A COMPARATIVE ANALYSIS BETWEEN THE MONTE CARLO METHOD AND THE DISCRETE ORDINATE METHOD APPLIED TO SOLVE THE RADIATIVE TRANSFER EQUATION

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Abstract. *In this work, a Monte Carlo formulation is employed to solve the Radiative Transfer Equation to one-dimensional slab condition. An isotropic scatter media with a collimated normal incident beam onto to the sample is analyzed. The self-emission is not considered. An uncertainty analysis is performed to show the influence of optical thickness, albedo, bundles number and the number of control volumes. A comparative analysis with Discrete Ordinate Method is also performed. A processing time analysis is presented for both methods.*

Keywords*: Monte Carlo Method, radiative transfer equation, radiation heat transfer.*

1. INTRODUCTION

In recent years, various works had had been carried on radiation heat transfer to diverse applications such as: combustion engines, boilers, furnaces, rocket engines and many other examples that involve high temperatures. Few analytical solutions are available in literature for these problems and the use of numerical methods is a powerful tool to study radiative transfer phenomena. One of these numerical methods is the model based on the Monte Carlo method. Since 1963, the Monte Carlo Method had been used to solve problems on radiation heat transfer (Yang *et al.*, 1995). In despite of an important number of Monte Carlo Method in radiative heat transfer problems founded in literature (Yang *et al.*, 1995) some analyses miss to be performed.

Until few years the Monte Carlo techniques had not been considered the anisotropy and the internal reflections. Prahl *et al.* (1989) considered these effects to analyze the laser incidence on the skin tissues. Churmakov *et al.* (2003) proposed an original technique to Monte Carlo simulation to analysis the skin tissues spatial fluorescence distribution. The advantage of Monte Carlo method is that even the most complicated problem (multidimensional or gas) may be solve with relative ease, in opposite of the finite difference or volume and finite element techniques that increase much more rapidly the complexity of formulation (Modest, 2001). Monte Carlo simulations can easily be applied for multidimensional geometries, not homogeneous, with time dependent boundary conditions, where others techniques are almost impossible to implement (Wong and Mengüç, 2002).

Ambirajan (1996) applied the Monte Carlo method in multiple scattering of a narrow light beam with normal incidence onto a narrow parallel slab. In this work it was observed that the Monte Carlo method is efficient until the second order scattering. Ruan, *et al.* (2002) developed a Monte Carlo method applied to the medium with nongray absorbing-emitting-anisotropic scattering particles and gray approximation.

Wang (1998) developed a modeling of diffuse reflectance of light in turbid slabs. He used a hybrid method which combines the Monte Carlo method and the diffusion theory, using the advantages of high precision to Monte Carlo method with the speed of diffusion theory.

Henson and Malalasekera (1997) realize a comparison of discrete transfer and Monte Carlo methods in three-dimensional nonhomogeneous isotropically scattering media. They founded good agreement with benchmark results. The average deviation between the two methods was less than 1.2%.

The Discrete Ordinates Method (DOM) was proposed by Schuster (1905) and Schwarzschild (1906) for studying radiative transfer in stellar atmospheres (Siegel and Howell, 2002). Until now days many works were proposed to improve this method and a review it can be found on Fiveland works (1985 and 1987). The DOM is a multi-flux method extension and it has been used to solve multi-dimensional radiative transfer problems.

In the sequence the two methods are used for the accomplishment a comparative analysis. The Discrete Ordinates Method (DOM) used with Finite Volume formulation (FVDOM) is compared to Monte Carlo method (MCM) to an isotropic scatter media and one-dimensional slab condition. An uncertainty analysis is performed to show the influence of optical thickness, albedo, bundles number and control volumes number. A processing time analysis is presented for both methods.

