ORBITAL MANEUVERS USING THE LAGRANGIAN POINTS

Regina Elaine Santos Cabette
recabette@uol.com.br
Antonio Fernando Bertachini de Almeida Prado
prado@dem.inpe.br
Instituto Nacional de Pesquisas Espaciais – INPE, Av. dos Astronautas, 1758 – São José dos Campos – SP – 12227-010

Abstract. The well-known Lagrangian points that appear in the planar restricted three-body problem are very important points for astronautical applications. They are five points of equilibrium in the equations of motion, what means that a particle located at one of those points with zero velocity will remain there indefinitely. The collinear points (L_1, L_2 and L_3) are always unstable and the triangular points (L_4 and L_5) are stable in the case studied in this paper (Earth-Sun system). They are all very good points to locate a space-station, since they require a small amount of ΔV (and fuel) for station-keeping. The triangular points are especially good for this purpose, since they are stable equilibrium points. In this paper, the elliptic planar restricted three-body problem is combined with numeric integration and the conjugate gradient method to solve the two point boundary value problem, that can be formulated as: "Find an orbit (in the elliptic three-body problem context) that makes a spacecraft to leave a given point A and goes to another given point B, arriving there after a specified time of flight". Then, by varying the specified time of flight it is possible to find a whole family of transfer orbits and study them in terms of the ΔV required, energy, initial flight path angle, etc. To solve this problem the following steps are used: i) Guess a initial velocity $\vec{V}_i$, so together with the initial prescribed position $\vec{r}_i$ the complete initial state is known; ii) Integrate the equations of motion from $t_0 = 0$ until t_f; iii) Check the final position $\vec{r}_f$ obtained from the numerical integration with the prescribed final position and the final time with the specified time of flight. If there is an agreement (difference less than a specified error allowed) the solution is found and the process can stop here. If there is no agreement, an increment in the initial guessed velocity $\vec{V}_i$ is made and the process goes back to step i). The method used to find the increment in the guessed variables ithe conjugate gradient method. This combination is applied to the search of families of transfer orbits between the Lagrangian points and the two primaries of the Earth-Sun system, with the minimum possible energy.

***Keywords**: Orbital Maneuvers, Lagrangian Points, Astrodynamics, Space Travel, Elliptic Restricted Three-Body Problem.*

1. INTRODUCTION

This paper has the goal of studying models and methods used for the calculation of optimal orbital trajectories, in the sense of using a small amount of fuel. It considers the transfer of the space vehicle between the Earth and Lagrangian Points in the Sun-Earth system. The Lagrangian Points are determined using the model of the circular restricted three-body problem, but the trajectories are studied with two options for the modeling of the dynamics: the three-dimensional circular restricted problem and the elliptic restricted three-body problem.

So, the problem is to transfer a space vehicle between two given points with the minimal possible amount of fuel. There are several important factors in a transfer of this type, like for example, the time used with the transfer, state of the vehicle, etc. Therefore, in this work, the amount of fuel is the critical element of the maneuvers, although the time required by the maneuver it is also verified.

2. THE PLANAR CIRCULAR RESTRICTED THREE-BODY PROBLEM

The circular planar restricted problem of three-bodies is defined as follows: two bodies revolve around their center of mass under the influence of their mutual gravitational attraction and a third body moves in the plane defined by the two revolving bodies. Therefore, the model assumes that two point masses (M_1 and M_2), called primaries, are orbiting their center of mass in circular orbits and a third body with negligible mass M_3 (not influencing the motion of the M_1 and M_2) is orbiting the primaries. The objective is to find the behavior of the third body M_3.

Being x,y a system of fixed coordinates (inertial) and $\bar{x}, \bar{y}$ a system that rotates with angular velocity n, the same of the motion of M_1 and M_2 and t^* is the time (M_1 and M_2 stay fixed in this system).

