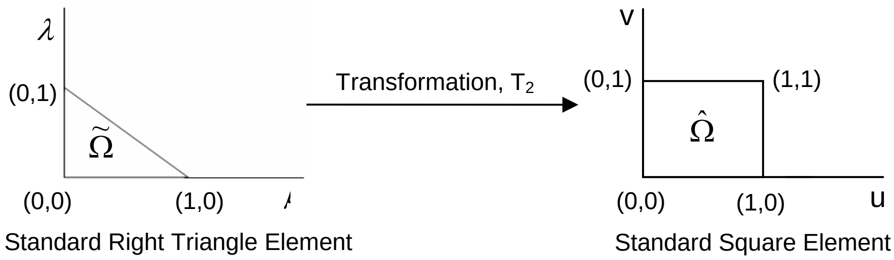


Figure 2. Coordinate system transformation in order to transform the Standard right triangle domain into one Standard square.

The Jacobean of $T_2$ transformation is

$$J_{T_2} \equiv \frac{\partial(\mu,\lambda)}{\partial(u,v)} \equiv \begin{bmatrix} \dfrac{\partial\mu}{\partial u} & \dfrac{\partial\lambda}{\partial u} \\ \dfrac{\partial\mu}{\partial v} & \dfrac{\partial\lambda}{\partial v} \end{bmatrix} = \begin{bmatrix} \left(1-\dfrac{v}{2}\right) & \dfrac{v}{2} \\ \dfrac{u}{2} & \left(1-\dfrac{u}{2}\right) \end{bmatrix} \Rightarrow det(J_{T_2}) = 1 - \frac{u+v}{2}, \tag{11}$$

Then the integral $I_{\tilde{\Omega}}$ in equation (9) can be changed as follows

$$I_{\tilde{\Omega}} \equiv \int_{\tilde{\Omega}} f(\mu(\mu,\lambda)d\mu = \int_{\hat{\Omega}} f(u,v)det(J_{T_2})dudv = \\ = \int_{\hat{\Omega}} f(u,v)(1-\frac{u+v}{2})dudv = \int_0^1\int_0^1 f(u,v)(1-\frac{u+v}{2})dudv. \tag{12}$$

In the right side of equation (12) the integrals intervals are [0,1] for both axes $u$ or $v$. This fact is incompatible with the interval definition for Legendre polynomials that is [-1,+1], so we do one last transformation $T_3$ in order to translate and scale the interval [0,1] into an interval [-1,+1]. To accomplish that we use the following linear formulae

$$u = \frac{1+\xi}{2}; \quad v = \frac{1+\eta}{2}. \tag{13}$$

As this transformation $T_3$ is linear, see Figure 3, thus the Jacobean is constant and equal to

$$J_{T_3} \equiv \frac{\partial(u,v)}{\partial(\xi,\eta)} \equiv \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \Rightarrow \det(J_{T_3}) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}. \tag{14}$$

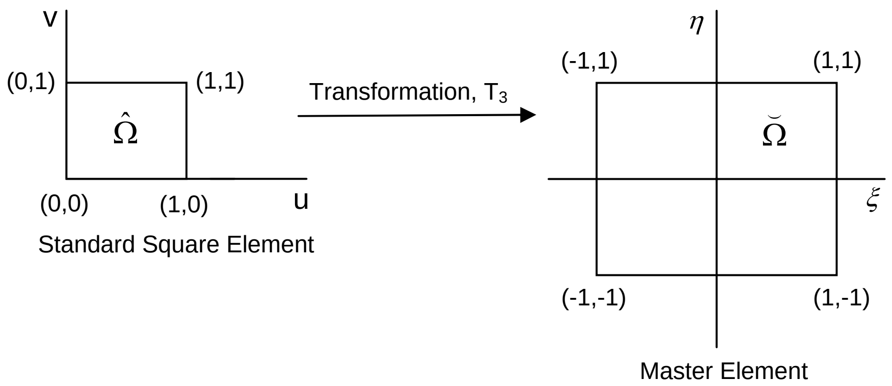


Figure 3. Linear transformation $T_3$ from the standard square element, $\hat{\Omega}$, into the master element, $\breve{\Omega}$.

