# A FINITE DEFORMATION VISCO-HYPOELASTICITY MODEL

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***Abstract.*** *The objective of this work is to propose a finite deformation visco-hypoelasticity model and a numerical scheme for the analysis of polimeric materials. The proposed rate form of constitutive equatios are formulated in terms of the Green-Naghdi rate Kirchhoff stress and the rate of the logarithm (Henchy) strain tensor. The integration of the rate constitutive equation yields an integro-diferential form of constitutive equation, which is formulated in terms of the rotated Kirchhoff stress tensor. The material is assumed to be isotropic and the kernel functions, associated with the shear and bulk modulus, are represented in a Prony series. The problem is formulated within a Total Lagrangian framework and is solved by the Galerkin finite element method. Finally a problem case is solved, and the proposed model, the robustness and performance of the algorithms employed are tested against exact solution of the problem.*

## 1. INTRODUCTION

The materials are termed viscoelastic because they exhibit both solid- and fluid-like behaviour, and amongst examples of such media we find concrete and the thermoplastic polymers. They are easily moulded, resistant to corrosion, lightweight and may be made very strong plastics are polymers and they consist of so called long chain molecules. The long chains, or backbones, are constructed by joining a great many hydrocarbon monomers together to form periodic sequences of specific arrangements of hydrogen and carbon atoms; this is then the long chain, the polymer. To get a qualitative picture of why this is so we need to distinguish the two basic types of polymer: Uncross linked, in these polymeric systems the long chains are joined by the intertwining alone, there are no chemical bonds between molecules and Cross linked, the molecules are not only tied together by the intertwining but also by chemical bonds between molecules.

Consider a thin uncross linked polymeric bar subjected to a tensile force, the bar will respond first by the molecules stretching throughout the volume, corresponding to an instantaneous elastic deformation, and then will begin to unravel, this is an internal flow. When the load is removed the molecules will allow some contraction as they attempt to return to their equilibrium length, but the flow deflection cannot recover since no flow of molecule over molecule can occur in the absence of a force. If now we imagine a cross linked polymeric bar subjected to the same loading there will be an elastic response and recovery in exactly the same way as before and an internal flow. However, this time when the force is removed the cross linking will produce a reverse internal flow allowing the bar to return to its original length. These are extreme cases: in practice it is possible for there to exist intermediate polymeric structures. We can intuitively classify an uncross linked polymer as a viscoelastic fluid since the deformation may continue indefinitely all the while the force is applied, on the other hand the cross linked polymer is a viscoelastic solid since the cross linking bonds will produce equilibriating forces when the strain exceeds a certain value.

Solid polymers can occur in the amorphous or crystalline state. Polymers in the amorphous state are characterized by a disordered arrangement of the macromolecular chains, which adopt conformations corresponding to statistical coils. The crystalline state is characterized by a long-range threedimensional order (order extending to distances of hundreds or thousands of times the molecular size of the repeating unit). The macromolecular chains in this state adopt fixed conformations such as planar zigzag, or helical, These chains are aligned parallel to each other, forming a compact packing that gives rise to a three-dimensional order. Many polymers have the capability to crystallize. This capability basically depends on the structure and regularity of the chains and on the interactions between them. The term “semicrystal1ine state” should be used rather than crystalline state, because regions in which the chains or part of them have an ordered and regular spatial arrangement coexist with disordered regions typical of the amorphous state.

Viscoelastic behavior is typical of a number of materials such as polymers and plastics, such it was said in above lines. These materials have memory, i.e. the stress depends on the entire history of the deformation and typically this memory fades with time. The stress therefore can be represented as a functional of the history of the deformation which, due to the requirement that the principle of material objectivity needs to be satisfied, leads to reasonably complex relations even in the “simplest” of constitutive relations. Later in this paper it describe a finite viscoelastic model based in Hencky strain measure in association with Boltzmann superposition principle used in small strain contexts to the case of large deformation. The extension to the large deformation case is achieved by considering a multiplicative decomposition of the deformation gradient and taking a logarithm strain measure with rotated Kirchhoff stress as the conjugated pair. Before presenting the model, it begin this paper by briefly discussing the basic physical phenomena of creep and stress relaxation, typical for viscoelastic materials. The next section starts by briefly discussing the two general categories of viscoelastic constitutive models; integral and differential models, before presenting the derivation of our model which is of the integral type. Finally the section discusses the application of this model in the uniaxial test.

## 2. CONSTITUTIVE LAWS

Viscoelastic media are characterized by two basic phenomena: creep and stress relaxation. We consider both cases below

### 2.1 Creep and Relaxation

Consider a simple uniaxial bar of a viscoelastic material, subjected to an instantaneously applied and sustained tensile load as shown in next figure.

