THE BOUNDARY ELEMENT METHOD APPLIED TO DYNAMICAL MODELLING OF FOLDED PLATES WITH REINFORCEMENTS

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Abstract. *A dynamic response of reinforced plate structures through the Boundary Element Method (BEM) have been carried out in the present investigation. The dynamic fundamental solutions of the plane stress elasticity and thin plates are used to transform the governing differential equations into displacements boundary integrals. Plane elements containing an association of flexural and in-plane mechanisms were adopted to represent a single folded plate called Macro-element. The assemblage of the Macro-elements is made by compatibility and equilibrium conditions at the element interfaces. Boundary conditions are imposed in the assembled structure. A well-posed problem presents four unknowns and, consequently, four integral equations are required for every node. The strategy is to use two integral equations for the membrane and the two integral equations for the thin plate at every node. Under the assumption of small deformation and small strain, the in-plane motion and the out-of-plane motion are not coupled. A non-singular BEM formulation is implemented. The geometry and the variables are discretized using linear elements. Two distinct collocation points for both the plate Boundary Integral Equation (BIE) and for the membrane BIE are required. For every boundary node four boundary integral equations are written. The stationary dynamic responses of the reinforced structures are characterized by its modal quantities that means, by its eigenfrequencies and eigenvalues. These quantities are obtained by analyzing the numerical Frequency Response Functions (FRF) where a harmonic force of constant amplitude excites the structure at a given point and the resulting displacement is measured at another point. From the resonances or peaks of the FRFs the operational eigenfrequencies may be determined. The operational eigenmodes (vibration mode shapes) are determined by calculating the folded plate structure displacement field at the determined operational eigenfrequencies. The propose scheme was shown to be capable of dealing with models subjected to different boundary conditions and out-of-plane loadings. The results agree very well with published results in the literature.*

Keywords*: Reinforced Panels, Folded Plates, Dynamic Analysis, Boundary Elements, Modal Data.*

1. INTRODUCTION

Reinforced structures systems are functional, efficient, economical and readily constructed of most common materials. Many structures taken out of a production line will exhibit variations on some or all of its geometrical, material and assembling properties and this dispersion will reflect in their behavior. Small changes of the physical properties can significantly affect the results, and the prediction error tends to increase when using models to perform a structural static and dynamic analysis. The static analysis of the reinforced plate system employed methods for the solution such as a methodology based on energy principles (Kukreti and Cheraghi, 1993), a semi analytical method (Mukhopadhay, 1994) or the differential quadrature method (Siddiqi and Kukreti, 1998). Also it is possible to model behavior this structures by the Finite Element Method (FEM) (Deb and Booton, 1988; Palani *et al.*, 1992), the Boundary Element Method (BEM) (Tanaka and Bercin, 1997; Sapountzakis and Katsikadelis, 2000; Tanaka *et al.*, 2000; Wen *et al.*, 2002; Oliveira Neto and Paiva, 2003) or a combination of these numerical methods (Ng and Cheung Xu, 1990). A rather limited amount of technical literature is available on the dynamic analysis of stiffened plate systems. On the other hand, a significant research effort is under way in both the academia and the industry to improve the numerical models and to develop new modeling methods for the dynamic analysis (Arruda and Ahmida, 2003). Finite and boundary elements have some limitations to obtain vibration responses at middle and upper frequency ranges due to the necessity of refining the mesh. Using finite elements fine meshes are required leading to very large algebraic systems. An alternative is posed by the BEM. If formulated with the proper auxiliary state, the BEM only requires boundary discretization, leading to considerable smaller algebraic systems. The direct boundary element sub region formulations based on Kirchhoff's plate theory has been applied to the dynamic analysis of thin-walled structures formed by assembling folded plate models using the so-called static fundamental solution (Tanaka *et al.*, 1988; Tanaka *et al.*, 1998). Assembled plate structures were also analyzed by BEM and comparisons with FEM are given to demonstrate the accuracy of this methodology (Dirgantara and Aliabadi, 2002). Another dynamic analysis of elastic plates reinforced with beams takes into account the resulting in plane forces and deformations in the plate as well as the axial forces and deformations in the beam, due to combined response of the system (Sapountzakis and Katsikadelis, 1999). The presented method employs the static solution similar to the models described previously. The consequence of these formulations is that the inertia forces lead to a domain integral. In these previous articles it was necessary to develop a procedure to deal with the domain integral. A way to derive the governing integral equation for the problem is to use a stationary dynamic fundamental solution as in Beskos (1987), Beskos (1991) and Beskos (1997). If this fundamental solution is applied the resulting integral equation requires only the discretization of the boundary of the single folded plate being analyzed.