Thus the integral $I_{\hat{\Omega}}$ in equation (13) can be modified to

$$I_{\hat{\Omega}} \equiv \int_{\hat{\Omega}} f(u,v)(1\text{-}\frac{u+v}{2})dudv = \int_{\breve{\Omega}} f(\xi,\eta)(1\text{-}\frac{2+\xi+\eta}{4})\frac{1}{4}d\xi d\eta \int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)(1\text{-}\frac{2+\xi+\eta}{4})\frac{1}{4}d\xi d\eta \tag{15}$$

If we do the following definition

$$g(\xi,\eta) \equiv \frac{1}{4}(1\text{-}\frac{2+\xi+\eta}{4})f(\xi,\eta), \tag{16}$$

then equation (15) becomes

$$I_{\breve{\Omega}} = \int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\eta)d\xi d\eta. \tag{17}$$

Considering $m$ – Gauss-Legendre integration points for the axis $\xi$ and $n$ – integration points for the axis $\eta$, thus the integral $I_{\breve{\Omega}}$ can be approximated as

$$I_{\breve{\Omega}} \cong \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j g(\xi_i,\eta_j) = \sum_{k=1}^{m\times n} w_k f(\xi_i,\eta_j); \quad k \equiv i \times j; \quad w_k \equiv \frac{1}{4}(1-\frac{2+\xi_i+\eta_j}{4})w_i w_j, \tag{18}$$

where $\xi_i$ and $w_i$, $i = 1,2,\ldots,m$ are the Gauss-Legendre integration points and weights for the $\xi$-axis, and $\eta_j$ and $w_j$, $j = 1,2,\ldots,n$ are the integration points and weights for the $\eta$-axis. It is interesting to remember that

$$f(\mu_k,\lambda_k) = f[(1\text{-}\frac{v_j}{2})u_i,(1\text{-}\frac{u_i}{2})v_j] = f[(1\text{-}\frac{1+\eta_j}{4})\frac{1+\xi_i}{2},(1\text{-}\frac{1+\xi_i}{4})\frac{1+\eta_j}{2}]. \tag{19}$$

Once obtained an approximation to the integral $I_{\breve{\Omega}}$ it is possible to obtain through equations (12), (15), (18) to obtain an approximation for the integral $I_{\Omega}$ as intended initially

$$I_{\Omega} = 2AI_{\tilde{\Omega}} = 2AI_{\hat{\Omega}} = 2AI_{\breve{\Omega}} = 2A\sum_{k=1}^{m\times n} w_k f(\mu_k,\lambda_k). \tag{20}$$

For the previous equation (20) the Gauss-Legendre points and weights for integration over a triangular domain were tabulated and are shown in Table 1.

Table 1. Gauss-Legendre points and weights for numerical integration.

| $\mu_k$ | $\lambda_k$ | $w_k$ |
|---|---|---|
| **m = n = 2** | | |
| 0.1889958E+00 | 0.1889958E+00 | 0.1971688E+00 |
| 0.7053418E+00 | 0.1279915E+00 | 0.1250000E+00 |
| 0.1279915E+00 | 0.7053418E+00 | 0.1250000E+00 |
| 0.4776709E+00 | 0.4776709E+00 | 0.5283122E-01 |
| **m = n = 3** | | |
| 0.1063508E+00 | 0.1063508E+00 | 0.6846439E-01 |
| 0.4718246E+00 | 0.8452624E-01 | 0.8563571E-01 |
| 0.8372983E+00 | 0.6270166E-01 | 0.3858025E-01 |
| 0.8452624E-01 | 0.4718246E+00 | 0.8563571E-01 |
| 0.3750000E+00 | 0.3750000E+00 | 0.9876543E-01 |
| 0.6654738E+00 | 0.2781754E+00 | 0.3782109E-01 |
| 0.6270166E-01 | 0.8372983E+00 | 0.3858025E-01 |
| 0.2781754E+00 | 0.6654738E+00 | 0.3782109E-01 |
| 0.4936492E+00 | 0.4936492E+00 | 0.8696116E-02 |
| **m = n = 4** | | |
| 0.6702145E-01 | 0.6702145E-01 | 0.2815038E-01 |
| 0.3185529E+00 | 0.5797526E-01 | 0.4538621E-01 |
| 0.6467312E+00 | 0.4617250E-01 | 0.3574555E-01 |
| 0.8982626E+00 | 0.3712631E-01 | 0.1512537E-01 |
| 0.5797526E-01 | 0.3185529E+00 | 0.4538621E-01 |
| 0.2755563E+00 | 0.2755563E+00 | 0.7123562E-01 |
| 0.5594389E+00 | 0.2194579E+00 | 0.5316166E-01 |
| 0.7770200E+00 | 0.1764613E+00 | 0.2096742E-01 |
| 0.4617250E-01 | 0.6467312E+00 | 0.3574555E-01 |
| 0.2194579E+00 | 0.5594389E+00 | 0.5316166E-01 |
| 0.4455469E+00 | 0.4455469E+00 | 0.3508770E-01 |
| 0.6188322E+00 | 0.3582546E+00 | 0.1132675E-01 |
| 0.3712631E-01 | 0.8982626E+00 | 0.1512537E-01 |
| 0.1764613E+00 | 0.7770200E+00 | 0.2096742E-01 |
| 0.3582546E+00 | 0.6188322E+00 | 0.1132675E-01 |
| 0.4975896E+00 | 0.4975896E+00 | 0.2100365E-02 |

