# A STOCHASTIC FINITE ELEMENT METHOD APPLIED FOR THE SOLUTION OF HEAT CONDUCTION PROBLEMS

**Maura Pauletto Taschetto, mauratascheto@hotmail.com**
**Marcelo Krajnc Alves, krajnc@emc.ufsc.br**
**Marcus Marinho, marcus@lva.ufsc.br**
Dep. Eng. Mecânica – Gmac, UFSC, CP 476, Florianópolis, SC, 88010-970, Brazil

**Claudio R. Ávila da S. Jr., avila@cefetpr.br**
Dep. Eng. Mecânica – Gmac, CEFET-PR, Curitiba, PR, Brazil

***Abstract.*** *The objective of this work is to propose a numerical proceedure for the solution of linear steady state heat conduction problems. One considers the thermal conductivity to be a non Gaussian second order stochastic processes with a known covariance function. The thermal conductivity process makes use of the Karhunen-Loeve expansion. In order to solve the stochastic differential equation, for the temperature random field, one employs a method based on Galerkin projections and extensions of Wiener's polynomial chaos. Specifically, one represents the stochastic temperature field with an optimum trial basis from the Askey family of orthogonal polynomials that reduces the dimensionality of the system and lead to an improvement of the convergence error. In order to validate the proposed proceedure one solves a simple heat conduction problem.*

***Keywords***: *Karhunen-Loeve expansion, stochastic differential equation, stochastic finite element method*

## 1. INTRODUCTION

A common practice in engineering is to analyze deterministic mathematical models. However, since such ideal models are not always adequate in modeling physical phenomena, the need to incorporate randomness is now clearly recognized, and there has been a growing interest in the applications of stochastic methods, Manas *et al.* (2001), Choi *et al.* (2004), Babuska *et al.* (2002) and Elman *et al.* (2005).

Probabilistic methods in engineering may be classified into two major categories: methods using a statistical approach and methods using a non-statistical approach. The statistical approach includes Monte Carlo simulation, Latin hypercube sampling, among others. These methods require sampling and estimation and are in general simple to apply. However, since the accuracy of the sampling techniques depends on the sample size, simulations can become prohibitively expensive. Thus, these methods are often used as the last resort leading most of the research effort to the development of non-statistical methods.

The most popular non-statistical method is the perturbation method. An inherent limitation of the perturbation method is that the uncertainties cannot be too large, i.e. variances of the random field cannot be too large compared with their mean values, e.g. typically less than 10%. Another approach is the Neumann expansion, which is based on the inverse of the stochastic operator in a Neumann series. This method too is restricted to small uncertainties and attempts have been made to couple it with the Monte Carlo simulation to obtain more efficient algorithms. Recently, a new non-statistical approach was proposed by Ghanem & Spanos that consists in the discretization of the random field in a polynomial chaos expansion method and has successfully applied it to various problems in mechanics. See Ghanem (1999), Ghanem and Kruger (1996) and Spanos and Ghanem (1989).

The polynomial chaos expansion is based on the homogeneous chaos theory of Wiener, see Wiener (1938), and is essentially a spectral expansion of the random variables. It allows high-order representation and promises fast convergence; coupled with Karhunen-Loeve decomposition for the input and Galerkin projection in random space, it results in computationally tractable algorithms for large engineering systems. More eficient Monte Carlo algorithms can also be designed when combined with the chaos expansion technique.

The classical polynomial chaos expansion is based on the Hermite polynomials in terms of Gaussian random variables. A more general framework, called the "generalized polynomial chaos" or the "Askey-chaos", was proposed in Xiu and Karniadakis (2002) and also applied by Silva (2006). Here the polynomials are chosen from the polynomials of the Askey-Wiener scheme, and may be applied to any type of random variables. Instead, the type of random variables is chosen according to the stochastic input and the weighting function of these random variables determines the type of orthogonal polynomials to be used as the basis in the random space. The convergence properties of different bases were studied in Xiu and Karniadakis (2002) and exponential convergence rate was demonstrated for model problems.