The present paper analysis the dynamic stationary of the reinforced panels subjected to time harmonic loadings using the BEM. In the propose methodology the panels are considered as assembled folded plate structure (Sanches et al., 2004). The formulation is built by coupling boundary element formulations of plate bending and two dimensional plane stress elasticity. This uncoupled system is joined to form a Macro-element. The plate structure is divided into several regions, and the equilibrium and compatibility equations along the interface boundaries are imposed. The boundaries are discretized into linear continuous and discontinuous isoparametric elements. Four displacement integral equations are written for every boundary node. The stationary dynamic responses are characterized by its modal quantities that means, by its eigenfrequencies and eigenvalues. These quantities are obtained by analyzing the numerical Frequency Response Functions of the reinforced structure. A harmonic force of constant amplitude excites the structure at a given point and the resulting displacement is measured (calculated) at another point. From the resonances or peaks of the FRFs the operational eigenfrequencies may be determined. The operational eigenmodes (vibration mode shapes) are determined by calculating the folded plate structure displacement field at the determined operational eigenfrequencies. A numerical example is presented to illustrate the proposed methodology. Different configurations of the reinforcements are used to simulate free and simply supported boundary conditions. The implementation is validated by comparison with numerical results determined in the literature. The results obtained are shown to be in good agreement with others formulations and the proposed scheme may be seen as an accurate methodology to analyze free and forced stationary vibrations of structures assembled by folded plates, as for example plate structures and also reinforced panels.

2. BOUNDARY INTEGRAL EQUATIONS

The boundary elements is a well established numerical technique in the academic community. Its formulation, based on the displacement boundary integral equation has been successfully applied to static and dynamic linear elastic problems. The procedure adopted to analyze stationary dynamic reinforced plates is obtained from the direct boundary integral equations to solve the plane stress elasticity and bending problems in each plane element (folded plate). The integral equations are independent due to the small strain hypotheses. Displacements and efforts of each plane element are combined to obtain the final equation using kinematics compatibility and equilibrium equations and the multi-region technique is employed with each plane element representing a region.

The plane element is referred to midline coordinates x_α and thickness coordinate x_3 under the local system coordinates. Throughout the paper, indicial notations are used. Latin indices taking values {1, 2, and 3} and Greek indices assuming the range {1, 2}. Figure 1 presents local reference axes for the plane element with domain Ω and contour Γ. In the dynamic plane stress elasticity problem, u_α represents the displacement values, $\sigma_{\alpha\beta}$ represents de stresses components and the boundary traction's t_α is related with the stress tensor and directions cosines through the boundary outward unit normal n. For the dynamic plate bending problem, q_α (q_1, q_2) represents the shear forces, $m_{\alpha\alpha}$ (m_{11}, m_{22}) represent the bending moments and $m_{\alpha\beta}$ (m_{12}, m_{21}) represent the twisting moments.

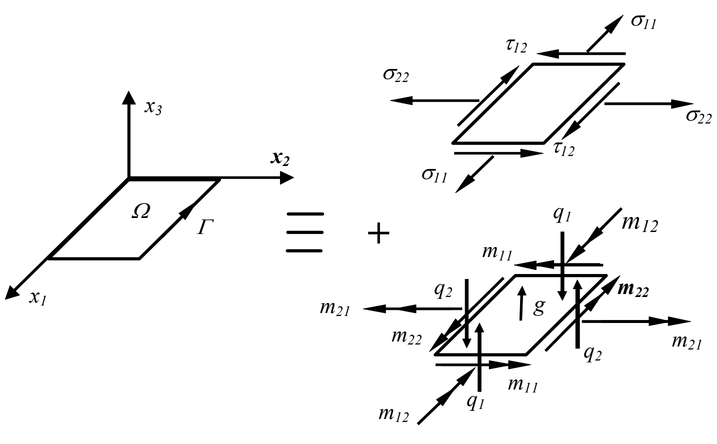


Figure 1. Analyzed domain stress and forces definitions.

The displacement boundary integral equation for the plane stress elasticity problem (membrane) and smooth boundaries is given by

$$\frac{1}{2}K_{\alpha\beta}(P)u_\beta(P) = \int_\Gamma U^*_{\alpha\beta}(P,Q)t_\beta(Q)d\Gamma(Q)$$
$$- \int_\Gamma T^*_{\alpha\beta}(P,Q)u_\beta(Q)d\Gamma(Q) + \int_\Omega U^*_{\alpha\beta}(P,Q)F_\beta(Q)d\Omega(Q) \tag{1}$$

where $K_{\alpha\beta}$ is equal to Kronecker delta for a smooth boundary, $d\Gamma$ and $d\Omega$ denote boundary and domain differentials, respectively; $u_\beta (Q)$ and $t_\beta (Q)$ are displacement and traction boundary values associated with a boundary point Q, respectively. The term $U_{\alpha\beta}^*(Q,P)$ represents a displacement fundamental solution and may be interpreted as the displacement at point Q in the direction α due to a harmonic unit point force applied at the point P in the direction β. Analogously the term $T_{\alpha\beta}^*(Q,P)$ represents the traction fundamental solution and may also be interpreted as the traction at point Q in the direction α due to a harmonic unit point load applied at P in the direction β. Considering that all variables are undergoing a time harmonic displacement, $u(t)=\hat{u}\ exp(i\ \omega\ t)$, with circular frequency ω, they are given by

$$U^*_{\alpha\beta} = \frac{1}{2\pi\rho\, c_2^2}\left[\psi\,\delta_{\alpha\beta} - \chi\, r,_\alpha\, r,_\beta\right] \tag{2}$$