The equations of motion of the third particle M_3 are given by (Szebehely, 1967):

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y \end{aligned} \tag{1}$$

$$\Omega = \frac{\left(x^2 + y^2\right)}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{2}$$

where:

$$\begin{aligned} r_1^{\,2} &= (x+\mu)^2 + y^2 \\ r_2^{\,2} &= (x-1+\mu)^2 + y^2 \end{aligned} \tag{3}$$

The canonical system of units is used, and it implies that:

1) The unit of distance is the distance between M_1 and M_2;
2) The unit of mass is $M = m_1 + m_2$;
3) The angular velocity of the motion of M_1 and M_2 is assumed to be one;
4) The gravitational constant G is one;
5) The period of the angular motion (M_1, M_2) is 2π .

The Jacobian integral may be obtained by multiplying the equations of motion Eqs. (1) by $\dot{x}$ (the first one) and $\dot{y}$ (the second one), adding the results and integrating with respect to the time. The result is:

$$\dot{x}^2 + \dot{y}^2 = 2\Omega(x,y) - C \tag{4}$$

2.1. The Lagrangian Points

The Lagrangian points that appear in this system are very important points for astronautical applications. They are five points of equilibrium in the equations of motion, what means that a particle located at one of those points with zero velocity will remain there indefinitely (Szebehely, 1967). Therefore, they are also called stationary points. Making the derivatives to obtain the positions of these points, we obtain:

$$\begin{aligned} \Omega_x &= x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0 \\ \Omega_y &= y\left(1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0 \end{aligned} \tag{5}$$

There are five solutions for this system of equations. The collinear points (L_1, L_2 and L_3) are always unstable and the triangular points (L_4 and L_5) are stable in the case studied in this paper (Earth-Sun system).

2.2. Extension to the Three-dimensional Problem

The equations of motion for the three-dimensional circular restricted problem can be obtained by adding the component out of the plane in the equations obtained above. All the hypotheses are the same. There are many reference systems that can be used in this problem (Szebehely, 1967). In the present research it is used the rotating system, similar to the plane case (Prado, 1996). In these conditions the equations of motion are given by:

$$\ddot{x} - 2\dot{y} = x - (1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x-1+\mu}{r_2^3} \tag{6}$$

$$\ddot{y} + 2\dot{x} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \tag{7}$$

$$\ddot{z} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \tag{8}$$

where r_1 and r_2 are the distances of M_1 and M_2, respectively.

2.3. Extension to the Elliptic Problem

In this situation the motion assumed for the two primaries are elliptic and not circular. The canonical system of units is maintained. Among the several systems of reference existent to describe that problem, we used in this research the inertial system and the pulsating-rotating system. In the fixed system the origin is placed in the mass center of the two massive bodies M_1 and M_2. The horizontal axis $\bar{x}$ is the line that connects M_1 and M_2 (in the initial instant) and the vertical axis $\bar{y}$ is perpendicular to the $\bar{x}$.

In this system M_1 and M_2 follow elliptic trajectories:

$$\bar{x}_1 = -\mu\, r \cos\nu\,, \tag{9}$$
$$\bar{y}_1 = -\mu\, r \sin\nu\,, \tag{10}$$
$$\bar{x}_2 = (1-\mu)\, r \cos\nu\,, \tag{11}$$
$$\bar{y}_2 = (1-\mu)\, r \sin\nu\,, \tag{12}$$

where r is the distance between the primaries, given by:

$$r = \frac{a\left(1-e^2\right)}{(1+e\cos\nu)}\,, \tag{13}$$

and ν is the true anomaly of M_2. The equations of motion of the particle in this system are:

$$\ddot{\bar{x}} = \frac{-(1-\mu)(\bar{x}-\bar{x}_1)}{r_1^3} - \frac{\mu\,(\bar{x}-\bar{x}_2)}{r_2^3} \tag{14}$$

$$\ddot{\bar{y}} = \frac{-(1-\mu)(\bar{y}-\bar{y}_1)}{r_1^3} - \frac{\mu\,(\bar{y}-\bar{y}_2)}{r_2^3} \tag{15}$$

where the two points present the second derivative with respect to the time, r_1 *e* r_2 are the distances between the space vehicle and M_1 and M_2, respectively, given by expressions:

$$r_1^2 = (\bar{x}-\bar{x}_1)^2 + (\bar{y}-\bar{y}_1)^2\,, \tag{16}$$
$$r_2^2 = (\bar{x}-\bar{x}_2)^2 + (\bar{y}-\bar{y}_2)^2\,. \tag{17}$$