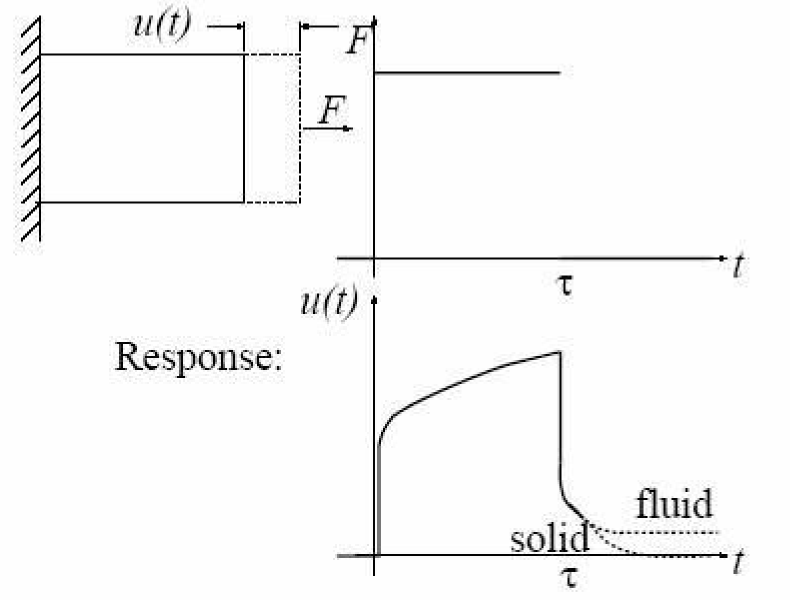


Figure 1. Schematic creep

Firstly imagine a qualitatively similar bar, again fixed at one end, to which an on/off axial step loading is applied at the other, that is, at $t_o = 0$ the loading increases instantaneously to the constant value $F$ and at $\tau > t_o$ $F$ is removed as illustrated in figure. The bar now responds by extending in length and we denote this extension by $u(t)$. At the instant $t_o+$ there is again linear elastic behaviour and $u(t_o+)$ is therefore given by Hooke's law, but over the range $t \in (t_o, \tau)$, the bar will continue to extend, this behaviour is termed creep. If for arbitrarily large $\tau$ this creep continues indefinitely then the material is a viscoelastic fluid, on the other hand if $u(t)$ approaches some constant value then it is a viscoelastic solid. Upon removal of the load $F$ at $t = \tau$ the material again displays elastic behaviour, there is an elastic recovery where $u$ instantaneosly snaps back to a lesser value. Thereafter is another creep phase during which the material attempts to return to its original configuration. It can make a solid fluid distinction: if, as $t \to \infty$ we have $u \to 0$ then the material is a solid but if $u \to constant \neq 0$ then there is a permanent set within the bar caused by irreversible molecular flow during the initial creep phase and the material is a fluid. The ability of a viscoelastic material to attempt to return to its original configuration even after inelastic deformation has taken place implies that it somehow has an internal record of its initial state. If this is the case then it seems plausible that it also keeps a record of all of its states up to the present time and for this reason viscoelastic materials are said to possess memory.

Now we think of the bar as being fixed rigidly at one end and, at some reference time $t = 0$, an instantaneous longitudinal displacement $u$, applied to the other as schematically shown in the figure on next page

In response an internal longitudinal stress $\sigma$ is set up within the bar which at the instant, $t = 0+$, is given by Hooke's law, and so the instantaneous response of the material is linear elastic. Over time however the stress decreases monotonically to either the constant, nonzero, value $\sigma_o$ or to zero. In the former case the material is termed a viscoelastic solid whilst in the latter a viscoelastic fluid. This phenomena is known as stress relaxation.