$$T^*_{\alpha\beta} = \frac{1}{2\pi}\left[\left(\frac{d\psi}{dr} - \frac{1}{r}\chi\right)\left(\delta_{\alpha\beta}\frac{\partial r}{\partial n} + r,_\beta\, n_\alpha\right)\right.$$
$$\left. -\frac{2}{r}\chi\left(n_\beta r,_\alpha - 2\, r,_\alpha\, r,_\beta\frac{\partial r}{\partial n}\right) - 2\frac{d\chi}{dr}r,_\alpha\, r,_\beta\frac{\partial r}{\partial n} + \left(\frac{c_1^2}{c_2^2} - 2\right)\left(\frac{d\psi}{dr} - \frac{d\chi}{dr} - \frac{1}{r}\chi\right)r,_\alpha\, n_\beta\right] \tag{3}$$

with

$$\psi = K_0(k_2 r) + \frac{1}{k_2 r}\left[K_1(k_2 r) - \frac{c_2}{c_1}K_1(k_1 r)\right] \quad \text{and} \quad \chi = K_2(k_2 r) - \frac{c_2^2}{c_1^2}K_2(k_1 r) \tag{4}$$

where K_0 and K_1 are the zero and one order modified Bessel function of second kind, r is the distance between load and displacement point, $k_1 = i\ (\omega /c_1)$ e $k_2 = i\ (\omega /c_2)$, $i = \sqrt{-1}$, $c_1 = (\lambda+2\mu/\rho)^{1/2}$, $c_2 = (\mu/\rho)^{1/2}$, ρ is the density and λ e μ are the Lamé's constants. The complete expressions for these fundamental solutions are listed by Domingues (1993).

The boundary integral representation of the displacement components for plate bending problems can be written as

$$\frac{1}{2}K(P)w(P) + \int_\Gamma\left[V_n^*(P,Q)w(Q) - M_n^*(P,Q)w,_n(Q)\right]d\Gamma(Q) + \sum_{i=1}^{Nc}R^*_{ci}(P,c)w_{ci}(P,c)$$
$$= \int_\Gamma\left[w^*(P,Q)V_n(Q) - w,^*_n(P,Q)M_n(Q)\right]d\Gamma(Q) + \sum_{i=1}^{Nc}w^*_{ci}(P,Q)R_{ci}(Q) + \int_\Omega w^*(P,q)g(q)d\Omega(q) \tag{5}$$

In the Equation (5), $K(P)$ is equal to *Kronecker* delta for a smooth boundary, w is the out-of-plane displacement, $w,_n$ is the rotation in the direction of outward normal to the boundary Γ, V_n is the equivalent shear, M_n is the bending moment and R_c is the corner reaction. The classical theory makes use of the equivalent shear (V_n) in boundary integrals and a corner reaction (R_c) at each corner when polygonal plates are considered,

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} = -D\left(w,_{\gamma\gamma\alpha}.n_\alpha + (1-\nu)w,_{nss}\right) \tag{6}$$
$$R_{ck} = \left(M^F_{ns} - M^B_{ns}\right)_k \tag{7}$$

where Q_n is the shear in the direction of outward normal and M_{ns} is the twisting moment in the direction normal and tangential to the boundary Γ. The expression (7) presents the corner reaction (R_c) at corner k as the difference between the twisting moments at the corner neighborhood on the forward side (M_{ns}^F) and the backward side (M_{ns}^B). Again, if that all variables are undergoing a time harmonic displacement, load g and deflections w will also vary harmonically and the fundamental solution of has the form (Vivoli and Filippi, 1974; Niwa et al. 1981)

$$w^{*} = -i\,C_{1}J_{0}(\eta r) + C_{1}Y_{0}(\eta r) + C_{2}K_{0}(\eta r) \tag{8}$$

$$w,_{n}^{*} = i\,C_{1}\eta J_{1}(\eta r)\cos\,\overline{\beta} - \eta\left[C_{1}Y_{1}(\eta r) + C_{2}K_{1}(\eta r)\right]\cos\,\overline{\beta} \tag{9}$$

$$\begin{aligned} M_{n}^{*} = & -i\left\{C_{1}\frac{D}{2}\left[1+\nu+(1-\nu)\cos\,2\overline{\beta}\right]\eta^{2}J_{0}(\eta r) - C_{1}D\,\eta\,(1-\nu)\frac{J_{1}(\eta r)}{r}\cos\,2\overline{\beta}\right\} \\ & + \frac{D}{2}\left\{\eta^{2}\left[1+\nu+(1-\nu)\cos\,2\overline{\beta}\right]\left[C_{1}Y_{0}(\eta r) - C_{2}K_{0}(\eta r)\right]\right. \\ & \left. - 2\eta\,(1-\nu)\frac{1}{r}\left[C_{1}Y_{1}(\eta r) + C_{2}K_{1}(\eta r)\right]\cos\,2\overline{\beta}\right\} \end{aligned} \tag{10}$$