In the rotating-pulsating system, the origin is placed in the mass center of the two massive primaries. The horizontal axis (x) is the line that connects the two primaries all time. It rotates with variable angular velocity to follow the trajectories of M_1 and M_2, so that the two massive primaries are always on this axis. The vertical axis (y) rotates with the same angular velocity to remain perpendicular to the horizontal axis. Besides this rotation, the system also pulses, so that, the positions of the massive primaries remain fixed. For this reason, it is necessary to multiply the unit of distance by the value of the instantaneous distance between the primaries (r). Then, the positions of the primaries are given by:

$$x_1 = -\mu\,,\ x_2 = 1-\mu\,,\ \ y_1 = y_2 = 0\,. \tag{18}$$

Therefore, the unit of distance is not constant, but it changes with the distance between the primaries. In this reference system, the equations of motion of the particle are given by:

$$\ddot{x} - 2\dot{y} = \frac{r}{p}\left(x - (1-\mu)\frac{x-x_1}{r_1^3} - \mu\frac{x-x_2}{r_2^3} \right) \tag{19}$$

$$\ddot{y} + 2\dot{x} = \frac{r}{p}\left(y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3} \right) \tag{20}$$

and there is also an equation that relates the time with the true anomaly of the primaries, given by:

$$\dot{t} = \frac{r^2}{p^{1/2}} \tag{21}$$

where the point present the derivative with respect the true anomaly of the primaries and:

$$p = a\left(1-e^2\right) \tag{22}$$

is the semi-lactus rectum of the ellipse.

The equations that relate the two systems are:

$$\bar{x} = rx\cos\nu - ry\sin\nu \tag{23}$$
$$\bar{y} = rx\sin\nu + ry\cos\nu \tag{24}$$
$$x = (\bar{x}\cos\nu + \bar{y}\sin\nu)/r \tag{25}$$
$$y = (\bar{y}\cos\nu - \bar{x}\sin\nu)/r \tag{26}$$

for the positions and:

$$\dot{\bar{x}} = \dot{x}r\cos\nu - \dot{y}r\sin\nu - \frac{x(1-e^2)\sin\nu}{(1+e\cos\nu)^2} - \frac{y(1-e^2)(e+\cos\nu)}{(1+e\cos\nu)^2} \tag{27}$$
$$\dot{\bar{y}} = \dot{x}r\sin\nu + \dot{y}r\cos\nu - \frac{y(1-e^2)\sin\nu}{(1+e\cos\nu)^2} + \frac{x(1-e^2)(e+\cos\nu)}{(1+e\cos\nu)^2} \tag{28}$$
$$\dot{x} = \frac{\dot{\bar{x}}\cos\nu}{r} + \frac{\dot{\bar{y}}\sin\nu}{r} - \frac{\bar{x}(\sin\nu + e\sin 2\nu)}{1-e^2} + \frac{\bar{y}(\cos\nu + 2e\cos^2\nu - e)}{1-e^2} \tag{29}$$
$$\dot{y} = \frac{\dot{\bar{y}}\cos\nu}{r} - \frac{\dot{\bar{x}}\sin\nu}{r} - \frac{\bar{y}(\sin\nu + e\sin 2\nu)}{1-e^2} - \frac{\bar{x}(\cos\nu + 2e\cos^2\nu - e)}{1-e^2} \tag{30}$$

for the velocities.

3. The Two Point Boundary Value Problem

The problem that is considered in the present paper can be formulated as:

”Find an orbit (in the three-body problem context) that makes a spacecraft to leave a given point A and goes to another given point B, arriving there after a specified time of flight".