## 2. THEORETICAL DEVELOPMENT

### 2.1. Strong formulation of the stochastic boundary value problem

Consider the conduction-convection fin problem illustrated in Figure 1,

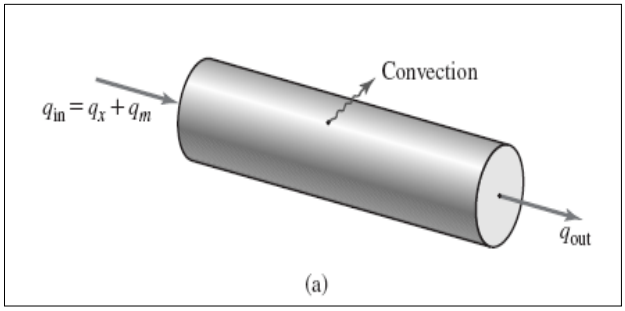


Figure 1. – Definition of the problem

where the *x*-direction is the bar longitudinal axis. Here, one assumes that at $x = 0$ the bar is subjected to a prescribed temperature, i.e. $T(0) = \overline{T}$ , and that the cross section at $x = L$ , is isolated, i.e. $q(L) = 0$.

By allowing the heat conductivity $k(x,\ \theta)$ to be a random field defined on a probability space $(\Omega,\ F,\ P)$, one obtains a stochastic boundary value problem (SBVP), the solution of which must then also be a random field. Thus, one seeks $T(x,\ \theta) : D \times \Omega \to \mathrm{R}$ such that, *P*-almost surely (*P*-a.s.), there holds

$$\frac{d}{dx}\left[k(x,\ \theta)\ \frac{dT(x,\ \theta)}{dx}\right] = \frac{hP_e}{A}\left(T(x,\ \theta) - T_f\right), \quad \text{for } (x,\ \theta) \in (0,\ L) \times \Omega\,, \tag{1}$$

subjected to the following boundary conditions:

$$T(0,\ \theta) = \overline{T} \text{ and } q(L,\ \theta) = -k\,\frac{dT}{dx}\Big|_{x=L} = 0\,. \tag{2}$$

Here, *A* is the cross sectional area of the bar, $P_e$ the perimeter of the cross section, $\rho$ the density of the solid, $c_p$ the heat capacity of the solid at constant pressure, *h* is the convection heat-transfer coefficient and $T_f$ the temperature of the fluid encompassing the fin, far from the thermal boundary layer. Moreover, for simplicity one prescribes deterministic boundary values only.

To obtain a well-posed problem, one further assumes that $k(x,\ \theta)$ is *P-a.s.* uniformly bounded above away from zero below, i.e., that there is $k^{\inf}$ and $k^{\sup} > 0$, so that $P\left(\theta \in \Omega \middle|\ k(x,\ \theta) \in \left[k^{\inf},\ k^{\sup}\right],\ \forall x \in D\right) = 1$.

Let $\mathbf{K}_T$ be the set of admissible temperature fields, which for a fixed $\theta \in \Omega$ is given by $\mathbf{K}_T = \left\{T(.,\ \theta) \in H^1(D) \middle|\ \text{with } T(0,\ \theta) = \overline{T}\right\}$ . Then, the subspace of admissible temperature variation $\mathbf{V}ar_T$ , for a fixed $\theta \in \Omega$ , is given by $\mathbf{V}ar_T = \left\{T(.,\ \theta) \in H^1(D) \middle|\ \text{with } T(0,\ \theta) = 0\right\} = \left\{T(.,\ \theta) \in H_o^1(D)\right\}$ . Moreover, for each fixed $x \in D$ , one assumes that $T(x,.) \in L_P^2(\Omega)$ .