2. RADIATIVE TRANSFER EQUATION

The Radiative Transfer Equation (RTE), which describes the variation of the spectral radiation intensity, I, (in a solid angle Ω, function of optical depth τ) in an absorbing-emitting-scattering medium, can be written as:

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = (1-\omega) I_o(T) + \frac{\omega}{2}\int_{-1}^{1} I(\tau,\mu') p(\mu',\mu) d\mu' \tag{1}$$

where ω is the albedo, p is the phase function, and I_o is Planck's blackbody function (in order to simplify the notations, the spectral subscript λ is not considered in the text). These properties are those of a pseudo-continuum medium equivalent, in terms of radiative transport, to the real dispersed material. The boundary conditions assumed a normally incident collimated beam onto the sample with bidirectional transmittance and reflectance measurements. The boundary conditions can be expressed like:

$$\begin{cases} I(\tau=0,\mu) = I_C; & \mu_o < \mu < 1 \\ I(\tau=0,\mu) = 0; & 0 < \mu < \mu_o \\ I(\tau=\tau_o,\mu) = 0; & -1 < \mu < 0 \end{cases} \tag{2}$$

where I_C is the radiative intensity of the incident beam with a divergence angle, θ_o ($\mu_o = cos\ \theta_o$).

The RTE in a scattering media has been studied, analytically and numerically in astrophysics, atmospheric, heat transfer and, more recently, in medical applications. However, analytical solutions are not always possible, and the numerical solutions must be employed. Some assumptions can be adopted to facilitate the solution of these problems, for example, homogeneous media, isotropic scattering, one-dimensional geometry, constant radiative properties, etc.

Numerical methods such as the DOM (Chandrasekhar, 1960) have been used on radiative transfer problems when the analytical solutions are not available. Statistical techniques like Monte Carlo method supply good approaches, but they present a high computational time (Churmakov *et al.*, 2003).

2.1. Discrete Ordinate Method

The DOM was initially used by Schuster (1905) and Schwarzschild (1906) for studying radiative transfer in stellar atmospheres (Siegel and Howell, 2002), and after, Chandrasekhar (1960) extended the formulation to astrophysics problems. Carlson and Lathrop in 1968 had developed a solution to the neutrons transport equation.

The majority of works use the RTE formularization presented by Chandrasekhar (1968) and Özisik (1973). These techniques of solution of the ETR can be found in Moura *et al.* (1997 and 1998). The RTE solution by DOM is constituted of two stages: i) an angular discretization, where the integral term of RTE is substituted by a radiative intensities weighted sum of the angular directions. In this way, the integro-differential equation is transform on a set of first-class ordinary partial equations; ii) a space discretization, considering control volume, for solution of partial equations. To a "cold" media (no self-emission media), Eq. (1) can be re-write as:

$$I_{i+1/2,j} = \frac{1}{(1+f\alpha_j)} \left[f\alpha_j \frac{\omega}{2\beta} \left[\sum_{n=1}^{N} w_n \left(p_{nj} I_{i+1/2,n} \right) \right] + I_{i,j} \right] \tag{2}$$

where $i+1/2$ represent the control volume center coordinate, f is the interpolation function that can be: upwind, linear, integral or exponential, w is the weight and α_j is:

$$\alpha_j = \frac{\Delta\tau_{i+1/2}}{\mu_j} \tag{3}$$

where $\Delta\tau$ is the optical thickness of the control volume.

2.2. Monte Carlo method

Radiative heat transfer by Monte Carlo method is based on probability concepts applied to the physical phenomena, such as: emission, reflection, and absorption of the photon.

Monte Carlo method use random techniques to determine the radiative transfer. In accordance with the CSEP (1995), the main components of Monte Carlo simulation are:

- Probability density function (pdf's): physical or mathematical system
- Random numbers Generators
- Sampling rule
- Scoring (or tallying): the interest results are stored
- Error estimation: a variance analysis must be performed to evaluate the error

- Parallelization and vectorization: algorithms to allow Monte Carlo method to be implemented with more efficiency and rapidity on advanced computers.