To solve this problem the following steps are used:

i) Guess a initial velocity $\vec{V}_i$, so together with the initial prescribed position $\vec{r}_i$ the complete initial state is known;

ii) Integrate the equations of motion from $t_0 = 0$ until t_f;

iii) Check the final position $\vec{r}_f$ obtained from the numerical integration with the prescribed final position. If there is an agreement (difference less than a specified error allowed) the solution is found and the process can stop here. If there is no agreement, an increment in the initial guessed velocity $\vec{V}_i$ is made and the process goes back to step i).

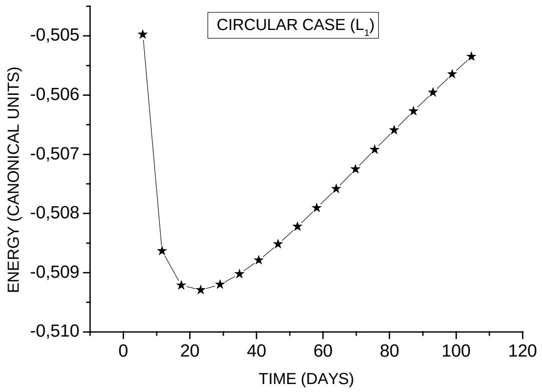
The method used to find the increment in the guessed variables is the conjugate gradient method (Prado, 1996; Press et alli, 1989). This combination is applied to the search of families of transfer orbits between the Lagrangian points and the two primaries of the Earth-Sun system, with the minimum possible energy.

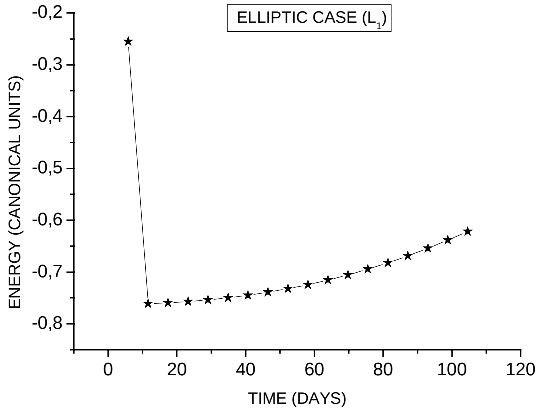
4. Numerical Results

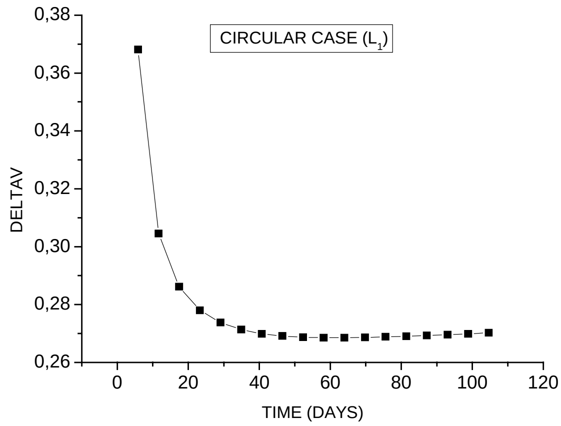
The results are organized in graphs of the energy and the velocity increment (ΔV) in the rotating system against the time of flight. To obtain the minimum values for the transfers, the maneuvers were parameterized in terms of the time of flight and then the minimum energy transfers for each time of flight were computed by the minimization method and compared, to obtain the global minimum.

