THREE APPROACHES TO THE DYNAMIC PROBLEM OF A PENDULUM AND A ROTATING FLEXIBLE BEAM

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Abstract. *The dynamic characteristics of a rotating beam play a significant role in the overall performance and design of various engineering systems, such as turbo-machinery, wind turbines, etc. In this work the dynamic problem of a flexible pendulum and a beam rotating around an axis perpendicular to its plane is addressed taking into account the gravitational effect. The dynamics of pendulums and rotating beams is studied according to three approaches, two of them related with the Strength of Materials theory (SM) and the other one with nonlinear Theory of Elasticity (TE). The stiffening effect due to the centrifugal forces is taken into account only in the case of the rotating beam subjected to high rotating speeds and neglected in the case of a pendulum under self-weight due to the low values of the speed, when addressed with SM theory. The derivation of the governing equations is done by superposition of the deformations to the rigid motion for the first approach (SM). The second approach with SM theory is through the application of Hamilton's principle. The* lagrangian *form is employed with the nonlinear TE. Comparison of the three models allows to determine advantages and drawbacks of each of them.*

Keywords: *rotating flexible beam, gravitational effect, stiffening effect*

1. Introduction

The dynamic modeling of beams rotating in its planes has been subject of many research works in the last decades due to its application in aerospace, aviation and robotic industries (see for instance Vetyukov *et al.*(2004), Al-Qaisia and Al-Bedoor (2005), Banarjee *et al.*(2006), among others). The stiffening contribution due to the centrifugal force in one of the major complexities when dealing with this problem.

The dynamic of a body that is subjected to large displacements may be dealt with the classical Theory of Elasticity (TE) for small deformations (Hunter, 1983), the Theory of Strength of Materials (TSM) (in which the stiffening effect due to centrifugal forces should be considered), finite elasticity of a "floating" system (Vetyukov *et al.*(2004), Fung (1968)) or finite deformations (Fung (1968), Truesdell (1960)). The particular case of a beam under a prescribed rotation of a pendulum and the derivation of the governing equations is addressed according to the following approaches: TSM with a "floating" frame, TSM *via* Hamilton's principle including the stiffening contributions and, finite elasticity (Fung, 1968).

A brief discussion is done on the type of constitutive equation (Fung, 1968 and Truesdell, 1960) using the Piola-Kirchoff stress tensor giving place to strongly non-linear equations that will be solved using the finite element methods. The boundary conditions are also discussed since the equations are stated in its *lagrangian* form. Also an analysis of the energy conservation is included (Lai *et al.*, 1993) which permits the control of the numerical solution convergence.

The results of the stiffening effect in the rotating beam and the pendulum motion found with TSM and nonlinear TE are compared.

2. Statement of the equations of motion

In this work the equations of motion of the a deformable pendulum and a rotating beam in its plane are stated and solved. Two approaches are dealt with and compared. The first one is given by the theory of Strength of Materials (TSM) in one dimension and the second by means of the theory of Elasticity (TE) in two and three dimensions. The latter is the most relevant in this paper.

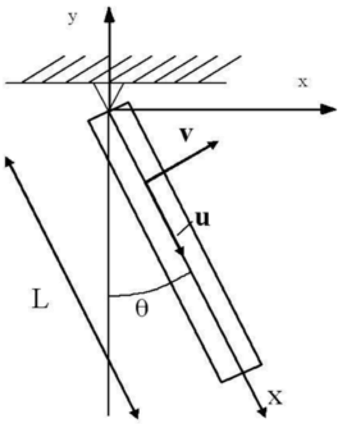

Figure 1. The pendulum and its displacement vectors.

2.1 Lineal one-dimensional model

Also in the TSM approach two types of model were stated. One of them by means of effect superposition (Hunter,1983) and the other one using Hamilton's principle.