Solving RTE by Monte Carlo method, radiative energy is not treated as a continuous energy flux, but is considered a pack of photons, each with a fixed amount of energy. To quantify the radiative energy attenuation in a monochromatic source is considered the Beer's law. The Beer's law expresses the attenuation of radiant energy inside a volume of a thickness, S. In the Monte Carlo method the Beer's law can be modify to express the radiative energy extinction probability emitted from a point in the media (or surface) and travel over the distance, S (Yang *et al.*, 1995) :

$$Rs = 1 - e^{KS} \tag{4}$$

where Rs is the random number, K is the extinction coefficient and S is trajectory distance by the photon until be absorbed or scattered.

If the value of S is bigger than the distance takes from the photon point emission the photon was absorbed or scattered. Whether each photon is absorbed or scattered it is determined by the albedo, ω, and the uniform random number for scattering albedo, R_ω. If R_ω is bigger than albedo the photon is absorbed, otherwise the photon is scattered (Brewster, 1992). The Monte Carlo algorithm for an isotropic media with a collimated beam incident beam onto a slab surface has this flow:

1. Determine the direction of propagation by the photon inside the solid angle of the incident beam onto the face of the slab.
2. Determine by Eq. (4) if it was absorbed, scattered or remaining in its trajectory.
3. If it was absorbed or kept the media, it is initiate a new photon analysis.
4. Verify the absorption or scattering criteria. If it was scattered, to choose a new direction (random choice) and to repeat step 2, considering the current position of the photon.
5. Repeat these steps for a number of photons to assure the accuracy.

3. RESULTS

A comparison between the Monte Carlo method and Discrete Ordinate method is presented in this section considering isotropic scattering and a normal incident beam onto a slab. Hemispherical transmittance and reflectance are used to compare the results. The hemispherical transmittance and reflectance to DOM are defined as:

$$T_{eh} = \frac{\sum_{\mu>0} w_n \mu_n I_{i,n}}{I_o \mathrm{d}\omega_o} \tag{5}$$

$$R_{eh} = \frac{\sum_{\mu<0} w_n \mu_n I_{i,n}}{I_o \mathrm{d}\omega_o} \tag{6}$$

where $I_o \mathrm{d}\omega_o$ is the incident radiative onto the slab with a slid angle, $\mathrm{d}\omega_o$.

The hemispherical transmittance and reflectance to MCM are defined as.

$$T_{eh} = \frac{Ejected\ photons number\ by\ face\ 1}{Total\ \ photons\ considered} \tag{7}$$

$$R_{eh} = \frac{Ejected\ pthotons\ number\ by\ face\ 0}{Total\ photons\ considered} \tag{8}$$

where *face 0* has an incident beam onto the face and *face 1* is the other face without incidence.

Although the DOM is a numerical method, consequently, an approached solution, it will be used as reference for the comparative analysis with MCM. For these specific proposed cases, analytical solutions are not available.

For all the cases, the DOM was solved using linear interpolation and Eq. (3) was used to not allow negative radiative intensities. For this purpose, the minimum control volumes number, function of the optic thickness, are presented in Tab. 1. The MCM use the same number of control volumes to respect the same conditions of analysis.

The quadrature used to MOD was the 24 directions Nicolau quadrature; propose by Nicolau (1994). This quadrature is particularly appropriate to considerer the divergence angle of the incident beam.

Figure 1 shows the hemispherical transmittance, hemispherical reflectance and the ratio between the incident and the transmitted intensity in the direction of the incident beam (normal) function of the optical thickness (τ_0). It is considered albedo equal to unity (ω=1). Figure 2 presents the same analysis, notwithstanding, using albedo 0.5. For the two cases, the MOD was used. As expected, the hemispherical transmittance and the radiative intensity in the incidence direction go to zero with the increase of the optical thickness.