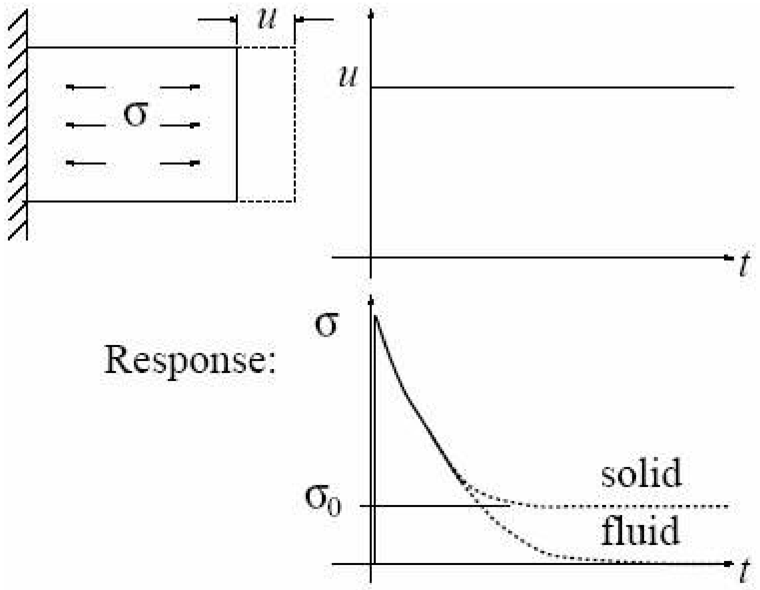


Figure 2. Schematic stress relaxation

### 2.2 Viscoelastic Constitutive Model

As we have seen for an elastic solid, the stress is determined by the deformation of the material relative to a fixed reference configuration. However, it is evident that several materials, exhibit viscoelastic characteristics, i.e. the magnitude of the measured stress depends on strain, strain rate, and time, (and temperature as well, although we will not consider it here), such that the stress in viscoelastic materials can be represented as a functional of the history of the deformation. In other words, these materials have memory: the stress depends on the entire history of the motion, and typically this memory fades with time. There are a number of approaches for constructing constitutive models for the large strain viscoelastic deformation. Mathematical relations which describe how stress can depend on the history of the deformation are either given in integral form or in differential form. In integral form the stress at time t is given in terms of an expression which involves an integral over previous times. The integral in such an expression is known as a history integral. In contrast, with a differential form of a constitutive model the history of the deformation is taken into account by certain ordinary differential equations which describe how certain quantities known as internal variables evolve in time. For a more detailed presentation of both ways of representing viscoelastic constitutive models see, for example, [1], [2], [3] and [4].

In order to generalize this constitutive equation to finite deformation problems, it is necessary to introduce some definitions

**Definition 2.1** *Let be $v$, $A$, and $D$ (first order, second order and fourth order tensor respectively), one defines the pull back or bar transformation as*

$$\bar{v}(t) = \Theta(t)^T v(t) \qquad \bar{A}(t) = \Theta(t)^T A(t)\Theta(t) \qquad \bar{D}(t) = \Theta(t)^T \Theta(t)^T D(t)\Theta(t)\Theta(t) \tag{1}$$

*and from continuum mechanics*

$$\bar{\mathbf{F}}(t) = \Theta(t)^T \mathbf{F}(t) \qquad \bar{\mathbb{D}}(t) = \Theta(t)^T \mathbb{D}(t)\Theta(t). \tag{2}$$

These pull back (rotation neutralized) quantities with $\Theta(t)$ are used to define a convenient framework to perform the integration of the constitutive model. Indeed, following the previous definition, the pull back Kirchhoff stress with $\Theta(t)$ can be defined as

$$\bar{\tau}(t) = \Theta(t)^T \tau(t)\Theta(t) \tag{3}$$

and one can show that

$$\dot{\bar{\tau}}(t) = \Theta(t)^T \check{\bar{\tau}}(t)\Theta(t) \tag{4}$$

where $\check{\bar{\tau}}$ denotes the Green-Naghdi rate of the kirchhoff stress, which is given by

$$\check{\bar{\tau}}(t) = \mathbf{R}\frac{d}{dt}(\mathbf{R}^T \tau \mathbf{R})\mathbf{R}^T = \dot{\tau} - \mathbf{\Xi}\tau + \tau\mathbf{\Xi} \tag{5}$$

where $\mathbf{\Xi} = \dot{\mathbf{R}}\mathbf{R}^T$. Let us define the rotation tensor $\Theta(t)$, as the solution to the following initial value problem:

**Problem 22** *Given $\mathbf{\Xi}(t)$, find $\Theta(t)$ that solves*

$$\dot{\Theta(t)}\Theta(t)^T = \mathbf{\Xi}(t); \tag{6}$$

$$\rightarrow \dot{\Theta(t)} = \mathbf{\Xi}(t)\Theta(t); \tag{7}$$

$$\Theta(0) = \mathbf{I} \tag{8}$$

*where $\mathbf{\Xi}(t) = \mathbf{R}\mathbf{R}^T$ (skew symmetric tensor)and $\mathbf{R}$ is determined from the polar decomposition of $\mathbf{F}$, i.e., $\mathbf{F} = \mathbf{R}\mathbf{U}$.*

By simple observation, one can verify that the solution to the initial value problem is given by

$$\Theta(t) = \mathbf{R}(t). \tag{9}$$

The Boltzmann superposition principle follows by assuming linearity the empirical observation that step discontinuities in strain are governed, at the instant of the step, by Hooke's law implies that, by approximating a smooth strain history with piecewise constant step functions we could employ a Boltzmann type of superposition of Hookean responses to derive the underlying constitutive law. Each component of the strain is now approximated with a piecewise continuous step function. For sufficiently smooth fields, the general linear constitutive equation for the linear viscoelastic (small deformation) solid is given by

$$\sigma_{ij}(t) = \mathbb{D}_{ijkl}(t-t_0)\varepsilon_{kl}(t_0) + \int_{t_0}^{t} \mathbb{D}_{ijkl}(t-\xi)\dot{\varepsilon}_{kl}(\xi)d\xi \tag{10}$$

or equivalently

$$\sigma_{ij}(t) = \mathbb{D}_{ijkl}(t_0)\varepsilon_{kl}(t-t_0) - \int_{t_0}^{t} \dot{\mathbb{D}}_{ijkl}(t-\xi)\varepsilon_{kl}(\xi)d\xi$$