Effect superposition The equations that govern the beam vibrations within the TSM are

$$E^*A\frac{\partial^2 u}{\partial X^2} - \rho A\frac{\partial^2 u}{\partial t^2} = f_1 \quad (1)$$
$$E^*I\frac{\partial^4 v}{\partial X^4} + \rho A\frac{\partial^2 v}{\partial t^2} = f_2 \quad (2)$$

where E^* is the modulus of elasticity, A is the cross-sectional area of the bar, ρ is the volumetric density, u y v are the longitudinal and transverse displacement respectively, $\mathbf{u} = (u, v)^T$ is the displacement vector and f_1 y f_2 are the eventual normal and shear applied forces (gravity force components). It is assumed that the body motion is a result of the superposition of a rigid motion and a small deformation. That is, let us suppose that the body (beam or pendulum) motion is given by the displacement vector

$$\mathbf{u} = \mathbf{u}_r + \mathbf{u}_d \quad (3)$$

where $\mathbf{u}_r(t) = (u_r(t), v_r(t))^T$ is the rigid part of the motion and $\mathbf{u}_d(X,t) = (u_d(X,t), v_d(X,t))^T$ is the part related with the deformation. After replacing in Eqs. 1 and 2 the following yields

$$E^*A\frac{\partial^2 u_d}{\partial X^2} - \rho A\left(\frac{\partial^2 u_r}{\partial t^2} + \frac{\partial^2 u_d}{\partial t^2}\right) = f_1 \quad (4)$$
$$E^*I\frac{\partial^4 v_d}{\partial X^4} + \rho A\left(\frac{\partial^2 v_r}{\partial t^2} + \frac{\partial^2 v_d}{\partial t^2}\right) = f_2 \quad (5)$$

Now $\mathbf{u}_r$ should be obtained from the equations governing the rigid motion to be replaced in Eqs. 4 and 5 in order to solve the problem. For instance, in the case of a pendulum of length L in y direction and gravity g, one obtains $\ddot{\theta} - \frac{3}{2L}g\sin\theta = 0$ and once solved $\theta(t)$, $u_r = X(\sin\theta - 1)$ $v_r = -X\cos\theta$ may be known.

Equations of motion vía Hamilton's principle The kinematic transformation equations (see Figure 1) are

$$x(X,t) = X\sin\theta + u(X,t)\sin\theta + v(X,t)\cos\theta \quad (6)$$
$$y(X,t) = -X\cos\theta - u(X,t)\cos\theta + v(X,t)\sin\theta \quad (7)$$

The following four energy contributions are introduced: W_1, is the axial and bending energy, W_2 is the energy generated by the axial stress state of the centrifugal effect due to the bending strain, K is the kinetic energy and P is the gravitational potential energy. Let $\sigma = \rho\omega^2(L^2 - X^2)/2$ be the stress due to centrifugal force, then

$$2W_1 = EA\int_0^L \left(\frac{\partial u}{\partial X}\right)^2 dX + EI\int_0^L \left(\frac{\partial^2 v}{\partial X^2}\right)^2 dX \quad (8)$$

$$2W_2 = \int_{(A)}\int_0^L \sigma \left(\frac{\partial v}{\partial X}\right)^2 dX\, dA \quad (9)$$

$$2K = \rho A\int_0^L (\dot{x} + \dot{y})^2 dX \quad (10)$$

$$P = -\rho g\int_{(A)}\int_0^L y\ dX\, dA \quad (11)$$

Consequently the *lagrangian* is $\mathcal{L} = K - (W_1 + W_2 + P)$ and with this, the Hamilton's principle

$$\delta\int_{t_1}^{t_2} \mathcal{L}dt = 0 \quad (12)$$

which gives place to the equations of motions where previously a non-dimensionalization was introduced as follows, $z = x/L\ \ (0 \le X \le L)\ ,\ (0 \le z \le 1)$

$$k_L\frac{\partial^2 u}{\partial z^2} - \left(\ddot{u} - \omega^2 u - 2\omega\dot{v}\right) = F(z,t) \quad (13)$$

$$k_v\frac{\partial^4 v}{\partial z^4} + \left(\dot{v} - \omega^2 v - 2\omega\dot{u}\right) + \omega^2\left(\frac{z^2-1}{2}\frac{\partial^2 v}{\partial z^2} + z\frac{\partial v}{\partial z}\right) = G(z,t) \quad (14)$$

where $k_L = \frac{E^*}{\rho L^2};\ \ k_V = \frac{E^* I}{\rho A L^4};\ F(z,t) = -\left(\omega^2 L z + g\cos\omega t\right);\ G(z,t) = -g\sin(\omega t)$