Table 1. Number of control volumes used to different optic thickness

Optical thickness, τ_0	**Number of control volumes**
0.1	3
1.0	19
5.0	91
10.0	180
20.0	359
30.0	538
50.0	895
75.0	1342
100.0	1789

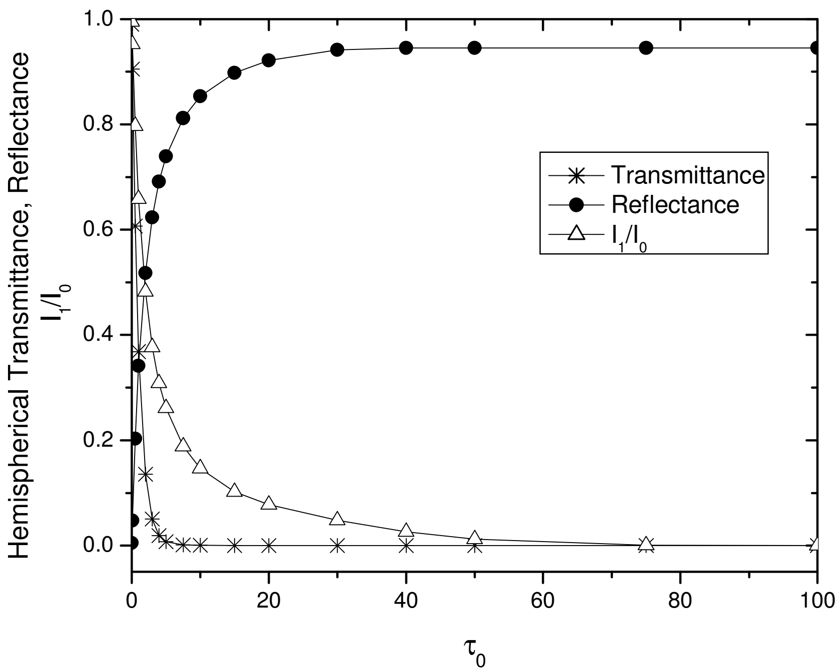

Figure 1. Hemispherical transmittance, hemispherical reflectance and the ratio between the incident and the transmitted intensity in the direction of the incident beam (normal), I_1/I_0 function of the optical thickness, τ_0 and ω=1.0

Figure 3 presents the percentage relative error in hemispherical transmittance, hemispherical reflectance and the ratio between the incident and the transmitted intensity in the direction of the incident beam (normal), I_1/I_0 function of the particle number, for unitary albedo and optical thickness (ω=1 e τ_o=1). From particle numbers more than 10^5 the results are less random behavior. Figure 4 presents a comparative analysis between the two methods to the processing time to different optical thickness, using a 3.0GHz PC computer. It was used ω=1 and 10^6 packages. As expected, the Monte Carlo method present a bigger processing time.