Those results were produced using a computational application developed and coded within a FORTRAN95 compiler.

## 3. RESULTS

### 3.1. Doing integration numerically

Here are presented some examples of function integration over triangular domains by using equation (20), developed in this work. It is done integration over two right triangles, one equilateral triangle, one isosceles triangle and one scalene triangle, respectively. All following results converged to the exact value, obtained by analytical integration, for all decimal places shown, and using up to four Gauss-Legendre integration points for each axis or up to sixteen points for the two directions.

- Right triangles with corners at (0,0), (1,0) and (0,1)

$$I = \int_0^1 \int_0^{1-y} x\sqrt{1-y}\; dxdy = \frac{1}{7} = 0.142857142 \tag{21}$$

$$I = \int_0^1 \int_0^{1-y} \sqrt{x+y}\ dxdy = \frac{2}{5} = 0.400000000 \qquad (22)$$

- Equilateral triangle with corners at (0,0), (0,2) and ($\sqrt{3}$,1)

$$I = \int_0^{\sqrt{3}} \int_{\frac{\sqrt{3}}{3}x}^{\frac{-\sqrt{3}}{3}x+2} e^{x+y}\ dydx = 9.763139379 \qquad (23)$$

- Isosceles triangle with corners at (-4,1), (-1,1) and (-2,5 ,-3)

$$I = \int_{-3}^{1} \int_{\frac{-3y-29}{8}}^{\frac{3y-11}{8}} (2x+y)^3\ dxdy = -1128 \qquad (24)$$

- Scalene triangle with corners at (-3,-2), (-2,1) and (5,-1)

$$I = \int_{-3}^{-2} \int_{\frac{x-13}{8}}^{3x+7} x^2(y+1)^3\ dydx + \int_{-2}^{-5} \int_{\frac{x-13}{8}}^{\frac{-2x+3}{7}} x^2(y+1)^3\ dydx = \frac{4301}{420} = 10.240476190 \qquad (25)$$

### 3.2. Integrating over meshes with triangular elements

Numerical integration of function over meshes is a crucial part of the spatial discretization process of a given problem by using the Finite Element Method. Such division of a problem domain into sub domains, generally, is done using triangular or quadrilateral finite elements. Now we show that the methodology developed in this paper works well to integrate function over triangular meshes.

Meshes used in this paper were produce using software developed and coded by Aparecido (2006).

#### 3.2.1. Mesh 1 – Unitary Square

Next is presented the mesh data and drawing. In the mesh data it is shown: total number of elements; total number of nodes; node number; x-coordinate value; y-coordinate value; element number; element type; and element nodes.

Table 2. Triangular mesh data.

Mesh=2D - GerMal2D v1.0.1
(Mesh number of elements; Mesh number of nodes)

| | |
|---|---|
| 8 | 9 |

(Node number; Node x-coordinate; Node y-coordinate)

| | | |
|---|---|---|
| 1 | 0,000000000000000E+000 | 0,000000000000000E+000 |
| 2 | 5,000000000000000E-001 | 0,000000000000000E+000 |
| 3 | 1,000000000000000E+000 | 0,000000000000000E+000 |
| 4 | 0,000000000000000E+000 | 5,000000000000000E-001 |
| 5 | 5,000000000000000E-001 | 5,000000000000000E-001 |
| 6 | 1,000000000000000E+000 | 5,000000000000000E-001 |
| 7 | 0,000000000000000E+000 | 1,000000000000000E+000 |
| 8 | 5,000000000000000E-001 | 1,000000000000000E+000 |
| 9 | 1,000000000000000E+000 | 1,000000000000000E+000 |