$$\begin{aligned} V_{n}^{*} = & \;iC_{1}D\left\{J_{1}(\eta r)\left[\eta^{3}\cos\,\overline{\beta} + \frac{\eta^{3}(1-\nu)}{2}\text{sen}\,2\overline{\beta}\,\text{sen}\,\overline{\beta} + \frac{2\eta(1-\nu)}{r}\left(\frac{\cos 3\overline{\beta}}{r} - \frac{\cos 2\overline{\beta}}{R}\right)\right]\right. \\ & \left. + (1-\nu)\eta^{2}J_{0}(\eta r)\left(\frac{\cos\,2\overline{\beta}}{R} - \frac{\cos\,3\overline{\beta}}{r}\right)\right\} - D\,\eta^{3}\left[C_{1}Y_{1}(\eta r) - C_{2}K_{1}(\eta r)\right]\cos\,\overline{\beta} \\ & + D(1-\nu)\left\{\frac{\eta^{2}}{r}\left[C_{1}Y_{0}(\eta r) - C_{2}K_{0}(\eta r)\right] - \frac{2\eta}{r^{2}}\left[C_{1}Y_{1}(\eta r) + C_{2}K_{1}(\eta r)\right]\cos\,3\overline{\beta}\right\} \\ & - D(1-\nu)\left\{\frac{\eta^{2}}{R}\left[C_{1}Y_{0}(\eta r) - C_{2}K_{0}(\eta r)\right] - \frac{2\eta}{rR}\left[C_{1}Y_{1}(\eta r) + C_{2}K_{1}(\eta r)\right]\cos\,2\overline{\beta}\right\} \\ & - \frac{D(1-\nu)}{2}\eta^{3}\left[C_{1}Y_{1}(\eta r) - C_{2}K_{1}(\eta r)\right]\text{sen}\,2\overline{\beta}\,\text{sen}\,\overline{\beta} \end{aligned} \tag{11}$$

with

$$C_{1} = \frac{1}{8\eta^{2}}, \qquad C_{2} = \frac{1}{4\pi\eta^{2}} \qquad \text{and} \qquad \eta^{4} = \frac{\rho h\,\omega^{2}}{D} \tag{12}$$

In Equations (8) to (12), J_0 e Y_0 are the zero order Bessel functions of the first and second kind, respectively, K_0 is the zero order modified Bessel function of the second kind, J_1 e Y_1 are the first order Bessel functions of the first and second kind, K_1 is the first order modified Bessel function of the second kind, respectively and β is the angle between r and n. The flexural rigidity D is equal to $Eh^3/[12(1-\nu^2)]$, E is the Young's modulus, ν is the Poisson's ratio, h is the thickness, ω is the circular frequency and ρh is the mass density per unit area.

2.1. Statement of the Model

The plane element presented in Figure 1, will be assembled by superposition of the membrane and thin plate effects to form the called Macro-element. The plane stress elasticity Boundary Integral Equation (1) representing the membrane may be discretized leading to the following algebraic system of equations:

$$\begin{bmatrix} H_{11}^{m} & H_{12}^{m} \\ H_{21}^{m} & H_{22}^{m} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} G_{11}^{m} & G_{12}^{m} \\ G_{21}^{m} & G_{22}^{m} \end{bmatrix}\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{13}$$

Analogously the BIE (5) describing the out-of-plane bending effect (thin plate) may be discretized as follows:

$$\begin{bmatrix} H_{11}^{p} & H_{12}^{p} \\ H_{21}^{p} & H_{22}^{p} \end{bmatrix}\begin{Bmatrix} w \\ w,_{n} \end{Bmatrix} = \begin{bmatrix} G_{11}^{p} & G_{12}^{p} \\ G_{21}^{p} & G_{22}^{p} \end{bmatrix}\begin{Bmatrix} V_n \\ M_n \end{Bmatrix} \tag{14}$$

In Equations (13) and (14) the upper indices m and p on the coefficient matrices H and G stand, respectively, for membrane and plate mechanisms. Furthermore u_1 and u_2 represent the in plane membrane displacements associated with the in plane tractions t_1 and t_2. The plate displacement normal to the x_1-x_2 plane is w and its derivative with respect to the boundary normal n is $w_{,n}$. The corresponding generalized forces are the shear forces V_n and the bending moment M_n. These Equations may be superposed to form the plane Macro-element in which membrane and bending mechanisms are uncoupled:

$$\begin{bmatrix} H_{11}^m & H_{12}^m & 0 & 0 \\ H_{21}^m & H_{22}^m & 0 & 0 \\ 0 & 0 & H_{11}^p & H_{12}^p \\ 0 & 0 & H_{21}^p & H_{22}^p \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ w \\ w,_n \end{Bmatrix} = \begin{bmatrix} G_{11}^m & G_{12}^m & 0 & 0 \\ G_{21}^m & G_{22}^m & 0 & 0 \\ 0 & 0 & G_{11}^p & G_{12}^p \\ 0 & 0 & G_{21}^p & G_{22}^p \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ V_n \\ M_n \end{Bmatrix} \tag{15}$$

The plane Macro element given by equation (15) is written in terms of a local coordinate system ($x_1' - x_2' - x_3'$), shown in Figure 2. To perform the coupling of distinct Macro elements it is necessary to transform this equation from a local to a global coordinate system. This is done by means of an intermediate coordinate system and a set of two coordinate transformation matrices (Palermo Jr., 1992). The local and global systems are presented in Figure 2. The x_3 axis represents the longitudinal length. An intermediate coordinate system was employed in the change of coordinates

$$x_l' = \bar{T}_{lj}\, x_j'' \qquad \text{and} \qquad x_i' = T_{il} x_l \tag{16}$$