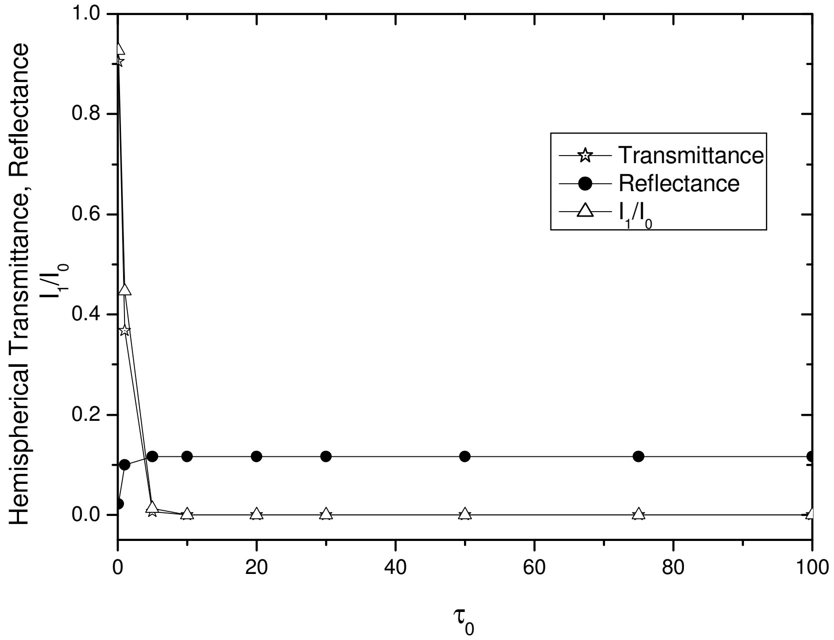

Figure 2. Hemispherical transmittance, hemispherical reflectance and the ratio between the incident and the transmitted intensity in the direction of the incident beam (normal), I_1/I_0 function of the optical thickness, τ_0 and ω=0.5

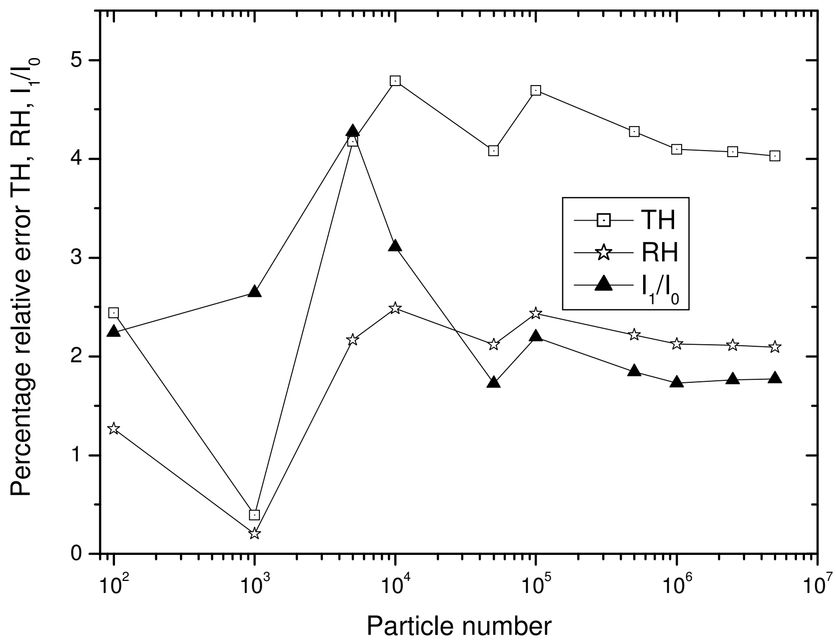

Figure 3. Percentage relative error in hemispherical transmittance, hemispherical reflectance and the ratio between the incident and the transmitted intensity in the direction of the incident beam (normal), I_1/I_0 , function of the particle number to ω=1 e τ_o=1