2.2 Two and three-dimensional model for finite deformations

2.2.1 Equations of motion

In this Section the equations of the elastic body continuum, in two or three dimensions for finite displacements and deformations are stated. That is, in these models no hypothesis is made with respect to the smallness of the deformations. The statement of these equations is made within the frame of the Mechanics of Continuum with the *lagrangian* or material representation presenting some advantages over the *eulerian* or spatial representation in the case of Mechanic of Solids problems. In turn if the problem of the continuum is given by the *eulerian* configuration, besides the equation of motion (Cauchy Eq.)

$$\nabla\cdot\sigma + \rho\mathbf{b} = \rho\mathbf{a} \quad (15)$$

the corresponding to the mass continuity should also be stated

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \quad (16)$$

where σ is the symmetric stress tension of Cauchy, ρ is the mass density, $\mathbf{b}$ are the body forces and $\mathbf{a}$ and $\mathbf{v}$ are the acceleration and velocity fields resp. It should be taken into account that both $\mathbf{a}$ and $\mathbf{v}$ are calculated as material derivatives which introduces a strong non-linearity in the differential equations. If the body is subjected to finite displacements in the space, the statement of the boundary equations is a problem of hard solution since the boundary position is one of the unknowns of the motion. Now, if the problem is given in the *lagrangian* or material reference, the equation of motion is the only to be solved.

$$\nabla\cdot\mathbf{P} + \rho_0\mathbf{b} = \rho_0\mathbf{A} \quad (17)$$

where $\mathbf{P}$ is the Piola - Kirchoff first stress tensor (Truesdell and Noll,1965), $\rho_0 = \rho(X, t_0)$ is the initial density (already known) and $\mathbf{A} = \frac{\partial\mathbf{V}}{\partial t} = \frac{\partial^2\mathbf{R}}{\partial t^2}$ ($\mathbf{R}$ is the position vector and $\mathbf{A}$ is the acceleration field that is simply the partial derivative of the velocity field. The boundary conditions are imposed over the initial boundary (already known), which together with the initial conditions and the equation of motion yields a closed problem. Both the boundary as well as any point position of the body will be known once solved the problem. All the non-linear problem is transfered to $\mathbf{P}$ tensor which besides being non-symmetric, is of hard physical interpretation.

The second Piola - Kirchoff stress tensor $\mathbf{S}$, which is symmetric, is given by $\mathbf{P} = \mathbf{FS}$ where $\mathbf{F}$ is the gradient of deformation, and then the equation of motion writes

$$\nabla\cdot(\mathbf{FS}) + \rho_0\mathbf{b} = \rho_0\mathbf{A} \quad (18)$$

2.2.2 First Piola - Kirchoff stress tensor

If $d\mathbf{f}$ is the force element that acts in the deformed elemental area dA, the following is true

$$d\mathbf{f} = \mathbf{t}dA \tag{19}$$

$$\mathbf{t} = \sigma\mathbf{n} \tag{20}$$

where σ is the Cauchy stress tensor and $\mathbf{n}$ is the unit vector normal to the elemental area dA. The same force $d\mathbf{f}$ may be referred to the non-deformed dA_0

$$d\mathbf{f} = \mathbf{t}_0 dA_0 \tag{21}$$

in which t_0 is the stress vector referred to the non-deformed area. Vector $\mathbf{t}_0$ is parallel to vector $\mathbf{t}$ but with different modulus since $d\mathbf{f} = \mathbf{t}_0 dA_0 = \mathbf{t}dA$. That is, to suppose the *same* force referred to different areas (deformed and non-deformed) gives place to stress vectors of the same direction and magnitudes proportional to the area change.

$$\mathbf{t}_0 = \mathbf{t}\frac{dA}{dA_0}. \tag{22}$$

The first Piola - Kirchoff tensor is defined form the relationship $\mathbf{t}_0 = \mathbf{PN}$, where $\mathbf{N}$ is the unit vector normal to area dA_0, analogously to the Cauchy tensor σ of $\mathbf{t}_0 dA_0 = \mathbf{t}dA$.

$$\mathbf{PN}dA_0 = \sigma\mathbf{n}dA \tag{23}$$

and since

$$dA\,\mathbf{n} = (\det\mathbf{F})\,dA_0\,(\mathbf{F}^{-1})^T\,\mathbf{N} \tag{24}$$

one obtains

$$\mathbf{P} = (\det\mathbf{F})\,\sigma\,(\mathbf{F}^{-1})^T. \tag{25}$$

that is the relationship between the Cauchy and the first Piola - Kirchoff tensors.