(Element number; Element type; Element nodes)

| | | | | |
|---|---|---|---|---|
| 1 | TRG01 | 1 | 5 | 4 |
| 2 | TRG01 | 1 | 2 | 5 |
| 3 | TRG01 | 2 | 6 | 5 |
| 4 | TRG01 | 2 | 3 | 6 |
| 5 | TRG01 | 4 | 8 | 7 |
| 6 | TRG01 | 4 | 5 | 8 |
| 7 | TRG01 | 5 | 9 | 8 |
| 8 | TRG01 | 5 | 6 | 9 |

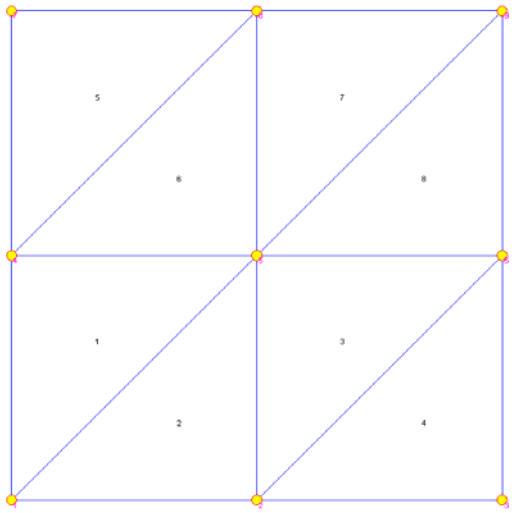

Figure 4. Triangular mesh drawing with 8 elements and 9 nodes.

Table 3. Results for integration over the elements of the Mesh 1.

| | $f(x,y) = e^{x+y}$ | | | | |
|---|---|---|---|---|---|
| | **Number of Gauss-Legendre points** | | | | |
| **Elements** | **m = n = 1** | **m = n = 3** | **m = n = 5** | **m = n = 10** | **m = n = 80** |
| **1** | 0,219381832 | 0,210419759 | 0,210419643 | 0,210419643 | 0,210419643 |
| **2** | 0,219381832 | 0,210419759 | 0,210419643 | 0,210419643 | 0,210419643 |
| **3** | 0,361699493 | 0,346923532 | 0,346923342 | 0,346923342 | 0,346923342 |
| **4** | 0,361699493 | 0,346923532 | 0,346923342 | 0,346923342 | 0,346923342 |
| **5** | 0,361699493 | 0,346923532 | 0,346923342 | 0,346923342 | 0,346923342 |
| **6** | 0,361699493 | 0,346923532 | 0,346923342 | 0,346923342 | 0,346923342 |
| **7** | 0,596341647 | 0,571980207 | 0,571979893 | 0,571979893 | 0,571979893 |
| **8** | 0,596341647 | 0,571980207 | 0,571979893 | 0,571979893 | 0,571979893 |

### 3.2.2. Mesh 2 – Annular Sector

This mesh has an annular sector shape with center at (2,2), inner radius equal to 10, outer radius equal to 20, anti-clock starting angle equal to 30º, and finishing angle equal to 120º, with 30 triangular elements and 24 nodes. Figure 5 show mesh aspect and some data. Global mesh data were omitted due to lack of space.

Table 4. Results for numerical integration over triangular elements of an annular sector shaped mesh.