### 2.2. Weak formulation of the stochastic boundary value problem

For the variational characterization of the SBVP, one chooses the set $\mathbf{K}_T \otimes L_P^2(\Omega)$ as the set of admissible random fields on *D* and now seeks $T(x,\ \theta) \in \mathbf{K}_T \otimes L_P^2(\Omega)$ so that

$$\left\langle a(T,\ v)\right\rangle = \left\langle l(v)\right\rangle \ \ , \forall v \in H_o^1(D) \otimes L_P^2(\Omega) \tag{3}$$

in which

$$\left\langle a(T,\ v)\right\rangle = \int_\Omega \int_D \left\{k(x,\ \theta)\ \frac{dT(x,\ \theta)}{dx}\frac{dv(x,\ \theta)}{dx} + \frac{hP_e}{A}T(x,\ \theta)\ v(x,\ \theta)\right\}\ dx dP(\theta) \tag{4}$$

and

$$\left\langle l(v)\right\rangle = \int_\Omega \int_D \frac{hP_e}{A}T_f v(x,\ \theta)\ dx dP(\theta) \tag{5}$$

where

$$\langle uv \rangle = \int_{\Omega}\int_{D} u(x,\ \theta)\ \ v(x,\ \theta)\ \ dxdP(\theta). \tag{6}$$

If one chooses the tensor product space $H_o^1(D) \otimes L_P^2(\Omega)$ as the function space of random fields on *D*, one may reformulate the SBVP in Eq. (3) as: Find $T_o(x,\ \theta) \in H_o^1(D) \otimes L_P^2(\Omega)$, $T(x,\ \theta) = \overline{T} + T_o(x,\ \theta)$, such that

$$\langle a(T_o,\ v) \rangle = \langle l(v) \rangle - \langle a(\overline{T},\ v) \rangle \ \ , \forall v \in H_o^1(D) \otimes L_P^2(\Omega) \tag{7}$$

where, since $\overline{T} = Cte$,

$$\langle a(\overline{T},\ v) \rangle = \int_{\Omega}\int_{D} \frac{hP_e}{A}\overline{T}\ \ v(x,\ \theta)\ \ dxdP(\theta). \tag{8}$$

The problem may again be rewritten as: Find $T_o(x,\ \theta) \in H_o^1(D) \otimes L_P^2(\Omega)$, so that

$$\langle a(T_o,\ v) \rangle = \langle l^*(v) \rangle \ \ , \forall v \in H_o^1(D) \otimes L_P^2(\Omega) \tag{9}$$

where

$$\langle a(T_o,\ v) \rangle = \int_{\Omega}\int_{D} \left\{ k(x,\ \theta)\ \ \frac{dT_o(x,\ \theta)}{dx}\frac{dv(x,\ \theta)}{dx} + \frac{hP_e}{A}T_o(x,\ \theta)\ \ v(x,\ \theta) \right\}\ \ dxdP(\theta) \tag{10}$$

and

$$\langle l^*(v) \rangle = \int_{\Omega}\int_{D} \frac{hP_e}{A}\left(T_f - \overline{T}\right)\ \ v(x,\ \theta)\ \ dxdP(\theta) \tag{11}$$

### 2.3. Representation of stochastic processes

Two of the most useful expansions for random processes are: the Karhunen-Loeve expansion and the polynomial Chaos expansion. The first requires the knowledge of the covariance structure of the process under consideration, while the second one is more general.

#### 2.3.1. The Karhunen-Loeve expansion

Let *D* be a compact subset of *R* and $\{k(x,\ \theta)\}_{x \in D}$ a stochastic field defined on a probability space $(\Omega,\ F,\ P)$ with values in *R*. This random field is a 1-parameter family on real valued $k(x,\ \theta)$, $(x,\ \theta) \in D \times \Omega$.