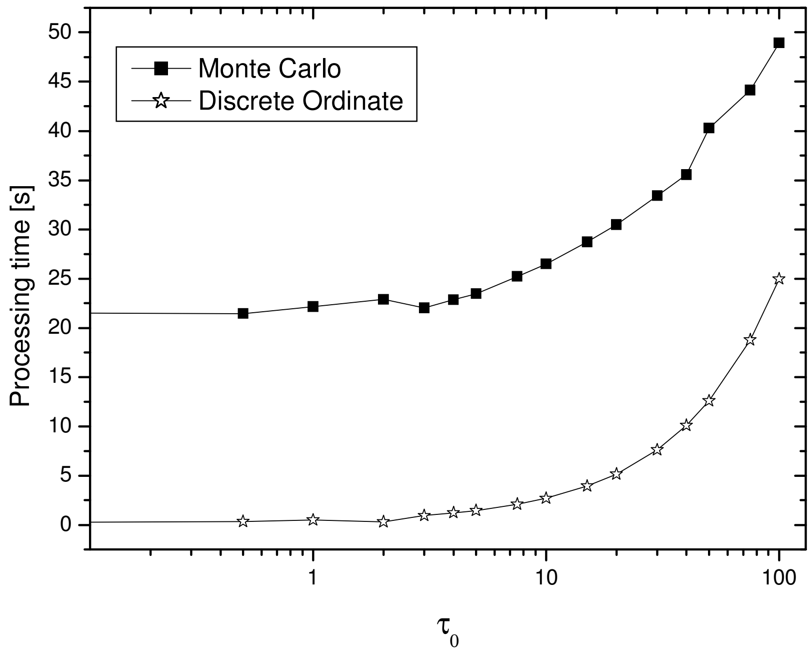

Figure 4. Comparative processing time analysis function of optical thickness. It was used ω=1 and 10^6 packages.

Figure 5 presents the absolute error in the ratio between the radiative intensity, I_1/I_0 , always using the DOM like reference. Increasing the optical thickness the absolute error of Monte Carlo method the Mount decreases, probably due to the great attenuation by the media.

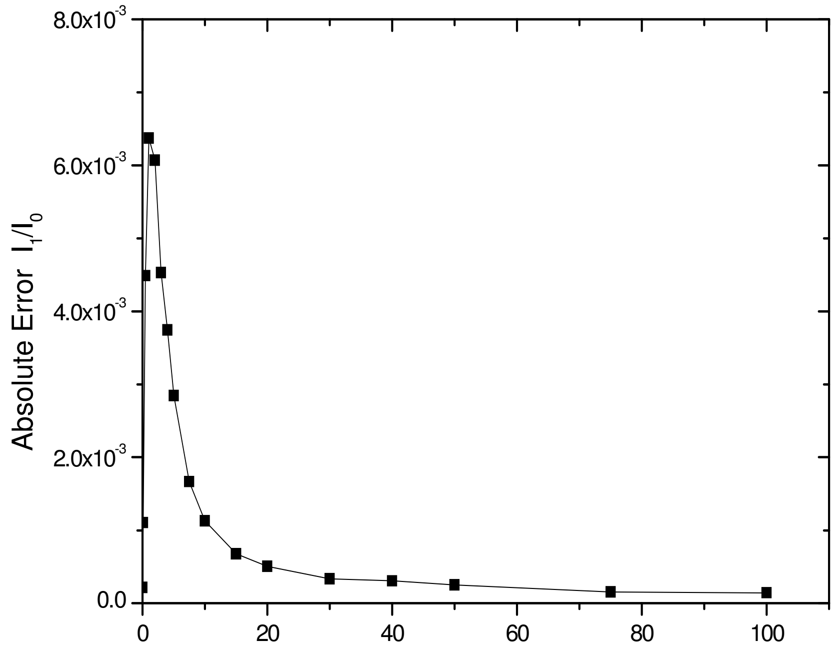

Figure 5. Absolute error in the ratio between the radiative intensity, I_1/I_0 , always using the DOM like reference. Increasing the optical thickness the absolute error of Monte Carlo method the Mount decreases, probably due to the great attenuation by the media. ω=1 and 10^6 packages

Figures 6 and 7 present the absolute error to hemispherical reflectance and hemispherical transmittance function of optical thickness, respectively. The Albedo is unitary, and 10^6 packages are used to Monte Carlo method. The absolute error is presented because values are close to zero (Figs. 1 and 2) and relative analyses will present false great errors.