| | $f(x,y) = (2x+y)^3$ | | | | |
|---|---|---|---|---|---|
| | **Number of Gauss-Legendre points** | | | | |
| **Elements** | **m = n = 1** | **m = n = 3** | **m = n = 5** | **m = n = 10** | **m = n = 80** |
| **1** | 98140,094587567 | 100088,120971646 | 100088,120971645 | 100088,120971645 | 100088,120971645 |
| **2** | 162348,062491771 | 159890,165718841 | 159890,165718840 | 159890,165718840 | 159890,165718840 |
| **3** | 176848,248156279 | 181009,320656773 | 181009,320656771 | 181009,320656771 | 181009,320656771 |
| **4** | 280283,380905540 | 277337,484854510 | 277337,484854508 | 277337,484854508 | 277337,484854508 |
| **5** | 295135,595971858 | 302995,483935827 | 302955,483935824 | 302955,483935824 | 302955,483935824 |
| **6** | 453007,308014356 | 449932,534730965 | 449932,534730961 | 449932,534730961 | 449932,534730960 |
| **7** | 464405,062915799 | 477842,602239938 | 477842,602239933 | 477842,602239933 | 477842,602239933 |
| **8** | 695299,773725571 | 692705,873971134 | 692705,873971127 | 692705,873971128 | 692705,873971127 |
| **9** | 697499,790018244 | 719103,854480729 | 719103,854480722 | 719103,854480723 | 719103,854480725 |
| **10** | 1023705,62175699 | 1022500,78093151 | 1022500,78093150 | 1022500,78093150 | 1022500,78093150 |
| **11** | 33837,1218891009 | 38399,0255259639 | 38399,0255259635 | 38399,0255259635 | 38399,0255259635 |
| **12** | 78486,9654792743 | 81666,0569996588 | 81666,0569996580 | 81666,0569996581 | 81666,0569996582 |
| **13** | 58375,0793615302 | 67085,772428400 | 67085,772428400 | 67085,772428400 | 67085,772428399 |
| **14** | 132413,118174273 | 138912,061702864 | 138912,061702863 | 138912,061702863 | 138912,061702863 |
| **15** | 94144,130780128 | 109308,550012289 | 109308,550012288 | 109308,550012288 | 109308,550012288 |
| **16** | 210161,816564474 | 221948,927176831 | 221948,927176828 | 221948,927176829 | 221948,927176830 |
| **17** | 144110,266967487 | 168753,017885327 | 168753,017885325 | 168753,017885325 | 168753,017885325 |
| **18** | 317871,938205747 | 337551,530994278 | 337551,530994275 | 337551,530994275 | 337551,530994277 |
| **19** | 211585,134961503 | 249547,583298554 | 249547,583298551 | 249547,583298551 | 249547,583298551 |
| **20** | 462385,326004949 | 493285,187546262 | 493285,187546257 | 493285,187546257 | 493285,187546256 |
| **21** | 2021,777556002 | 3884,188185626 | 3884,188185626 | 3884,188185626 | 3884,188185626 |
| **22** | 11431,7592114370 | 44563,9188452999 | 44563,9188452997 | 44563,9188452997 | 44563,9188452997 |
| **23** | 2732,530823015 | 5928,060465467 | 5928,060465467 | 5928,060465467 | 5928,060465467 |
| **24** | 17401,719793995 | 22889,544900026 | 22889,544900026 | 22889,544900026 | 22889,544900026 |
| **25** | 3574,312846116 | 8681,747469442 | 8681,747469442 | 8681,747469442 | 8681,747469442 |
| **26** | 25392,235070887 | 34332,682164398 | 34332,682164398 | 34332,682164398 | 34332,682164398 |
| **27** | 4560,791272289 | 12298,259689498 | 12298,259689498 | 12298,259689498 | 12298,259689498 |
| **28** | 35807,537228552 | 49587,319776376 | 49587,319776376 | 49587,319776376 | 49587,319776376 |
| **29** | 5706,275535407 | 16945,512182855 | 16945,512182855 | 16945,512182855 | 16945,512182855 |
| **30** | 49088,313676790 | 69417,173015240 | 69417,173015240 | 69417,173015240 | 69417,173015240 |

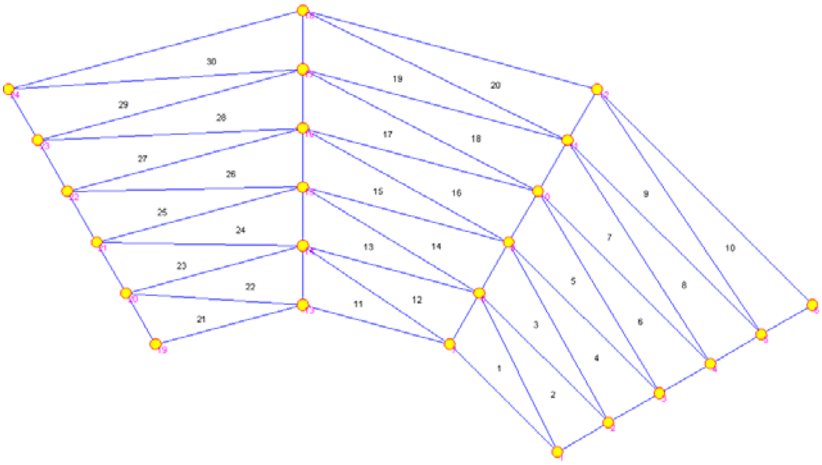

Figure 5. Annular sector mesh with 30 elements and 24 nodes.