- Assumption I: One assumes that $\{k(x,\ \theta)\}_{x \in D}$ is a second-order random field, i.e., $E\left[ k(x,\ \theta)^2 \right] \le \infty, \forall x \in D,$ where *E*[.] denotes the mean or ensemble average.
- Assumption II: One also assumes that $\{k(x,\ \theta)\}_{x \in D}$ is continuous in quadratic mean, i.e., $(k(x+h,\ \theta) - k(x,\ \theta))^2 \to 0, h \to 0.$

Under assumptions I and II and due to the symmetry and the positive definiteness of $C(x_1,\ x_2)$, the auto-covariance function $C(x_1,\ x_2)$ on the field $\{k(x,\ \theta)\}_{x \in D}$ defines a continuous self-adjoint Hilbert-Schmidt operator *Q* on the Hilbert space $L_2(D, R)$:

$$(Qf)\ \ (x_2) = \int_{\mathbf{D}} C(x_1,\ x_2)\ \ f(x_1)\ \ dx_1, \text{ for } f \in L_2(D, R). \tag{12}$$

This operator has countable number of eigenvalues $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge .....\lambda_n \ge ....$, and the associated eigenfunctions are solutions of the integral equation

$$(Qf_n)\ \ (x_2) = \lambda_n\ \ f_n(x_2) \tag{13}$$

and constitute an orthonormal basis $\{f_n\}$ of $L_2(D,R)$, i.e.,

$$\langle f_n,\ f_m\rangle_{L_2(D)} = \delta_{nm}. \tag{14}$$

The random field $\{k(x,\ \theta)\}$ can be expanded in terms of the eigenfunctions $f_n$ as

$$k(x,\ \theta) - \langle k(x,\ \theta)\rangle_P = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\xi_n(\theta)\ \ f_n(x) \tag{15}$$

with

$$\langle k(x,\ \theta)\rangle_P = E[k(x,\ \theta)] = \int_\Omega k(x,\ \theta)\ \ dP(\theta) \tag{16}$$

in which $\langle k(x,\ \theta)\rangle_P$ is the mean function of the stochastic process/field $k(x,\ \theta)$ and $\xi_1(\theta)$, $\xi_2(\theta)$, $\xi_3(\theta)$,.. $\xi_n(\theta)$,.., are uncorrelated random variables.

By definition, the covariance function is bounded, symmetric, and positive definite. As a result, one may show that it has the spectral decomposition, which is given by

$$C(x_1,\ x_2) = \sum_{n=1}^{\infty}\lambda_n f_n(x_1)\ \ f_n(x_2) \tag{17}$$

Moreover, defining $\bar{k}(x) = \langle k(x,\ \theta)\rangle_P$, one may show that

$$k(x,\ \theta) = \bar{k}(x) + \sum_{n=1}^{\infty}\sqrt{\lambda_n}\xi_n(\theta)\ \ f_n(x) \tag{18}$$

where,

$$\langle \xi_n(\theta)\rangle_P = 0,\quad \langle \xi_n(\theta)\ \ \xi_k(\theta)\rangle_P = \delta_{nk}, \tag{19}$$

and

$$\xi_n(\theta) = \frac{1}{\sqrt{\lambda_n}}\int_D \alpha(x,\ \theta)\, f_n(x)\ \ dx. \tag{20}$$

In this work one employs the following covariance function $C(x_1,\ x_2)$ given by

$$C(x_1,\ x_2) = \sigma_b \exp\left(\frac{-|x_2 - x_1|}{b}\right) \tag{21}$$

in which *b* is associated with the correlation length. A covariance function $C(x_1,\ x_2)$ is denoted homogeneous and isotropic if $C(x_1,\ x_2) = C(r)$ where $r = |x_2 - x_1|$. Some typical values for *b* are: *b*=0.1 (lower curve), *b*=1.0 (middle curve) and *b*=2.0 (upper curve) as depicted in Figure 2, for $\sigma_b$ =1.

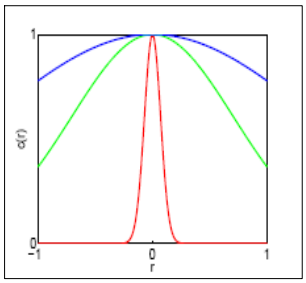


Figure 2. – Dependence of the Covariance function with respect to the correlation length *b*

Notice that the steeper a bilinear form decays to zero as a function of one of its arguments, the more terms are needed in its spectral representation in order to reach a present accuracy. Noting that the Fourier transform operator yields a spectral representation, it may be concluded that the faster the autocorrelation function tends to zero, i.e. the smaller is *b*, the broader is the corresponding spectral density, and the larger the number of requisite terms necessary to adequately represent the underlying random process by the Karhunen-Loeve expansion.