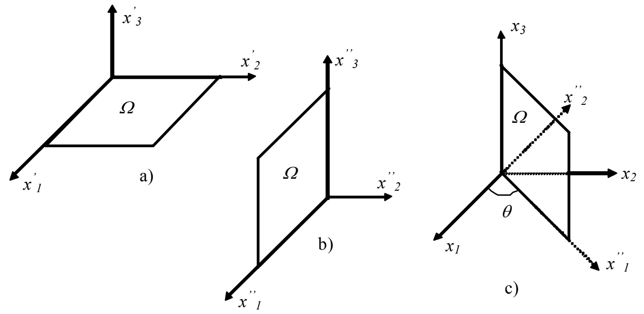


Figure 2. Coordinates - a) local, b) intermediate, c) global

The local system is noted by x_i', the intermediate system by x_i'' and the global system by x_i. The transformation matrices are given by

$$\bar{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad T = \begin{bmatrix} \cos\theta & sen\theta & 0 \\ -sen\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

It is notice that the direction of the bending moment and normal rotation vectors at the Macro-element side should be changed to obtain the global matrices when their directions do not agree with the global coordinate's system direction. This procedure was presented by Palermo Jr. (1992).

2.2. Multi-region Formulation of the Macro-elements

After the Macro-element equations have been written in terms of the global coordinate system the assemblage may take place. The interface boundaries between Macro-elements must be parallel to a single axis. In a global coordinate system this axis is called x_3 as shown in Figure 3.

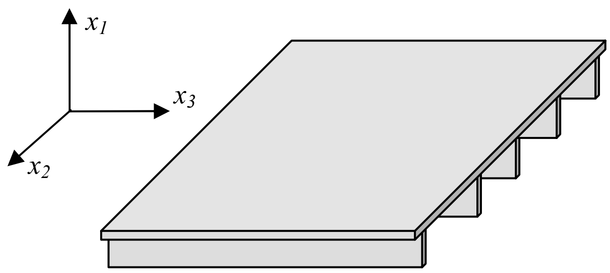


Figure 3: Global coordinate system and Macro-elements interface boundaries.

Figure 3 also shows a plate structure with reinforcements. It can be noticed that the reinforcements are all aligned to the x_3 axis. The vector of generalized displacements and forces may now be sub-divided into ones belonging or not to a common interface. For the case of two Macro elements Ω_1 and Ω_2, shown in Figure 4, the individual equations for every Macro-element may be written as

$$\begin{bmatrix} H_{11}^1 & H_{1i}^1 \\ H_{i1}^1 & H_{ii}^1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_i^1 \end{Bmatrix} = \begin{bmatrix} G_{11}^1 & G_{1i}^1 \\ G_{i1}^1 & G_{ii}^1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_i^1 \end{Bmatrix} \quad , \quad \begin{bmatrix} H_{11}^2 & H_{1i}^2 \\ H_{i1}^2 & H_{ii}^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_i^2 \end{Bmatrix} = \begin{bmatrix} G_{11}^2 & G_{1i}^2 \\ G_{i1}^2 & G_{ii}^2 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_i^2 \end{Bmatrix} \tag{18}$$

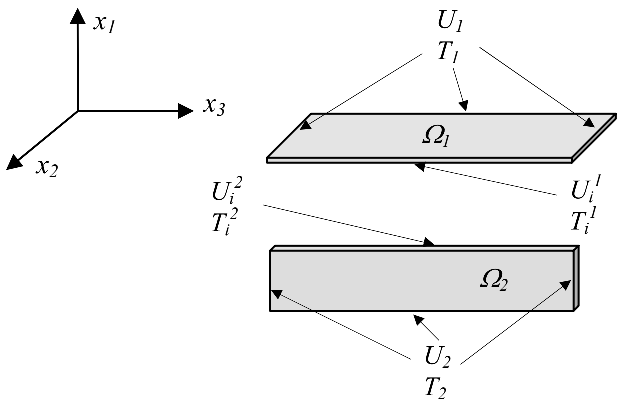


Figure 4. Displacements and forces in two Macro-elements.

The coupling of the Macro-elements is performed by considering kinematics compatibility and equilibrium at the interface nodes. Considering T the vector of external loads applied at the elements interface, compatibility and equilibrium are given by

$$U_i^1 = U_i^2 = U_i \tag{19}$$

$$T_i^1 + T_i^2 + T = 0 \tag{20}$$

After Equations (19) and (20) have been applied to Equation (18) the basic system of equation for two coupled macro-elements are given by

$$\begin{bmatrix} H_{11}^1 & H_{1i}^1 & 0 & -G_{1i}^1 & 0 \\ H_{i1}^1 & H_{ii}^1 & 0 & -G_{ii}^1 & 0 \\ 0 & H_{ii}^2 & H_{i2}^2 & 0 & -G_{ii}^2 \\ 0 & H_{2i}^2 & H_{22}^2 & 0 & -G_{2i}^2 \\ 0 & 0 & 0 & I & I \end{bmatrix} \begin{Bmatrix} U_1 \\ U_i \\ U_2 \\ T_i^1 \\ T_i^2 \end{Bmatrix} = \begin{bmatrix} G_{11}^1 & 0 & 0 \\ G_{1i}^1 & 0 & 0 \\ 0 & 0 & G_{i2}^2 \\ 0 & 0 & G_{22}^2 \\ 0 & I & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T \\ T_2 \end{Bmatrix} \tag{21}$$

where, U_1 and U_2 are generalized displacement vectors (bending and stretching) related to sub-regions Ω_1 and Ω_2; respectively. T_1 and T_2 are the corresponding generalized forces. The displacement vector U_i and the corresponding forces vector T_i stands for the values at the interface; T_i^1 and T_i^2 represent forces vectors at the interfaces for each one of the Macro-elements and I is the identity matrix.