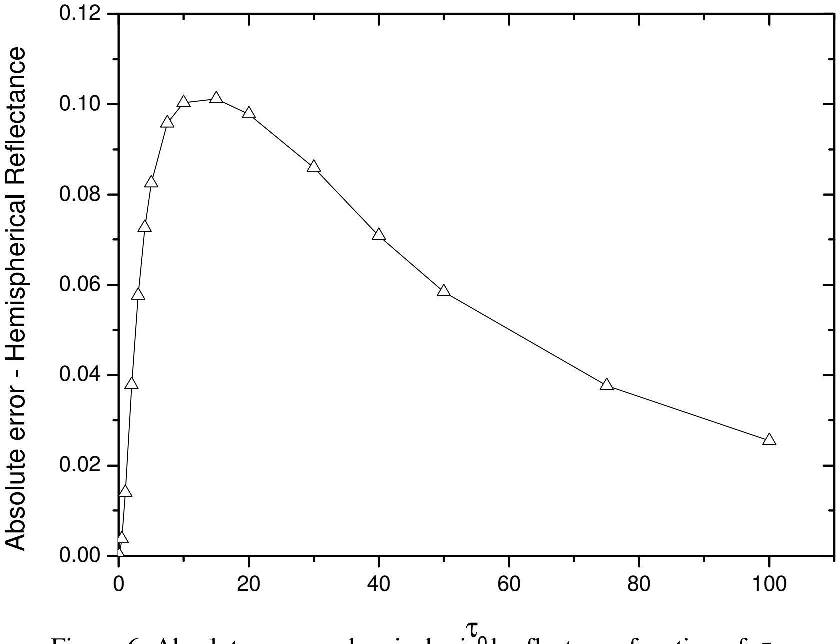

Figure 6. Absolute error on hemispherical reflectance function of τ_0.

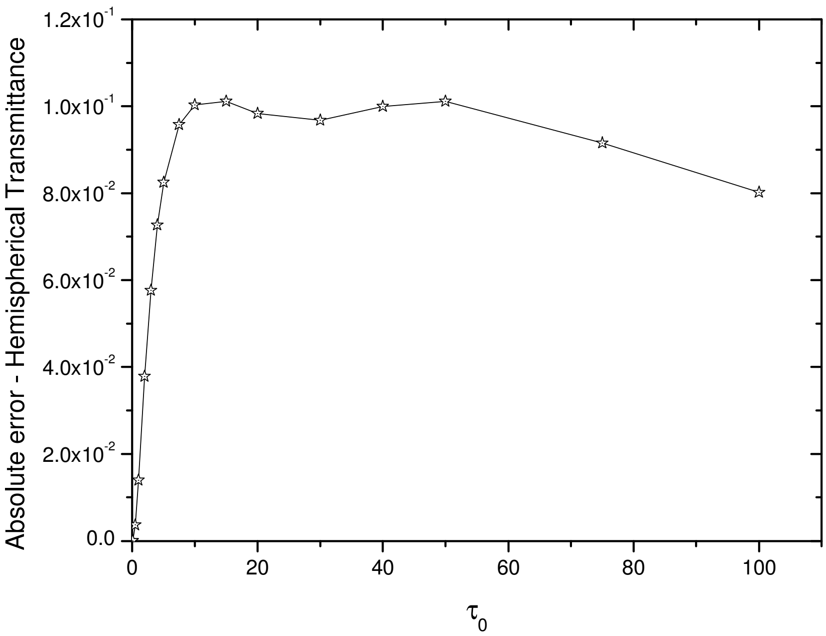

Figure 7. Absolute error on hemispherical transmittance function of τ_0.

4. CONCLUSION

In this work a comparative study of the Monte Carlo and Discrete Ordinate methods to the radiative transfer problem in anisotropic one-dimensional geometry was presented. Considering the Ordinate Discrete method like a benchmark, tests are performed, changing the optical thickness and the albedo. Hemispherical transmittance and reflectance are used to compare the results. It was possible to verify the efficiency and the limitations of each method. Monte Carlo method presents better results when it works with a significant number of packages, however a sensible increase in the processing time will result. Analysis to an anisotropic media will be improved in the Monte Carlo algorithm to the future works. The use of other boundary conditions will be performed to use the analytical benchmark solutions.