Considering now a viscoelastic isotropic material, where

$$\mathbb{D}(t) = 2\,G(t)\,\mathbb{I}_{dev} + K(t)\,(\mathbf{I}\otimes\mathbf{I}) \tag{11}$$

in which

$$\mathbb{I}_{dev} = \left\{\mathbb{I} - \frac{1}{3}(\mathbf{I}\otimes\mathbf{I})\right\} \tag{12}$$

where $\mathbf{I}$ is the second order identity tensor and $\mathbb{I}$ is the fourth order identity tensor and the kernel functions are represented in terms of Prony series, which assumes that:

$$G(t) = G_\infty + \sum_{i=1}^{n_G} G_i \exp\left(-\frac{t}{\tau_i^G}\right), \tag{13}$$

$$K(t) = K_\infty + \sum_{i=1}^{n_K} K_i \exp\left(-\frac{t}{\tau_i^K}\right), \tag{14}$$

in which $G_\infty$ and $G_i$ are shear elastic moduli, $K_\infty$ and $K_i$ are bulk elastic moduli and $\tau_i^G$ and $\tau_i^K$ are the relaxation times for each Prony component. An a alternative formulation for the quasi-static rate model is the constitutive equation given in

$$\varepsilon(t) = \mathbb{C}(t-t_0)\sigma(t_0) + \int_{t_0}^{t} \mathbb{C}(t-\xi)\dot{\sigma}(\xi)d\xi \tag{15}$$

in which

$$\mathbb{C} = \frac{1}{9K}(\mathbf{I}\otimes\mathbf{I}) + \frac{1}{2G}\mathbb{I}_{dev} \tag{16}$$

or

$$\mathbb{C} = \frac{B}{9}(\mathbf{I}\otimes\mathbf{I}) + \frac{J}{2}\mathbb{I}_{dev} \tag{17}$$

where $B = \frac{1}{K}$ is the bulk compliance function and $J = \frac{1}{G}$ is the shear compliance function with the kernel functions are represented in terms of Prony series, which assumes that:

$$J(t) = J_0 + \sum_{i=1}^{n_j} J_i \left(1 - \exp\left(-\frac{t}{\tau_i^J}\right)\right); \tag{18}$$

$$B(t) = B_0 + \sum_{i=1}^{n_B} B_i \left(1 - \exp\left(-\frac{t}{\tau_i^B}\right)\right) \tag{19}$$

in which $J_0$ and $J_i$ are shear compliance moduli, $B_0$ and $B_i$ are bulk compliance moduli and $\tau_i^J$ and $\tau_i^B$ are the relaxation times for each Prony component.

### 2.3 The Hencky Model

The Hencky's logarithmic strain measure model was proposed in 1938 to study elastic behaviour of rubbers at some simple finite deformation modes.The Hencky's logarithmic strain or natural strain has inherent advantages over other strain measures in his study of a priori constitutive inequalities and treated the Hencky strain, its rate and its work-conjugate stress as basic measures for strain, strain rates and stresses. The Hencky's logarithmic tensor $\mathbf{E}$, based in Lagrangean formulation, can be defined in following way:

$$\begin{aligned}\mathbf{E}(\mathbf{x}_o, t) :&= \frac{1}{2}\ln\left(\mathbf{C}(\mathbf{x}_o, t)\right); \\ &= \ln\left(\mathbf{U}(\mathbf{x}_o, t)\right).\end{aligned} \tag{20}$$

where $\mathbf{U}$ is the symetric positive defined second order tensor from polar decomposition of $\mathbf{F} = \mathbf{RU}$ (gradient of the deformation function). From the spectral decomposition:

$$\mathbf{U}(\mathbf{x}_o, t) = \sum_{i=1}^{n} \sqrt{\Lambda_i}\left(\Phi_i \otimes \Phi_i\right) \tag{21}$$

with $\{\Phi_i\}_{i=1}^n$ e $\{\Lambda_i\}_{i=1}^n$ are the eigenpairs of $\mathbf{U}$. Let the Cauchy $\sigma$ stress tensor in reference configuration $\sigma$ and the Kirchhoff stress $\tau = J\sigma$, $J = \det(\mathbf{F})$, and

$$J = \frac{\rho_o}{\rho} \tag{22}$$

where $\rho_o$ and $\rho$ being the reference and the current mass densities, then, as pointed out by the famous Hill's work in 1978, the rate of stress work per unit of mass which is invariant under a change of strain measure and the reference configuration is used to generate stress measures conjugate to the family of strain measures