### 2.3.2. Discretization of the Karhunen-Loeve expansion

The discrete solution of the weak form of the integral equation may be stated as: Determine a non zero function $f \in V^m$, $V^m = span\left\{\varphi_j\left(x\right) \;\; , j=1, \;\; m\right\} \subset L_2\left(D,R\right)$, such that, there is a $\lambda \in R$, which satisfies

$$\int_D\int_D C\left(x_1,\;\; x_2\right) \;\; f\left(x_1\right) \;\; \hat{f}\left(x_2\right) \;\; dx_1dx_2 = \int_{\mathbf{D}} \lambda \;\; f\left(x_2\right) \;\; \hat{f}\left(x_2\right) \;\; dx_2, \text{ for every } \hat{f} \in V^m \tag{22}$$

Now, since $f \in V^m$ and $\hat{f} \in V^m$, then

$$f\left(x\right) = \sum_{j=1}^{m} a_j \varphi_j\left(x\right) \text{ and } \hat{f}\left(x\right) = \sum_{i=1}^{m} b_i \varphi_i\left(x\right) \tag{23}$$

Here, one notice that, since $\hat{f} \in V^m$ is arbitrary, then one must have that $\mathbf{b} \in R^m$ is an arbitrary vector. Thus, replacing Eq. (23) into Eq. (22) one derives the following discrete problem: Determine the non zero vectors $\mathbf{a} \in R^m$, such that there is a real $\lambda$ that solves

$$\left[\left[P\right] - \boldsymbol{\lambda}\left[M\right]\right] \;\; \mathbf{a} = \mathbf{0} \tag{24}$$

in which

$$\left[P\right]_{ij} = \int_D\int_D C\left(x_1,\;\; x_2\right) \;\; \varphi_i\left(x_2\right) \;\; \varphi_j\left(x_1\right) \;\; dx_1dx_2 \tag{25}$$

and

$$\left[M\right]_{ij} = \int_D \varphi_i\left(x_2\right) \;\; \varphi_j\left(x_2\right) \;\; dx_2 . \tag{26}$$

From the above results, one may approximate the stochastic process associated with the conductivity $k\left(x,\;\; \theta\right)$ by a truncated series and consider the following finite sum

$$k\left(x,\;\; \theta\right) = \bar{k}\left(x\right) + \sum_{n=1}^{m} \sqrt{\lambda_n}\, \xi_n\left(\theta\right) f_n\left(x\right) \tag{27}$$

### 2.4. Representation of the uncertainty: Homogeneous Chaos

In this work, the modeling of the uncertainty over the thermal conductivity is done by using uniform random variables. In general, there is not complete information about the probabilistic uncertainty. Normally, in engineering, one knows only the statistical moments of first and second order. The type of the density of probability function is determined based on experience, observation of the random process or heuristically. When the uncertainty is described by a set of random variables and their joint probability density function one obtains a complete characterization of the mathematical model of the uncertainty. Here, we make use of the Askey-Wiener scheme in order to represent the uncertainty in the construction of the solution space for the stochastic heat conduction problem.

It is clear from the preceding discussion that the implementation of the Karhunen-Loeve expansion requires the knowledge of the covariance function of the process being expanded. This implies that the expansion can be used for the random coefficients in the operator equation. However, it cannot be used for the solution process $T\left(x,\;\; \theta\right)$, since its covariance function and therefore the corresponding eigenvalues and functions are not known. An alternative expansion is needed which circumvents this problem. Such an expansion could involve a basis of known random functions with deterministic coefficients to be found by minimizing some norm of the error resulting from a finite representation. This is done with the use of the Askey-Wiener scheme.