5. REFERENCES

Ambirajan, A., Look JR, D.C., 1996, "A Backward Monte Carlo Estimator for the Multiple Scattering of a Narrow Light Beam", Journal of Quantitative Spectroscopy and Radiative Transfer, Vol. 56, No 3, September 1996 , pp. 317-336.

Brewster, M.Q., 1992, "Thermal radiative transfer and properties". New York: John Wiley and Sons, 543p..

Carlson, B.G. and Lathrop, K.D., 1968, "Transport Theory - The Method of Discrete Ordinates, In: Computing Methods in Reactor Physics". Edited by H. Greespan, C. N. Kleber and D. Okrent. New York: Gordon and Breach, p. 171-266.

Chandrasekhar, S., 1960, "Radiative Transfer". New York : Dover Publication, 393p.

Churmakov, D. Y., Meglinski, I. V., Piletsky, S. A. and Greenhalgh, D. A., 2003, "Analysis of skin tissues spatial fluorescence distribution by the Monte Carlo simulation". J Phys D Appl Phys, v. 36, n. 14, p. 1722-1728.

Computational Science Education Project (CSEP),1995, "Eletronic Book – Introduction to Monte Carlo Methods and Random Number Generators". (fonte:http://csep1.phy.ornl.gov)

Fiveland, W.A., 1987, "Discrete ordinate methods for radiative hear transfer in isotropically and anisotropically scattering media". J. of Heat Transfer, Vol. 109, pp. 809-812.

Fiveland, W.A., 1985, "Discrete ordinate methods for radiative hear transfer in isotropically and anisotropically scattering media". ASME HTD. Vol. 49, pp.1-8.

Henson and Malalasekera, 1997, "Comparison of Discrete Transfer and Monte Carlo Methods for Radiative Heat Transfer in Three-Dimensional Nonhomogeneous Scattering Media", Numerical Heat Transfer, Part A, 32:19-36.

Lathrop, K. D., 1969, "Spatial Differencing of the Transport Equation: Positivity vs Accuracy", Journal of Computational Physics, vol.4, pp. 475 - 498.

Modest, M.F., 2001, "Radiative Heat Transfer", 2° Edition, New York: McGraw-Hill, 832p.

Moura, L.M, Baillis, D. and Sacadura, J.F., 1997, "Analysis of the discrete ordinate method: angular discretization". 14th Brazilian Congress of Mechanical Engineering, CD-ROM, December 8-12th, COB1425, Bauru, Brasil.

Nicolau, V.P., Raynaud, M. and Sacadura, J.-F., 1994, "Spectral radiative properties identification of fiber insulating materials". Int. J. Heat Mass Transfer, Vol. 37, pp. 311-324.

Özisik, M.N., 1973, "Radiative Transfer and interaction with conduction and convection", John Wiley and Sons, New York, 575p

Prahl, S. A., Keijzer, M., Jacques, S. L. and Welch, A. J., 1989, "A Monte Carlo model of light propagation in tissue". SPIE, Proc. Dosimetry of Laser Radiation In Medicine And Biology, v. IS 5, p. 102-111.

Ruan, L.M., Tan, H. P. and Yan, Y. Y., 2002, "A Monte Carlo method applied to the medium with nongray absorbing-emittieng-anisotropic scattering particles and gray approximation", Harbin: Numerical Heat Transfer, part A.

Yang, W.-J.; Hiroshi T. and Kazuhiko K., 1995, "Radiative Heat Transfer by the Monte Carlo Method". San Diego: Academic Press, 215 p.

Wang, L. V., 1998, "Rapid modeling of diffuse reflectance of light in turbid slabs". J Opt. Soc. Am A, v. 15, No 4, p. 936-944.

Wong, B. T. and Mengüç, M. P., 2002, "Comparison of Monte Carlo techniques to predict the propagation of a collimated beam in participating media". Lexington: Numerical Heat Transfer, part B.

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