$$\dot{W} = \frac{1}{\rho}(\sigma : \mathbf{D}) = \frac{1}{\rho_o}(\tau : \mathbf{D}) = \frac{1}{\rho_o}\left(\mathbf{P} : \dot{\mathbf{F}}\right) = \frac{1}{2\rho_o}\left(\mathbf{S} : \dot{\mathbf{C}}\right) \tag{23}$$

in which $\mathbf{D}$ is the infinitesimal deformation rate, $\mathbf{P}$ is the first Piola-Kirchhoff stress, $\dot{\mathbf{F}}$ is the gradient of the deformation rate function, $\mathbf{S}$ is the second Piola-Kirchhoff stress and $\dot{\mathbf{C}}$ is the right Cauchy-Green strain rate tensor. Then

**Theorem 23** *The rotated Kirchhoff stress tensor* $(\bar{\tau})$ *forms the conjugated stress-tensor pair with the Hencky's logarithmic strain measure* $(\mathbf{E})$, *supposed the isotropy, i.e.*

$$\begin{aligned}\dot{W} &= \frac{1}{\rho_o}\bar{\tau} : \dot{\mathbf{E}}; \\ &= \frac{1}{\rho_o}\left((\mathbf{R}^T\tau\mathbf{R}) : \frac{\partial}{\partial t}(\ln(\mathbf{U}))\right).\end{aligned} \tag{24}$$

By a similar way it follows in this unrotated configuration the constitutive equation may be rewritten as

$$\bar{\tau}(t) = \bar{\mathbb{D}}(t)\mathbf{E}(0) + \int_0^t \bar{\mathbb{D}}(t-\xi)\dot{\mathbf{E}}(\xi)d\xi \tag{25}$$

and assuming that the material is isotropic, then $\bar{\mathbb{D}} = \mathbb{D}$, as a result

$$\bar{\tau}(t) = \mathbb{D}(t)\mathbf{E}(0) + \int_0^t \mathbb{D}(t-\xi)\dot{\mathbf{E}}(\xi)d\xi. \tag{26}$$

## 3. THE FINITE VISCOELASTIC PROBLEM

The approach used here is the total Lagragian formulation. Considering the reference configuration let be $\Omega_o$, defined at $t_o$, a bounded domain with a Lipschitz boundary $\partial\Omega_o$ subjected to a prescribed body force $\mathbf{b}$ defined on $\Omega_o$, a prescribed surface traction defined on $\Gamma_o^t$ and a prescribed displacement defined on $\Gamma_o^u$, where $\partial\Omega_o = \overline{\Gamma_o^u \cup \Gamma_o^t}$ and $\Gamma_o^u \cap \Gamma_o^t = \varnothing$. Taking the motion function $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$ is such that

$$\mathbf{x} = \varphi\left(\mathbf{x}_o, t\right) = \varphi_t\left(\mathbf{x}_o\right) \qquad \therefore \qquad \mathbf{x}_o = \varphi_t^{-1}\left(\mathbf{x}\right).$$

Thus the displacement field is defined as:

$$\mathbf{x} = \mathbf{u}\left(\mathbf{x}, t\right) + \mathbf{x}_o \qquad \therefore \qquad \mathbf{u}_o\left(\mathbf{x}_o, t\right) = \varphi_t\left(\mathbf{x}_o\right) - \mathbf{x}_o = \mathbf{x} - \mathbf{x}_o = \mathbf{x} - \varphi_t^{-1}\left(\mathbf{x}\right) = \mathbf{u}\left(\mathbf{x}, t\right).$$

Thus it is possible announce the problem in the reference configuration as:

**Problem 3.1** *For each $t \in [0, t_f]$ determine $\mathbf{u}_o\left(\mathbf{x}_o, t\right)$ that solves the following boundary value problem stated as*

$$\begin{aligned} \operatorname{div}\mathbf{P}\left(\mathbf{x}_o, t\right) + \rho_o\left(\mathbf{x}_o\right)\bar{\mathbf{b}}\left(\mathbf{x}_o, t\right) = 0 \qquad & in\ \Omega_o \\ \mathbf{P}\left(\mathbf{x}_o, t\right)\mathbf{n}_o\left(\mathbf{x}_o, t\right) = \bar{\mathbf{t}}_o\left(\mathbf{x}_o, t\right) \qquad & on\ \Gamma_o^t \\ \mathbf{u}_o\left(\mathbf{x}_o, t\right) = \bar{\mathbf{u}}_o\left(\mathbf{x}_o\right) \qquad & on\ \Gamma_o^u. \end{aligned} \tag{27}$$