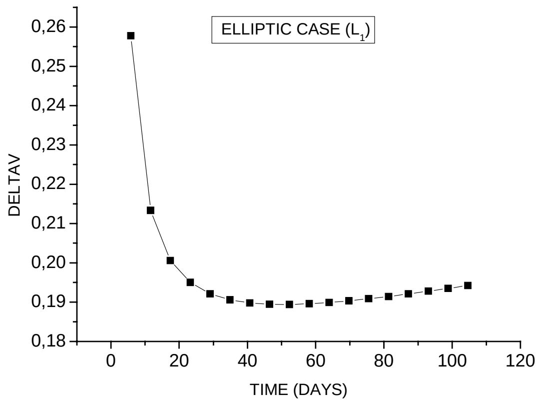
- Trajectories to the L_1

The point equilibrium L_1 is the collinear Lagrangian point that exists between the Sun and the Earth, it is located about 1,496,867 km from the Earth. The results presented below shows the transfer to the Lagrangian point L_1 that was found in this research. For the circular problem, the local minimum for a transfer to the L_1 occurs for a time of flight close to 23 days, requires an energy E = -0.50929. For the elliptic problem (eccentricity e = 0.3), the local minimum for a transfer to the L_1 occurs for a time of flight close to 12 days, requires an energy E = -0.76098.



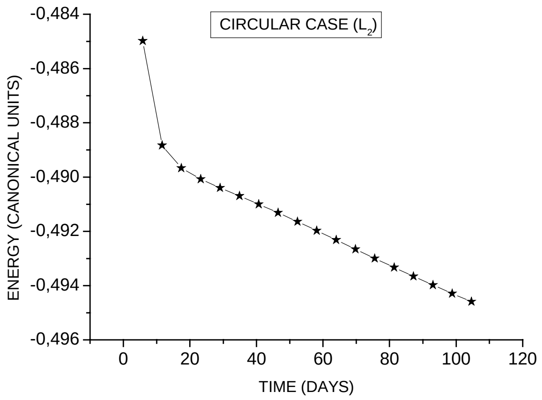


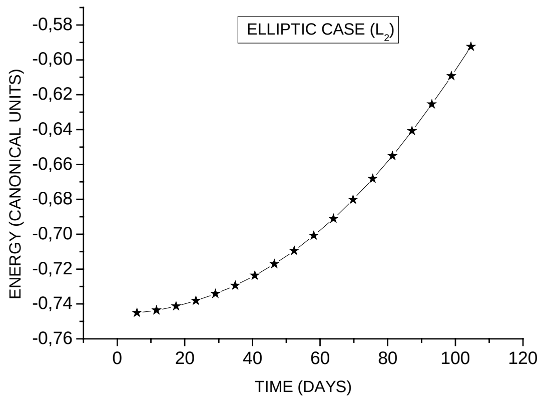


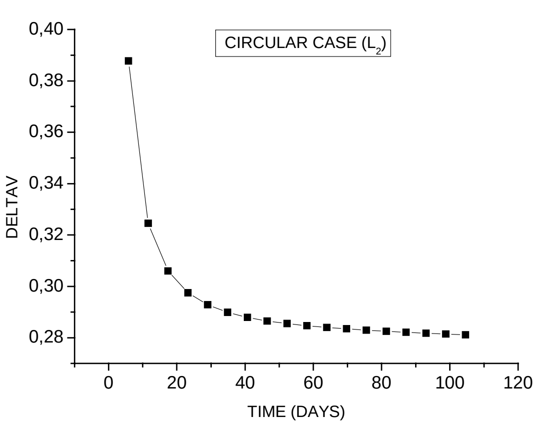


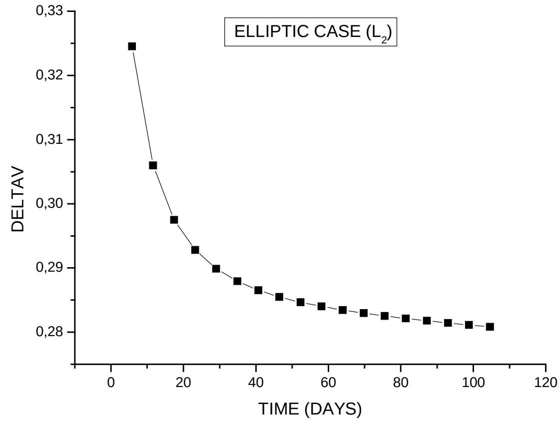
- Trajectories to the L_2

The point equilibrium L_2 is the collinear Lagrangian point that exists behind the Earth, it is located about 1,506,915 km from the Earth. The results presented below shows the transfer to the Lagrangian point L_2 that was found in this research. For the circular problem, the local minimum for a transfer to the L_2 occurs for a time of flight close to 105 days, requires an energy E = -0.49459. For the elliptic problem (eccentricity e = 0.3), the local minimum for a transfer to the L_2 occurs for a time of flight close to 6 days, requires an energy E = -0.745.