### 3.2.3. Mesh 3 – Quadrilateral Polygon

Mesh 3 is quadrilateral polygon with corners at (-5,6); (5,-6); (-8,12) and (2,9). Such domain was divided into 24 elements having 20 nodes. Mesh data were also omitted due to its big extension. Following we show the mesh drawing in Figure 6.

Table 5. Numerical values for integration over the triangular elements with the quadrilateral mesh.

| | $f(x,y)=(2x+y)^3$ | | | | |
|---|---|---|---|---|---|
| | **Number of Gauss-Legendre points** | | | | |
| **Elements** | **m = n = 1** | **m = n = 3** | **m = n = 5** | **m = n = 10** | **m = n = 80** |
| **1** | -9,740950584 | -41,533852358 | -41,533852358 | -41,533852358 | -41,533852358 |
| **2** | -1,098930358 | -29398480902 | -29398480902 | -29398480902 | -29398480902 |
| **3** | 390,429317368 | 546,383751085 | 546,383751085 | 546,383751085 | 546,383751085 |
| **4** | 400,082270869 | 669,701620370 | 669,701620370 | 669,701620370 | 669,701620370 |
| **5** | 8956,551444124 | 8854,747137225 | 8854,747137225 | 8854,747137225 | 8854,747137225 |
| **6** | 5922,472035443 | 7196,984736689 | 7196,984736689 | 7196,984736689 | 7196,984736689 |
| **7** | -0,02513122558 | -11,329832175 | -11,329832175 | -11,329832175 | -11,329832175 |
| **8** | 72,703402519 | 142,965695529 | 142,965695529 | 142,965695529 | 142,965695529 |
| **9** | 5959,573825412 | 6436,834516059 | 6436,834516059 | 6436,834516059 | 6436,834516059 |
| **10** | 11043,146947790 | 10928,773804615 | 10928,773804615 | 10928,773804615 | 10928,773804615 |
| **11** | 75096,779265792 | 71315,516261574 | 71315,516261574 | 71315,516261574 | 71315,516261574 |
| **12** | 86411,279197092 | 83060,203079789 | 83060,203079788 | 83060,203079788 | 83060,203079788 |
| **13** | 9,447358131 | 213,178387225 | 213,178387225 | 213,178387225 | 213,178387225 |
| **14** | 1026,827333450 | 1287,180457899 | 1287,180457899 | 1287,180457899 | 1287,180457899 |
| **15** | 27099,827008459 | 28003,457156033 | 28003,457156032 | 28003,457156032 | 28003,457156033 |
| **16** | 56363,440552322 | 52748,995891204 | 52748,995891203 | 52748,995891203 | 52748,995891203 |
| **17** | 275374,093240490 | 261721,678452331 | 261721,678452329 | 261721,678452329 | 261721,678452329 |
| **18** | 370522,000745491 | 346592,230016641 | 346592,230016637 | 346592,230016637 | 346592,230016638 |
| **19** | 119,832275390 | 966,194907407 | 966,194907407 | 966,194907407 | 966,194907407 |
| **20** | 4569,449475288 | 5070,311235894 | 5070,311235894 | 5070,311235894 | 5070,311235894 |
| **21** | 78841,103722466 | 80279,503624133 | 80279,503624132 | 80279,503624132 | 80279,503624132 |
| **22** | 170570,998408566 | 157316,158895763 | 157316,158895762 | 157316,158895762 | 157316,158895762 |
| **23** | 710583,852502895 | 677803,214178248 | 677803,214178241 | 677803,214178241 | 677803,214178243 |
| **24** | 1024583,82502750 | 952341,698359746 | 952341,698359737 | 952341,698359738 | 952341,698359740 |

Integration over mesh elements was accurate and the results shown in Tables 3-5 have very good convergence. So, the methodology developed here has a great potential as an integration tool to be applied to several kind of problems in which is necessary integration over triangular domains. Meshes presented in this work are small just for example purpose, but this methodology is scalable up to meshes with millions of elements. The mesh generator developed by Aparecido (2006) and used in this work has a friendly graphic interface that allows a comprehensive visualization of the mesh and an easy understanding of elements and nodes relationship.