with $\bar{\mathbf{b}}\left(\mathbf{x}_o, t\right) \in L^2\left(\Omega_o\right)$ for each $t \in [0, t_f]$ and $\bar{\mathbf{u}}\left(\mathbf{x}_o\right) \in H_{00}^{\frac{1}{2}}\left(\Gamma_o\right)$ for each $t \in [0, t_f]$. Let us define now the following sets, for each time $t \in [0, t_f]$

$$\begin{aligned} Kin_u^o &= \left\{\mathbf{u}_o : \Omega_o \to \mathbb{R}^3 \mid \mathbf{u}_o \in H^1\left(\Omega_o\right), \mathbf{u}_o\left(\mathbf{x}_o, t\right) = \bar{\mathbf{u}}_o\left(\mathbf{x}_o\right) \text{ at } \mathbf{x}_o \in \Gamma_o^u\right\} \\ Var_u^o &= \left\{\hat{\mathbf{v}} : \Omega_o \to \mathbb{R}^3 \mid \hat{\mathbf{v}} \in L^2\left(\Omega_o\right), \hat{\mathbf{v}}\left(\mathbf{x}_o\right) = 0 \text{ at } \mathbf{x}_o \in \Gamma_o^u\right\} \end{aligned}$$

and it has the weak form of the prblem

**Problem 3.2** *Determine $\mathbf{u}_o\left(\mathbf{x}_o, t\right) \in Kin_o^u$, for each $t \in [0, t_f]$, such that*

$$\int_{\Omega_o} \mathbf{P} \cdot \nabla\hat{\mathbf{v}} d\Omega_o = \int_{\Omega_o} \rho_o \mathbf{b} \cdot \hat{\mathbf{v}} d\Omega_o + \int_{\Gamma_o^t} \mathbf{t}_o \cdot \hat{\mathbf{v}} d\Omega_o \qquad \forall\ \hat{\mathbf{v}} \in Var_o^u \tag{28}$$

*or denoting, for each $t \in [0, t_f]$,*

$$\mathcal{F}\left(\mathbf{u}_o; \hat{\mathbf{v}}\right) = \int_{\Omega_o} \mathbf{P} \cdot \nabla\hat{\mathbf{v}} d\Omega_o = \int_{\Omega_o} \rho_o \mathbf{b} \cdot \hat{\mathbf{v}} d\Omega_o + \int_{\Gamma_o^t} \mathbf{t}_o \cdot \hat{\mathbf{v}} d\Omega_o$$

*the problem can be stated as*

**Problem 3.3** *Determine $\mathbf{u}_o\left(\mathbf{x}_o, t\right) \in Kin_o^u$, for each $t \in [0, t_f]$, such that*

$$\mathcal{F}\left(\mathbf{u}_o; \hat{\mathbf{v}}\right) = 0 \qquad \forall\ \hat{\mathbf{v}} \in Var_o^u. \tag{29}$$

### 3.1 Incremental Formulation

To solve the non linear system 29, it poupose the Newton Method, let

$$\mathbf{u}_{n+1}^0 = \mathbf{u}_n, \qquad k = 0$$

that is, $k$ will denote the iteration of the Newton process with $k = 0$ being the initial value assumed as the last converged incremental of $\mathbf{u}$, i.e., $\mathbf{u}_n$. Thus, at the *k-th* iteration

$$\mathbf{u}_{n+1}^{k+1} = \mathbf{u}_{n+1}^k + \Delta\mathbf{u}_{n+1}^k.$$

With the objective of determining $\Delta\mathbf{u}_{n+1}^k$ it is necessary impose

$$\mathcal{F}\left(\mathbf{u}_{n+1}^{k+1}; \hat{\mathbf{v}}\right) = 0 \qquad \forall\ \hat{\mathbf{v}} \in Var_o^u.$$

Now, considering $\mathcal{F}\left(\cdot, \cdot\right)$ to be sufficiently smooth

$$\mathcal{F}\left(\mathbf{u}_{n+1}^{k+1}; \hat{\mathbf{v}}\right) = \mathcal{F}\left(\mathbf{u}_{n+1}^k + \Delta\mathbf{u}_{n+1}^k; \hat{\mathbf{v}}\right) = 0. \tag{30}$$

Expanding $\mathcal{F}\left(\mathbf{u}_{n+1}^{k}+\Delta\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right)$ in Taylor series at $\mathbf{u}_{n+1}^{k}$

$$\mathcal{F}\left(\mathbf{u}_{n+1}^{k}+\Delta\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right)\simeq\mathcal{F}\left(\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right)+D\mathcal{F}\left(\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right)\left[\Delta\mathbf{u}_{n+1}^{k}\right] \tag{31}$$

hence

$$D\mathcal{F}\left(\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right)\left[\Delta\mathbf{u}_{n+1}^{k}\right]=-\mathcal{F}\left(\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right). \tag{32}$$