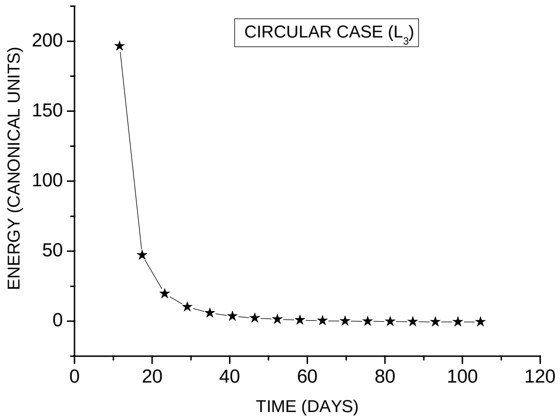


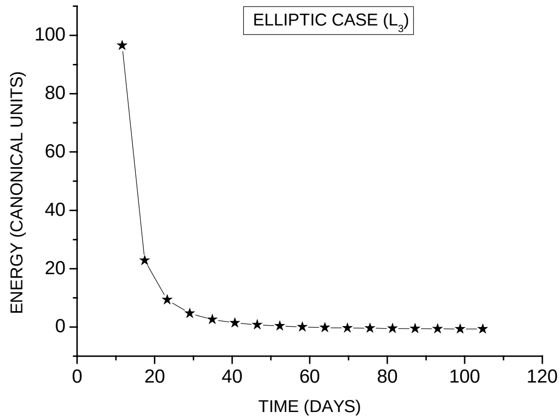


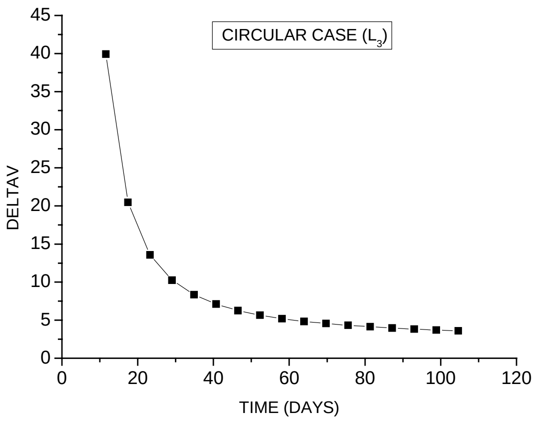


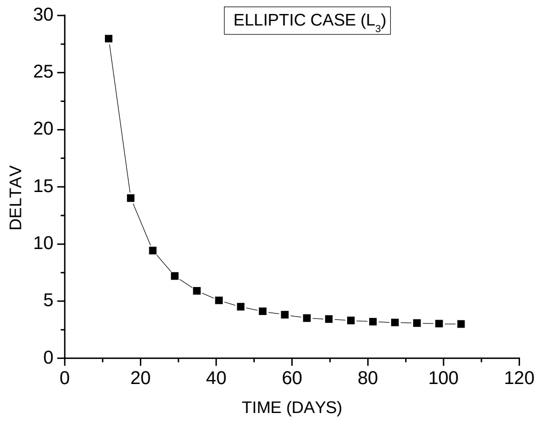
- Trajectories to the L_3

The point equilibrium L_3 is the collinear Lagrangian point that exists on the opposite side of the Sun (when compared to the position of the Earth), it is located about 149,595,740 km from the Sun. The results presented below shows the transfer to the Lagrangian point L_3 that was found in this research. For the circular problem, the local minimum for a transfer to the L_3 occurs for a time of flight close to 105 days, requires an energy E = -0.56587. For the elliptic problem (eccentricity e = 0.3), the local minimum for a transfer to the L_3 occurs for a time of flight close to 105 days, requires an energy E = -0.69794.