2.3. Discrete BEM Formulation of the Macro-elements

The Macro-elements coupled by Equation (21) were discretized by rectilinear boundary elements described by linear shape functions. Considering B_1 and B_2 the initial and final coordinates of the elements, the element geometry may be expressed in terms of intrinsic coordinates, ς:

$$b(\varsigma) = B_1 \frac{1-\varsigma}{2} + B_2 \frac{1+\varsigma}{2} \tag{22}$$

This same interpolation is used for the field variables of the boundary elements possessing no corners, leading to an isoparametrical formulation. For elements with corners the field variables were discretized by discontinuous elements. The corner nodes were displaced towards the interior by one fourth of the element length ($0.25L_e$). Four integral equations were written for every boundary node. The collocation points were placed outside the Macro-element domains. When collocation point P is placed outside the domain ($P \notin \Omega$), the integration free-term disappears. Moreover, the corner reactions R_{ck} can be written in terms of neighbor node rotations using a finite difference scheme. Although this is the correct way to treat corner reactions, in the present implementation these terms were neglected. A final algebraic system $[A]\ \{X\} = \{B\}$ is obtained once the equations are assembled and the prescribed boundary conditions applied. The solution of this system, vector X, contains all unknown boundary quantities. The system matrix $[A\ (\omega)]$ contain circular frequency dependent terms. After the vector X is determined, the displacement at the assembled domains may be readily obtained by the non-singular integrations with Gauss quadrature.

3. NUMERICAL EXAMPLE

This section presents an attempt to demonstrate the previously described strategy to analyze reinforced panels structures. A computer program has been developed and numerical integration techniques have been adopted for the boundary elements. A representative example has been studied to demonstrate the efficiency of the methodology. In all cases treated, the numerical results have been obtained using twelve Gauss point integration.

3.1. Plate structure with reinforcement

Consider a reinforced structure composed of the two joined rectangular plates and a reinforcement, simply supported (SS) in two edges $x_3 = 0$ and $x_3 = a_p$, and freely supported (F) at the remaining edges, $x_1 = 0$ and $x_1 = b_p$, where length a_p = 18 m and width b_p = 9 m, as shown in Figure 5. Sapountzakis and Katsikadelis (2000) have analyzed the same structure for plate thickness and reinforcement dimension using a software package.

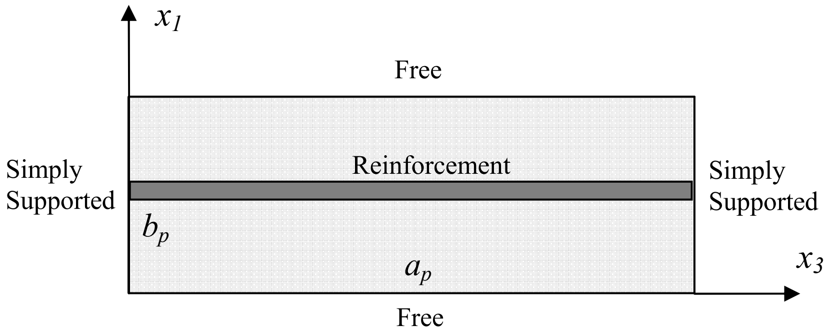


Figure 5. Plan of a reinforced structure subjected to concentrated time-harmonic load.

The plates and reinforcement are excited by a concentrated point force and the frequency of excitation is continuously changed within a pre-established range. Each assembled region (plates and reinforcement) is made of same constitutive properties with Young's modulus $E_p = E_r = 3.0 \times 10^7$ kN/m^2, Poisson's ratio $\nu = 0.154$. The thickness of two plates is h = 0.20 m. The reinforcement is of rectangular cross section 1.0 m × h_r placed along its axis of symmetry, as shows in Fig. 5. Computations by the BEM are carried out for the following two boundary discretizations using linear elements: Mesh 1: 30 boundary elements per Macro-element (region), as shown Figure 6, and Mesh 2: 60 boundary elements per macro-element (region).

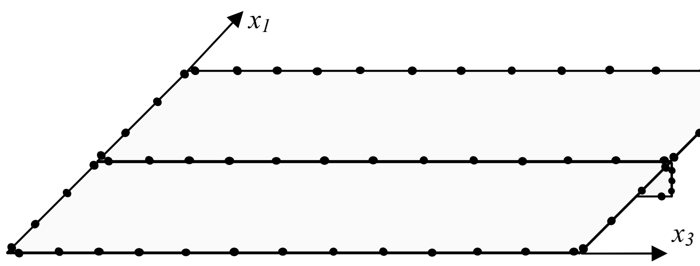


Figure 6. Boundary discretization: Mesh 1.