2.2.3 Second Piola - Kirchoff tensor

On the other hand, let us suppose that the elemental force vector $d\mathbf{f}$ is the result of the transformation of other elemental force vector $d\mathbf{f}_0$ (with respect to the non-deformed body) by means of the gradient of deformation tensor $\mathbf{F}$ (in the same way that the displacement vectors $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ are transformed).

$$d\mathbf{f} = \mathbf{F}d\mathbf{f}_0 \tag{26}$$

with $d\mathbf{f}_0 = \mathbf{t}^* dA_0$ and let us assume that this new stress vector is related to the normal vector through a new tensor $\mathbf{t}^* = \mathbf{S}^*\mathbf{N}$,. Then

$$d\mathbf{f} = \mathbf{F}(\mathbf{t}^* dA_0) = \mathbf{F}\,\mathbf{S}^*\mathbf{N}\,dA_0 \tag{27}$$

and since $d\mathbf{f} = \mathbf{t}dA$, the following stands $\mathbf{t}dA = \mathbf{F}\,\mathbf{S}^*\mathbf{N}\,dA_\mathbf{0}$ and after a new transformation of dA, the next yields

$$\mathbf{F}\,\mathbf{S}^*\mathbf{N}\,dA_0 = \sigma\mathbf{n}dA \tag{28}$$

$$\mathbf{F}\,\mathbf{S}^*\mathbf{N}\,dA_0 = \sigma((\det\mathbf{F})\,dA_0\,(\mathbf{F}^{-1})^T\,\mathbf{N}) \tag{29}$$

That is, $\mathbf{F}\,\mathbf{S}^* = (\det\mathbf{F})\,\sigma\,(\mathbf{F}^{-1})^T = \mathbf{P}$. Then $\mathbf{S}^* = \mathbf{S}$;, second Piola - Kirchoff tensor. Then, to suppose that the elemental force is transformed as the displacement vectors by means of the $\mathbf{F}$ gives place to the second Piola - Kirchoff tensor as the stress tensor. It can be shown that the latter is symmetric.

2.2.4 Constitutive equation

If the Hookean material lineal model is abandoned and one introduces in the finite deformation model, the type of constitutive equations is a universe of large variety (Truesdell and Noll, 1965 and Fung,1968). Truesdell classifies three large families, all of them consistent with the linear elasticity, *elastic*, *hyperelastic and hypoelastic* materials. The simpler ones to model are the *elastic* materials that are given in the following form

$$\sigma = g(\mathbf{e}) \tag{30}$$

$$\mathbf{S} = g'(\mathbf{E}) \tag{31}$$

where $\mathbf{e}$ and $\mathbf{E}$ are the *Eulerian* (of *Almansi*) and *Lagrangian* (of *Green*) are the finite deformation tensors. Analogously with linear elasticity, in this work it is assumed (Fung,1968)

$$\mathbf{S} = \lambda\, tr(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} \tag{32}$$

in which λ and μ are Lame's constants, $\lambda = \nu E^*/(1+\nu)(1-2\nu)$, $\mu = E^*/2(1+\nu)$ and E^* and ν are the modulus of elasticity and the Poisson's coefficient, resp. Eq. (32) is also known as St. Venant–Kirchhoff material model (Truesdell and Noll, 1965) that, obviously, in the limit for infinitesimal deformation, leads to the Hooke's law for elastic homogeneous and isotropic bodies, $(\mathbf{S} \to \sigma = \lambda\, tr(\varepsilon)\mathbf{I} + 2\mu\varepsilon)$. Additional alternative to possible constitutive equations may be read in Filipich and Rosales, 2000.

2.2.5 Boundary conditions

As was mentioned before, the main advantage of the statement in *lagrangian* coordinates is given by the simple consideration of the boundary conditions imposed over the known non-deformed body.