Now it has

1. The initial value is the same from the last convergent value
$\mathbf{u}_{n+1}^{0}=\mathbf{u}_{n}, \qquad k=0$
and this value is actualized
$\mathbf{u}_{n+1}^{k+1}=\mathbf{u}_{n+1}^{k}+\Delta\mathbf{u}_{n+1}^{k}$ (33)
2. The $\Delta\mathbf{u}_{n+1}^{k}$ is got by the solution of this equation
$\int_{\Omega_o}\mathbb{A}\left(\mathbf{u}_{n+1}^{k}\right)\nabla\left(\Delta\mathbf{u}_{n+1}^{k}\right)\cdot\nabla\hat{\mathbf{v}}\,d\Omega_o=-\mathcal{F}\left(\mathbf{u}_{n+1}^{k};\hat{\mathbf{v}}\right)$

where $\mathbb{A}\left(\mathbf{u}_{n+1}^{k}\right)$ represents the tangent modulus

$$\left[\mathbb{A}\left(\mathbf{u}_{n+1}^{k}\right)\right]_{ijkl}=\left.\frac{\partial P_{ij}}{\partial F_{kl}}\right|_{\mathbf{u}_{n+1}^{k}}. \tag{34}$$

in which

$$[\mathbf{P}]_{ij}=P_{ij}; \tag{35}$$

$$[\mathbf{F}]_{kl}=F_{kl}. \tag{36}$$

### 3.2 NUMERICAL EXAMPLE

In this example, one is considered the plane strain state. The body is a rectangular shape, in which the dimentions are width $1mm$ and height $2mm$. Let be presented the following reference configuration $\Omega_0=\{(x,y)\,|0\le x\le 1$ and $0\le y\le 1\}$ under simety considerations and for the quasi-static problem one has the boundary conditions: $u(0,y,t)$ =0, $v(0,y,t)$ =0, $u(1,y,t)$ =$0.1t\,mm$ and the other conditions are tension free for $t\in[0,1]$ (3). In this section one demonstrates the efficiency of the numerical algorithm (code) by comparing the known solutions against artificial exact solution. Invariably one design the loads and tractions so that the exact stretch have the form $x(t)=\lambda_1(t)x_o$ and $y(t)=\lambda_2(t)y_o$. By a simple computation from the boundary condition and supposing $G(t)=151989e^{-0.001t}+877289e^{-0.01t}+677823\,(Pa)$ one has that this block is composed by a polymer that has the following ralaxation function for $K$

$$K(t)=101330e^{-0.001t}+584860e^{-0.01t}+451880\,(Pa) \tag{37}$$

where the time scale is taken in days. In next page, one presents the time evolution of $\sigma_{xx}$ in the body (constant profile along entire body). Where one can observe the evolution of stress $\sigma_{xx}$ component with the time for one day obeservation (4). For this analysis, one used a mesh with two six points triangular elements. Other meshes are tested but anyone changes are detected in $\sigma_{xx}$ profile. The developed code is written in Fortran 90 and for post processing, it's used the GID 8.0 software. The global tolerance is $10^{-6}$. Note that the numerical results are equal to the exact solution of the problem for each time step. The last figure shows the displacement profile on the deformed body (5).

## 4. REFERENCES

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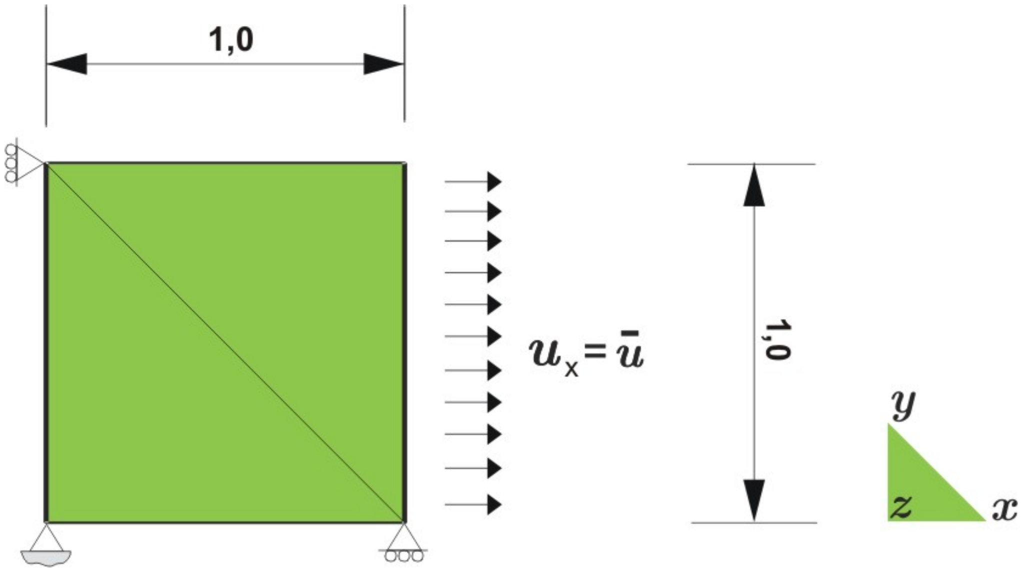