Take initially the two plates loaded by a unit harmonic normal excitation on the interface between the two plates at distances x_1 = 4.5 m and x_3 = 9 m (x_1-x_3 plane) and later the same configuration for the two plates plus reinforcement are analyzed. The values of the initial eigenfrequencies of the structure are reproduced in Table 1. These values are compared with the values presented by Sapountzakis and Katsikadelis (2000).

Table 1. Natural frequency of the structure no reinforcement and with reinforcement 1.0 m × h_r.

Results	No reinforcement			1.0 × 0.20			1.0 × 0.40		
	BEM		Sapountzakis and Katsikadelis (2000)	BEM		Sapountzakis and Katsikadelis (2000)	BEM		Sapountzakis and Katsikadelis (2000)
	Mesh 1	Mesh 2		Mesh 1	Mesh 2		Mesh 1	Mesh 2	
Frequency (Hz)	-	-	-	1.416	1.416	-	2.470	2.470	-
	2.119	2.119	3.127	9.501	9.501	8.571	9.853	9.853	9.430
	10.556	10.556	9.435	10.556	10.556	9.430	13.016	13.016	13.906
	11.610	11.610	12.751	21.101	21.101	21.954	22.508	22.508	22.168
	22.508	22.508	22.182	26.374	26.374	22.168	30.944	30.944	27.879
	28.834	28.834	29.729	32.350	32.350	33.326	32.350	32.350	35.233
	32.350	32.350	33.314	41.489	41.489	41.608	43.599	43.599	41.608
	41.138	41.138	41.624	43.599	43.599	42.310	45.005	45.005	47.314

The $FRF_{24\text{-}24}$ (Mesh 1) and $FRF_{42\text{-}42}$ (Mesh 2) that means the frequency response functions obtained by exciting the node number 24 and 42 and calculating the response at the same point are shown in Figures 7, 8 and 9. The three configurations are presented, structure no reinforcements and with reinforcements 1.0 × 0.20 and 1.0 × 0.40 m. In these Figures the resonances and anti-resonances can be clearly recognized. The system operational eigenfrequencies (natural frequencies) are determined from the frequencies at which resonances in the FRFs occur. Note that the present model give very near values corresponding to symmetric and antisymmetric modes.

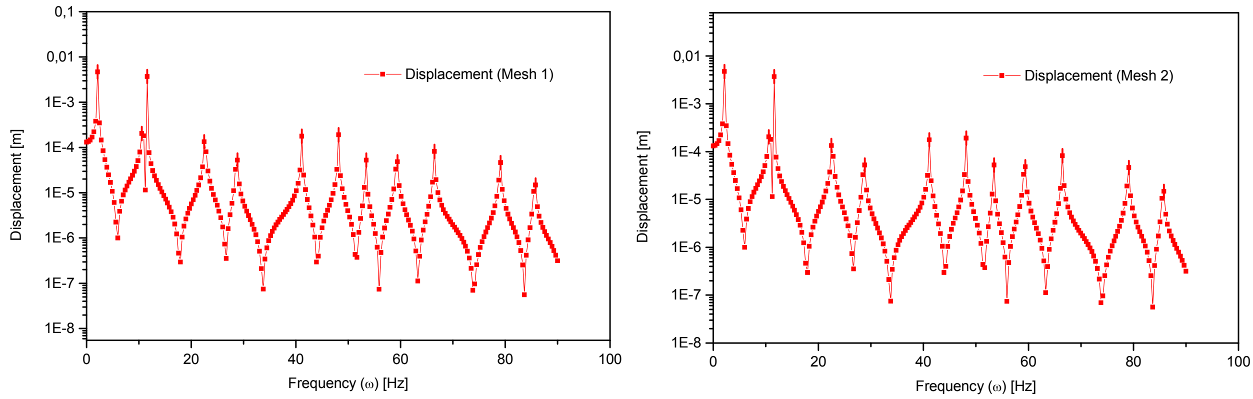


Figure 7. FRFs for the structure no reinforcements: Mesh 1 and 2.

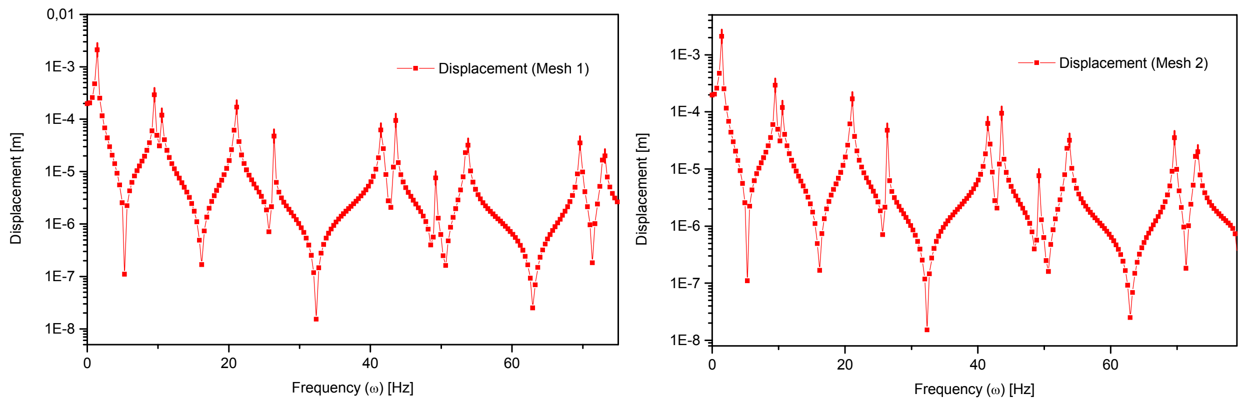


Figure 8. FRFs for the structure with reinforcement 1.0 × 0.20: Mesh 1 and 2.