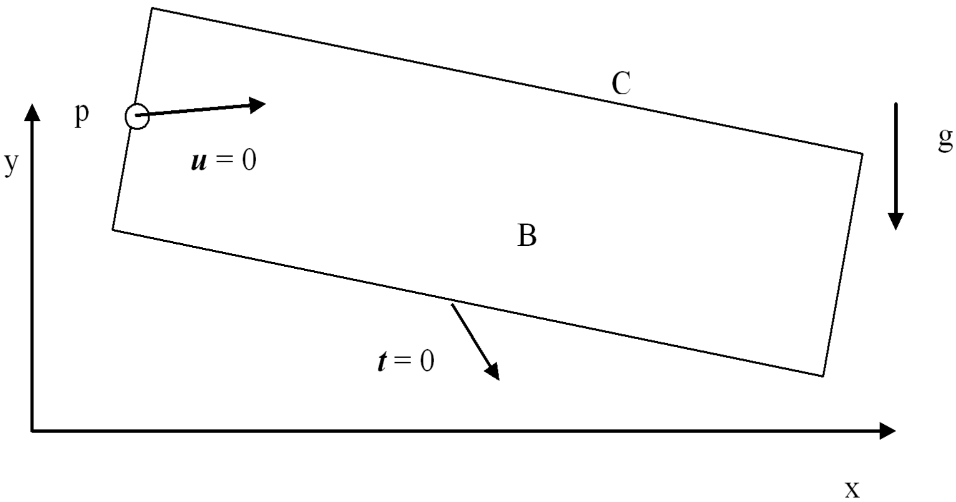

Figure 2. Pendulum scheme in the non-deformed configuration.

That is, once the pendulum boundary is defined, a null condition is imposed for the stress in all boundary point with exception to the pivot (p) and null displacement on the pivot (p). In this work the pendulum is assumed to be 5 m long and having a square cross-sectional area of $0.1m^2$ (see Figure 2).

$$\mathbf{t}_0 = 0 \tag{33}$$
$$\mathbf{u}(\mathbf{p}) = 0 \tag{34}$$

In the rotating beam case, the rotational velocity is assumed constant to compare with the one-dimensional theory. The boundary conditions are again, null stress at all boundary points except for the clamped side and null displacement at this side.

2.2.6 Simulations and results

The pendulum The numerical simulations are carried out using the finite element method by means of the software FlexPDE 5. Quadratic elements are used in the spatial domain in all the non-linear TE simulations. When dealing with the linear TSM cubic elements were considered. Temporal integration was done using the Gear method (second-order implicit Backward Difference Formula). Example 1 deals with the pendulum with $E^* = 4 \cdot 10^7\ N\ m^{-2},\ \nu = 0.3$ and density $\rho = 7850\ kg/m^3$. The pendulum is freed from the horizontal position with both null deformation and velocity.

Figure 3 shows 11 superpositions of the pendulum for the first second of motion, corresponding to the TE model and Fig. 4 depicts 11 superpositions that were found with the TSM model. The number of finite elements and the time step are adjusted in order to attain an error less that 1% in $t = 1$. The energy variation of the pendulum (non-lineal elastic model)

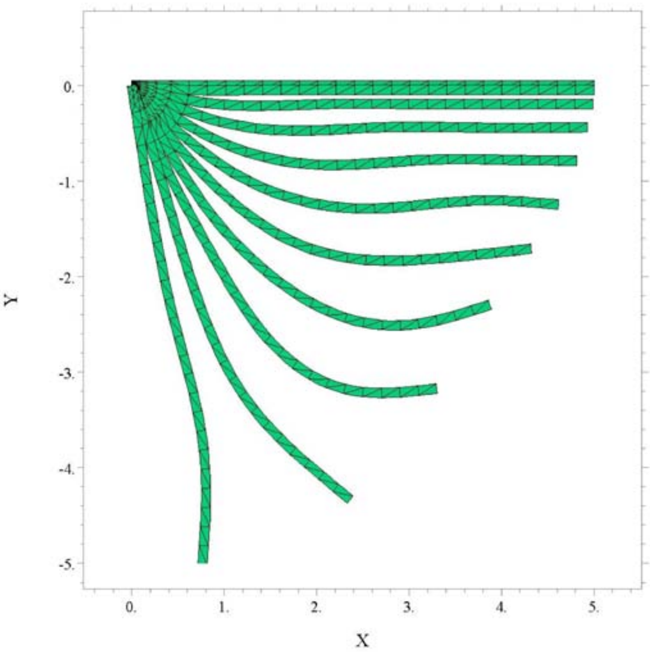

Figure 3. Sequence of the pendulum motion. Example 1. First second (TE).