### 2.4.1. Definitions and properties

Let $\{\xi_i(\theta)\}_{i=1}^{\infty}$ be a set of random variables. Consider the space $\hat{\Gamma}_p$ of all polynomials in $\{\xi_i(\theta)\}_{i=1}^{\infty}$ of degree not exceeding *p*. Let $\Gamma_p$ represent the set of all polynomials in $\hat{\Gamma}_p$ orthogonal to $\hat{\Gamma}_{p-1}$. Finally, let $\overline{\Gamma}_p$ be the space spanned by $\Gamma_p$. Then the subspace $\overline{\Gamma}_p$ of $\Omega$ is called the *p*-th Homogeneous Chaos, and $\Gamma_p$ is called the Polynomial Chaos of order *p*.

Based on the above definitions, the Polynomial Chaos of any order *p* consist of all orthogonal polynomials of order *p* obtained from a linear combination of the random variables $\{\xi_i(\theta)\}_{i=1}^{\infty}$. Notice that, since random variables are themselves functions, it becomes clear that Chaos Polynomials are therefore functionals.

The set of Polynomial Chaos is a linear subspace of the space of square-integrable random variables in $\Omega$, and is a ring with respect to the functional multiplication $\Gamma_p(\xi(\theta))\ \ \Gamma_l(\xi(\theta)) = \Gamma_l(\xi(\theta))\ \ \Gamma_p(\xi(\theta))$. In this context, square integrability must be constructed to be with respect to the probability measure defining the random variables. Denoting the Hilbert space spanned by the set $\{\xi_i(\theta)\}_{i=1}^{\infty}$ by $W(\boldsymbol{\xi})$, the resulting ring is denoted by $\Phi_{W(\boldsymbol{\xi})}$, and is called the ring of functions generated by $W(\boldsymbol{\xi})$. Then, it can be shown that under some general conditions, the ring $\Phi_{W(\boldsymbol{\xi})}$ is dense in the space $L_2(\Omega)$, Kakutani, 1961. This means that any square-integrable random function ($\Omega \to R$) can be approximated as closely as desired by elements from $\Phi_{W(\boldsymbol{\xi})}$. Thus, any element $\beta(\theta)$ from the space $L_2(\Omega)$ admits the following representation,

$$\beta(\theta) = \sum_{p\geq 0} \sum_{n_1+n_2+\ldots+n_r=p} \sum_{\rho_1,\ldots,\rho_p} a^{n_1,\ldots,n_r}_{\rho_1,\ldots,\rho_p} \Gamma_p\left(\xi_{\rho_1}(\theta),\ldots..\xi_{\rho_p}(\theta)\right) \tag{28}$$

where $\Gamma_p(\circ)$ is the Polynomial Chaos of order *p*. The superscript $n_i$ refers to the number of occurrences of $\xi_{\rho_i}(\theta)$ in the argument list for $\Gamma_p(\circ)$. The form of the coefficients appearing in Eq. (28) can be rewritten as

$$\beta(\theta) = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(\xi_{i_1}(\theta),\ \xi_{i_2}(\theta)) + \ldots \tag{29}$$

In this equation, the symbol $\Gamma_n(\xi_i(\theta))$ denotes the Polynomial Chaos of order *n* in the variables $(\xi_{i_1}(\theta),\ldots,\ \xi_{i_n}(\theta))$. The upper limits on the summations in Eq. (29) reflect the symmetry of the Polynomial Chaoses with respect to their arguments. Moreover, Eq. (29) may be rewritten in the form

$$\beta(\theta) = \sum_{j=1}^{\infty} \hat{a}_j \Psi_j(\boldsymbol{\xi}(\theta)), \tag{30}$$

where there is a one-to-one correspondence between the functionals $\Psi[\circ]$ and $\Gamma[\circ]$, and also between the coefficients $\hat{a}_j$ and $a_{i_1 i_2}$ appearing in Eq. (30).