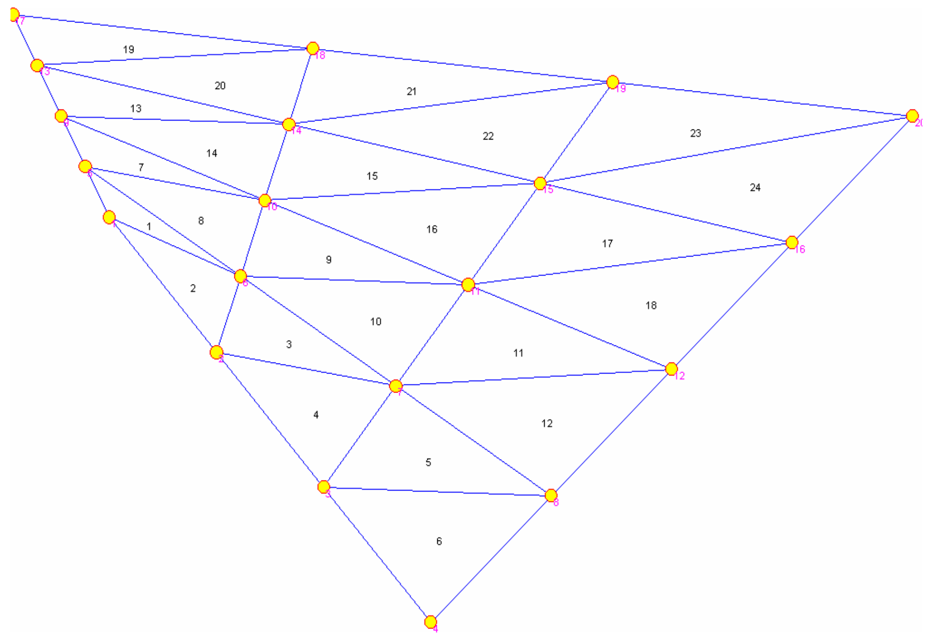


Figure 6. Quadrilateral mesh with 24 elements and 20 nodes.

## 4. CONCLUSIONS

Numerical solution of integrals is an issue that is necessary in several applications. When the integrand function do not allow an analytic integration; when the domain geometry has great complexity, and so son, a way to accomplish such task is numerical integration. There are in the related literature some integration methods that were developed aiming to solve a variety of different kinds of integration. In this work we tried to develop a methodology that has a strong theoretical basis and is suitable algorithm to computational implementation.

Integrands that appear in Section 3.1 were chosen in order to let know the reader about the analytic solution of each integral as well to verify that the numerical solution obtained using the formulation (20) developed in this paper provides results that agreed very well with the analytic ones. For that agreement it was necessary just to use up to four Gauss-Legendre integration points for each direction or up to sixteen points for the two-dimensional domain.

Integrals applied over meshes and presented in Section 3.2 also presented very good results when done using the Gauss-Legendre Quadrature methodology developed in this work for integration over triangular domains. Several others cases were tested and our experience show that is possible to obtain convergence with up to eight decimal places using just three Gauss-Legendre integration points for each direction or nine integration points for two axes. Using eighty Gauss-Legendre points is not necessary we showed it just for testing and documenting. High degree of Gauss-Legendre integration is necessary just for very complex function defined over big domains. For simple function or for small domains integration degrees of 3, 4 or 5, generally, are enough. This fact is enough to set viable this methodology in applications to solve Fluid Mechanics and Heat Transfer, among others problems, by the Method of Finite Elements. Also, this methodology can be used in any method in which is needed integration over triangular domains. Gauss-Legendre points and weights here developed and shown were derived directly from the classic and well-posed Gauss-Legendre points and weights for one dimensional integration problems, thus avoiding other techniques that may need new ones weights and integration points for integration over two dimensional domains. We believe that this technique can be successfully extended to three or higher dimensionality integrations.

Jiang (1992), Tang, Cheng and Tsang (1995), Winterscheidt and Surana (1993), Codina (1998) and several others authors validated the Finite Element Method obtaining good results when solving Fluid Mechanics and Heat Transfer problems. Finally, this methodology developed here has shown to be simples, robust and reliable to accomplish integrations over two dimensional triangular domain with accuracy and, relative, low computational costs. Also, this technique is very useful working together the Finite Element Method, providing a tool to compute several integrations that appear during application of such method.

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## 6. RESPONSIBILITY NOTICE

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