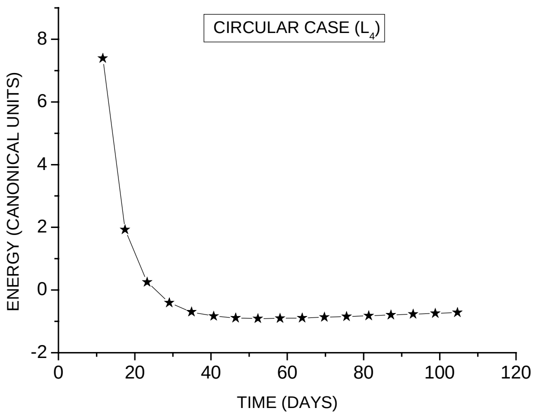


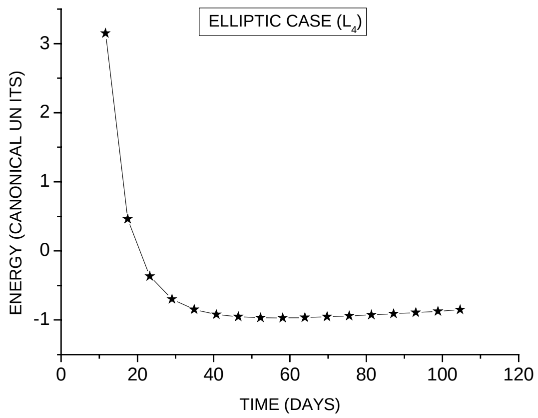


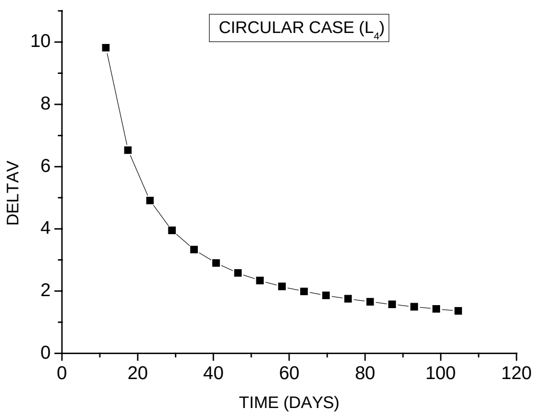
- Trajectories to the L_4

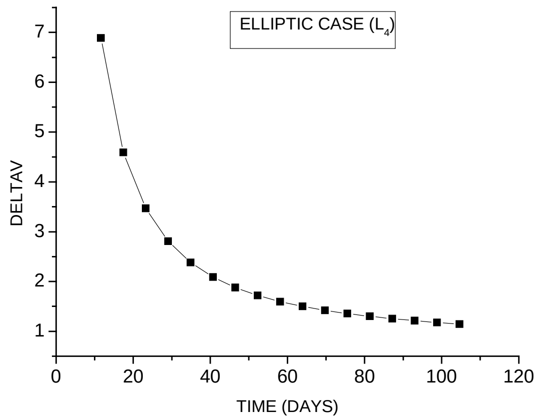
The point equilibrium L_4 is one of the triangular Lagrangian points. It is located on the third vertice of the equilateral triangle formed with the Sun and the Earth, in the semi-plane of positive y. It has the stability property makes the fuel required for station-keeping almost zero, therefore, it is an excellent location for a space station. The results presented below shows the transfer to the Lagrangian point L_4 that was found in this research. For the circular problem, the local minimum for a transfer to the L_4 occurs for a time of flight close to 52 days, requires an energy

E = -0.90708. For the elliptic problem (eccentricity e = 0.3), the local minimum for a transfer to the L_4 occurs for a time of flight close to 58 days, requires an energy E = -0.96855.