Now, as defined above, each Polynomial Chaos is a function of the infinite set $\{\xi_i(\theta)\}_{i=1}^{\infty}$, and is therefore an infinite dimensional polynomial. In practice, however, this infinite set has to be replaced by a finite dimensional one. As a result, it's natural to introduce the concept of a finite dimensional Polynomial Chaos. In particular, the *m*-dimensional Polynomial Chaos of order *p* is the subset of the Polynomial Chaos of order *p*, as defined above, which is a function of only *m* uncorrelated random variables $\xi_i(\theta)$. In this case, one has

$$\beta(\theta) \simeq \sum_{j=1}^{\infty} \overline{a}_j \Psi_j(\boldsymbol{\xi}(\theta)), \quad \text{with } \boldsymbol{\xi}(\theta) \in R^m. \tag{31}$$

In this case, $\beta(\theta) \in \Theta^m$, in which $\Theta^m$ is the subspace spanned by the polynomials of $\xi(\theta) \in R^m$ of all orders. Notice that $\lim_{m\to\infty} \Theta^m \equiv \Theta^\infty$, which is dense in $L_2(\Omega)$. However, $\dim\left\{\Theta^m\right\} = \infty$. As a result, in order to make it computable, a second approximation is done. Here, one denotes $\Theta^{m,\,p}$ to be the subspace spanned by all polynomials of order less or equal to *p*, of $\xi(\theta) \in R^m$. Consequently $\dim\left\{\Theta^{m,\,p}\right\}$ is Finite and $\lim_{m,\,p\to\infty} \Theta^{m,\,p} \equiv \Theta^\infty$, which is dense in $L_2(\Omega)$.

### 2.4.2. Askey-Wiener Scheme

The Askey-Wiener scheme is a generalization of the chaos polynomials, also known as Wierner-Chaos polynomials. The theorem of Cameron-Martin (1947), shows that these polynomials form a base, of a dense subspace of random variables of second order in $L_2(\Omega)$.

The Askey-Wiener scheme consists of a family of subspaces generated by orthogonal polynomials obtained, in general, as the solution of ordinary differential equations. Among them are the Hermite, Laguerre, Jacobi and Legendre polynomials. By taking the limit of a given set of parameters, the Askey-Wiener scheme establishes relations among the subspaces generated by the above orthogonal polynomials. Each subspace generated by these polynomials form a complete set in $L_2(\Omega)$. Figure 3 depicts the relation among the polynomials belonging to the Askey-Wiener scheme.

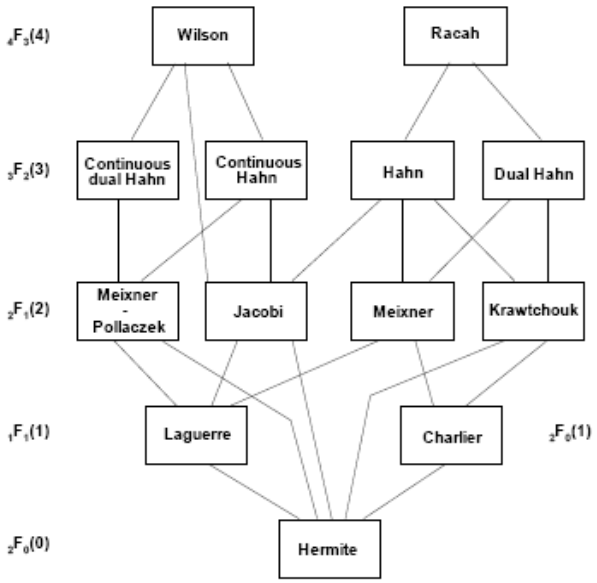


Figure 3. - Relation among the polynomials in the Askey-Wiener scheme

The orthogonality of these polynomials is defined with respect to a given weight function, which is given by the density of the probability function. For example, for Gaussian random variables, one obtains the Hermite polynomials. A Table (1) shows the correspondence among the subsets of polynomials of the Askey-Wiener scheme and their associate density of probability function.