is shown in Fig. 5. It may be observed that the total energy remains constant, necessary condition for the numerical solution since we are dealing with a conservative system. The total energy is the sum of the kinetic, elastic strain and potential energies,

$$E = T + Ue + Ug \tag{35}$$

with

$$T = \frac{1}{2}\int \rho_0 \mathbf{V}\cdot\mathbf{V}\, dV \tag{36}$$

$$Ug = g\int \rho_0 y\, dV \tag{37}$$

$$\dot{U}e = \int \frac{\rho}{\rho_0} tr\left(\mathbf{S}\,\frac{D\mathbf{E}}{Dt}\right) dV \tag{38}$$

where g is the gravity acceleration and y is the material component along the height of the position vector. Consider that all the integrals are made w.r.t. the non-deformed configuration *lagrangian* with which the material derivatives are coincident with the partial ones. $\frac{D\mathbf{E}}{Dt} = \frac{\partial \mathbf{E}}{\partial t}$

Rotating beam When the pendulum is clamped at one end and subjected to a constant angular velocity $\omega$the boundary conditions are the ones stated above. It is known that the vibration frequencies of a rotating beam increases as ω does. In the lineal one-dimensional theory (TSM) the stiffening effect of a rotating beam is due to the contribution of the second order work done by the axial stress caused by the centrifugal force over the bending deformation. For the general case of the dynamic of the elastic body considering finite deformation it is not necessary to introduce additional terms in the equation of motion since they are general. Obviously the modal superposition is not valid, since the equation of motion with large deformations are not linear. Notwithstanding, the stiffening effect that is evident in the linear model frequencies, may be observed when the Fourier transform of the dynamic response is found for any point of the rotating body.

Figure 6 depicts the variation of the vibration frequency when the rotating velocity is increased. The TSM results are shown in full lines and the non linear TE in dashed lines. In this case the material is steel: $E^* = 2.1 \cdot 10^{11} Nm^{-2}$. and $\rho = 7850\, kg$ and $\nu = 0.3$.

3. FINAL COMMENTS

The dynamic of a flexible pendulum was addressed with the Strength of Material (SM) theory and the finite Theory of Elasticity (TE). The SM approach was performed with two models, superposition of motions and Hamilton's principle.

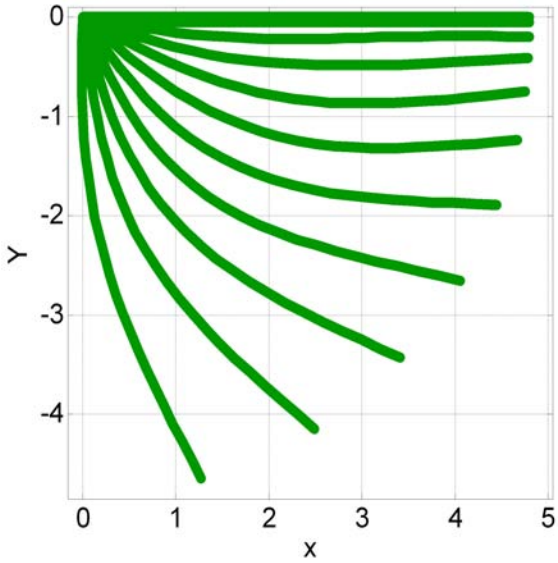

Figure 4. Sequence of the pendulum motion. Example 1. First second. TSM

The latter provides for the stiffening effect. However, numerical instability problems prevented numerical simulations. In the case of the superposition model, the equations are partially coupled. That is, only the deformation equations are coupled with the rigid motion but the rigid motion equations are uncoupled with the deformations. This derived from the type of construction. The nonlinearity is only present in the rigid body motion. On the other hand, when applying Hamilton's principle, fully coupled equations arise. The latter is a consistent approach.

The stiffening effect due to the centrifugal forces is considered only for the rotating beam since the pendulum is subjected to low rotational speeds and consequently such effect is negligible.

The finite elasticity approach yielded similar results to the ones obtained with SM theory (first approach). However, in the above illustration, since a low value of modulus of elasticity E^* was chosen, the resulting deformations were not very small, then the response is not identical.