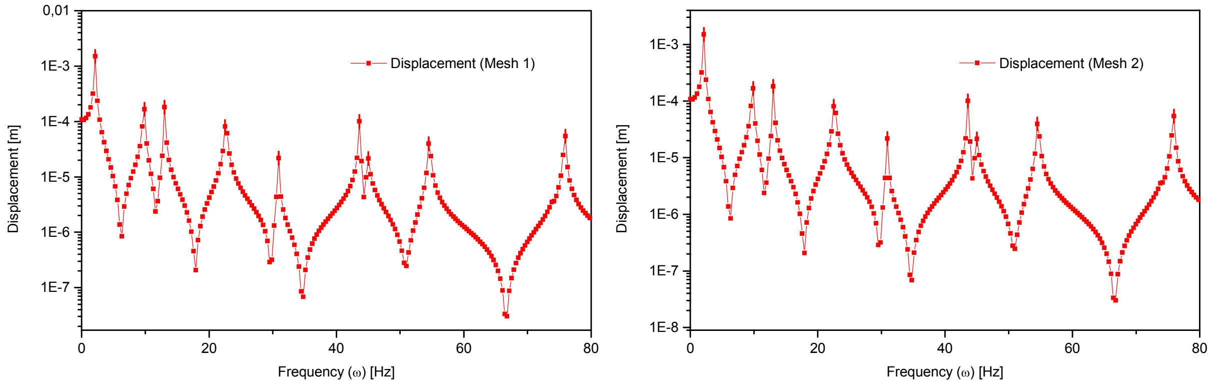
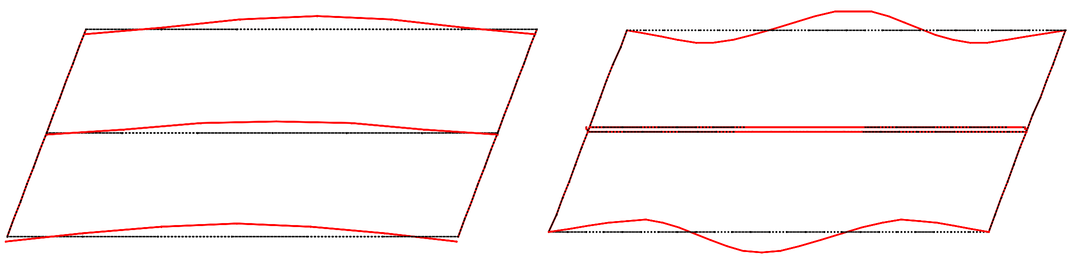


Figure 9. FRFs for the structure with reinforcement 1.0 × 0.40: Mesh 1 and 2.

The additional modal quantity necessary to characterize the dynamic behavior of the reinforced structure is given by the eigenmodes or the natural modes of vibration. Figure 10a show the boundary displacements corresponding to the first eigenmode of the two plate structure, $\omega = 2.119$ *Hz*. Figure 10b show the boundary displacements corresponding to the third eigenmode of the reinforced structure $\omega = 21.101$ *Hz*.



a) First eigenmode, $\omega = 2.119$ *Hz*. b) Third eigenmode, $\omega = 21.101$ *Hz*.

Figure 10. Operational eigenmodes: a) two plates, b) reinforced structure.

In Figure 10 the operational modes are obtained by calculating the displacement field at boundary of the structure at each resonance frequency present in the FRF. In all configurations, external boundary nodes of the structure in x_1 direction are simply supported.

4. CONCLUDING REMARKS

The stationary dynamic analysis of structures with reinforcements has been studied. A version of the Boundary Element Method is implemented in a computer program to analyze the behavior of the reinforced plate structures. The dynamic stationary fundamental solution is used to transform the differential equation governing the thin plate and membrane behavior into a boundary-only integral equation. The boundary integral equation is discretized using linear continuous and discontinuous linear elements. Four displacement integral equations are written for every boundary node. The collocation points of the integral equations are placed outside de plate domain, leading to a non-singular Boundary Element formulation. The proposed scheme is used, exemplarily, to obtain modal data, that is, eigenfrequencies and operational eigemodes of a reinforced plate structure. Frequency Response Functions may be determined for every boundary or domain point of the model. In the reported example, the FRF of a node on a free boundary is used to recover eigenfrequencies. The eigenfrequencies are determined from the resonance of the FRF. At these resonance frequencies the displacement fields of the structure furnish the operational eigenmodes. The presented results agree very well with a numerical solution presented in the literature. It is shown that there is some discrepancy in the eigenfrequencies obtained on the base in the model for the reinforcement-plate system. However, the methodology may be seen as an accurate model to analyze free and forced stationary vibrations of the reinforced plate structures have in view that the procedure require the discretization of the boundary only.

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