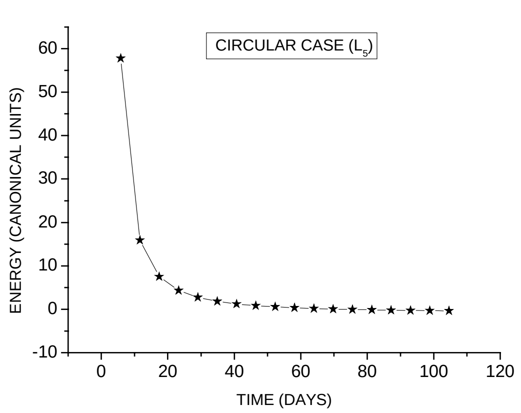


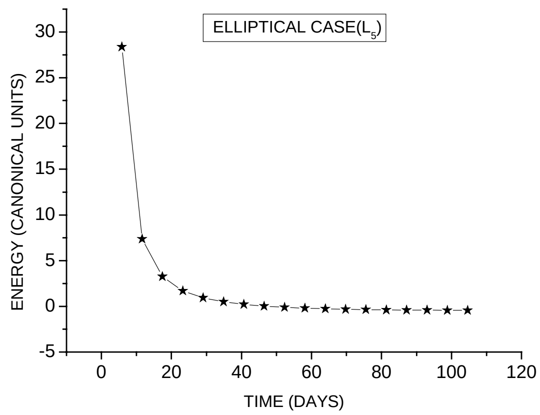


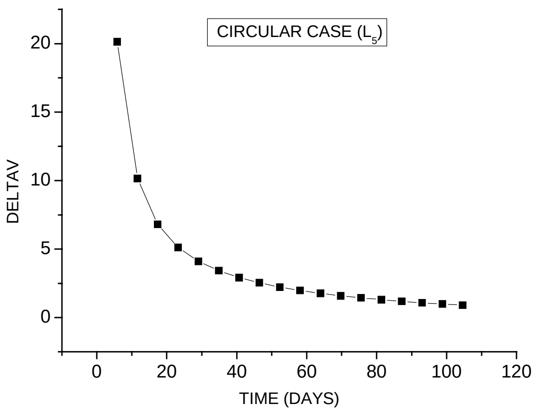


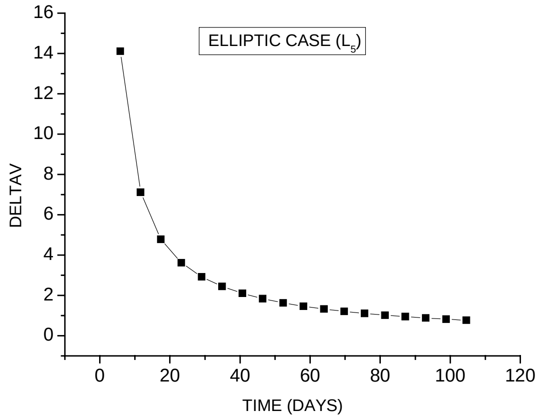
- Trajectories to the L_5

The point equilibrium L_5 is the other triangular Lagrangian point. It is located on the point symmetric to L_4, in the semi-plane of negative y. It is also stable and an important point for the same reasons that L_4. The results presented below shows the transfer to the Lagrangian point L_5 that was found in this research For the circular problem, the local minimum for a transfer to the L_5 occurs for a time of flight close to 105 days, requires an energy E = -0.35012. For the elliptic problem (eccentricity e = 0.3), the local minimum for a transfer to the L_4 occurs for a time of flight close to 105 days, requires an energy E = -0.43628.









5. Conclusions

In this paper, the restricted three-body problem is used to find families of transfer orbits between the Earth and all the five Lagrangian points that exist in the Earth-Sun system.

Two models were used, considering or not the eccentricity of the primaries. From the results, it is clear that the inclusion of the eccentricity can help mission designers to save fuel. The differences are large enough to be used in favor of the mission.

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