Also the dynamics of a beam rotating with high speed was studied. The stiffening effect in the SM model makes it possible to find almost coincident values of frequencies found via finite TE. It may be concluded that not only the stiffening contribution is considered but also it is the correct one.

In the conservative pendulum case, the total energy composed of the gravitational, strain and kinetic parts remains constant in time. This is useful to check that the numerical integration scheme do not introduce damping or instabilities.

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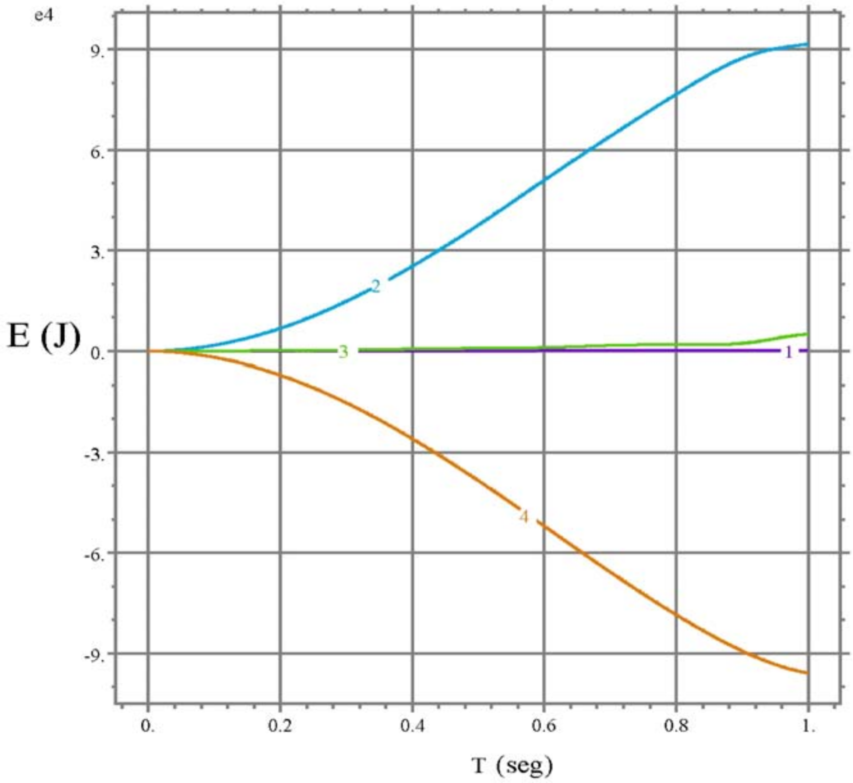

Figure 5. Energy as function of time for the pendulum of Example 1. (1) Total energy, (2) kinetic energy, (3) strain energy and (4) gravitational potential energy.

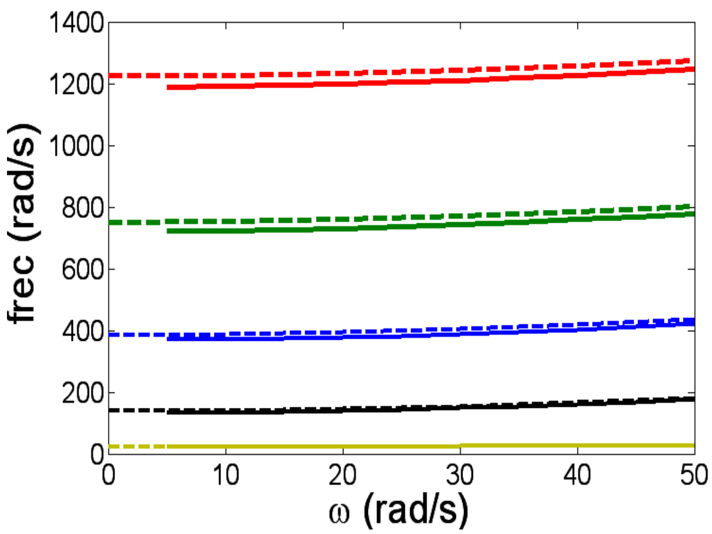

Figure 6. Variation of the first six frequencies with to ω. - - TE, — TSM.