Figure 3. Traction test

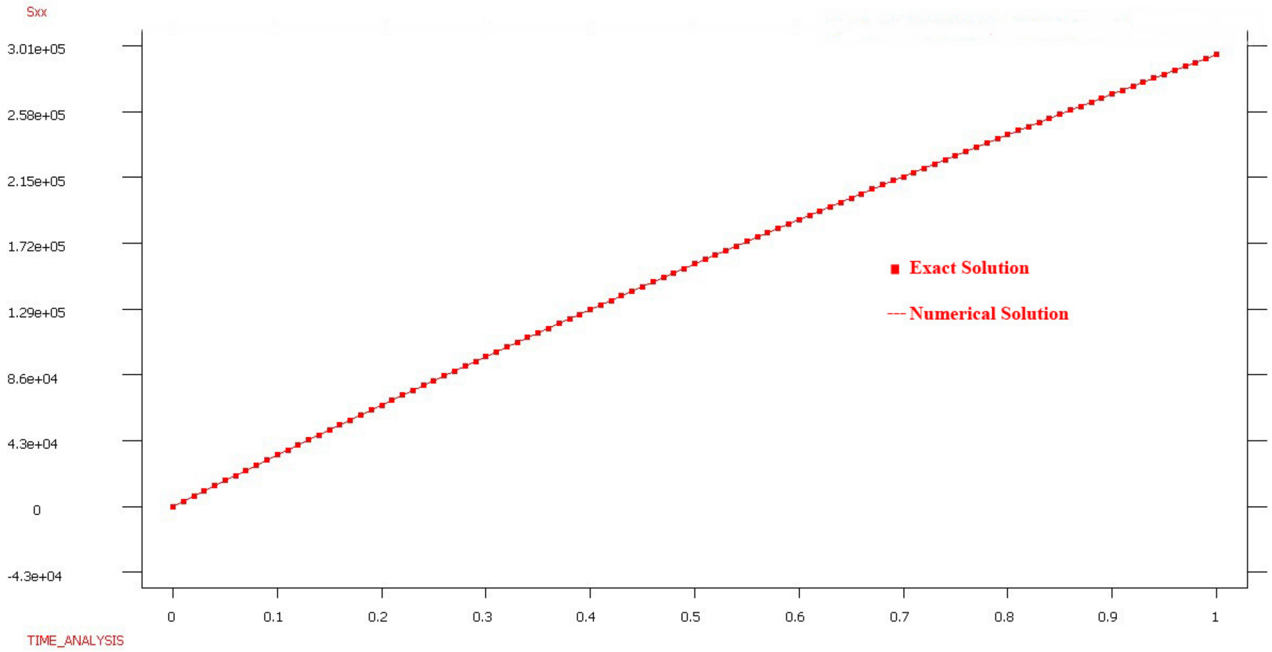

Figure 4. Tension (Pa) vs. time (days)

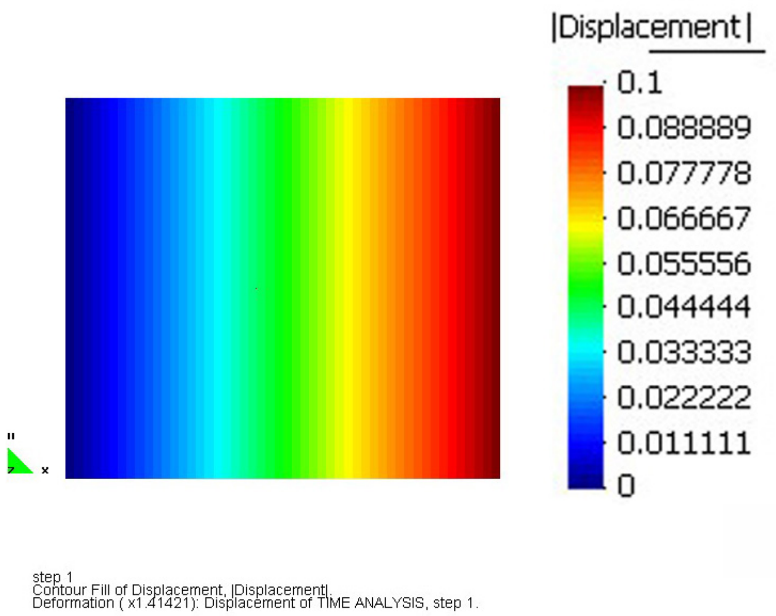


Figure 5. Displacement for $t = 1$ day