Table 1 – Relation among the type of random variables and the polynomials from the Askey-Wiener scheme

| Random variables | polynomial | support |
|---|---|---|
| Gaussian | Hermite | (-∞,+∞) |
| Gamma | Laguerre | [0, +∞) |
| Beta | Jacobi | [a,b] |
| Uniform | Legendre | [a,b] |

### 2.5. The Stochastic Finite Element Method (SFEM)

The SFEM in its current form was first introduced by Ghanem and Spanos. Although both the term and the idea of incorporating randomness in a finite element formulation have a longer history, this probably constitutes the first systematic Galerkin approximation in deterministic and random variables.

In this work, one follows the ideas proposed by Ghanem and Spanos and apply the Galerkin method in order to determine approximate solutions to the stochastic boundary value problem. The temperature random field is assumed to be given by

$$T(x,\ \theta) = \overline{T} + T_o(x,\ \theta) \text{ with } T_o(x,\ \theta) = \sum_{i=1}^{q} p_i(x)\ \ \Psi_i(\xi(\theta)) \tag{32}$$

where $\Psi_i(\theta) \in \Theta^{m,\ p} \subset L_2(\Omega)$ and $p_i(x) \in Var_T^n \subset H_o^1(D)$, in which $Var_T^n = span\left\{\phi_i(x)\ \ , i = 1,...n \middle|\ \ \phi_i(x) \in H_o^1(D)\right\}$ and the polynomial functions $\Psi_i(\theta)$ are defined in terms of the Askey-Wiener scheme. Notice that, given the number *m* of the term used in the Karhunen-Loeve expansion, and the *p* of Homogeneous Chaos used, *q* may be determined by $q = 1 + \sum_{s=1}^{p} \frac{1}{s!} \Pi_{r=0}^{s-1}(m+r)$. Now, since $p_i(x) \in Var_T^n \subset H_o^1(D)$ one has

$$p_i(x) = \sum_{j=1}^{n} a_{ij} \phi_j(x) \tag{33}$$

in which $a_{ij}$ are the unknown coefficients to be determined. Introducing Eq. (33) into Eq. (32) gives

$$T_o^{qn}(x,\ \theta) = \sum_{s=1}^{q} \sum_{j=1}^{n} a_{sj} \phi_j(x)\ \ \psi_s(\xi(\theta)). \tag{34}$$

Replacing Eq. (34) into Eq. (9) leads to the discrete stochastic boundary value problem that consist in the determination of $T_o^{nq}(x,\ \theta) \in Var_T^n \otimes \Theta^{m,\ p}$, so that

$$\left\langle a\left(T_o^{nq},\ v\right)\right\rangle = \left\langle l^*(v)\right\rangle\ \ , \forall v \in Var_T^n \otimes \Theta^{m,\ p} \tag{35}$$

where

$$\left\langle a\left(T_o^{nq},\ v\right)\right\rangle = \int_\Omega \int_D \left\{ k(x,\ \theta)\ \ \frac{dT_o^{nq}(x,\ \theta)}{dx} \frac{dv(x,\ \theta)}{dx} + \frac{hP_e}{A} T_o^{nq}(x,\ \theta)\ \ v(x,\ \theta) \right\}\ \ dxdP(\theta) \tag{36}$$

and

$$\left\langle l^*(v)\right\rangle = \int_\Omega \int_D \frac{hP_e}{A}\left(T_f - \overline{T}\right)\ \ v(x,\ \theta)\ \ dxdP(\theta) \tag{37}$$

in which $\Theta^{m,\ p}$ is the subspace generated by all polynomials of $R^m$ of degree $n \le p$, with $\dim\left(\Theta^{m,\ p}\right) = q$.

Defining $\mathbf{U}_s = \sum_{j=1}^{n} a_{sj} \mathbf{e}_j$, where $\left\{\mathbf{e}_l,\ \ l = 1,..n\right\}$ is a cartesian base system in $R^m$, and denoting

$$K_{kisj} = \int_\Omega \int_D \left\{ k(x,\ \xi)\ \ \psi_s(\xi)\ \ \psi_k(\xi)\ \ \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx}\ \ + \frac{hP_e}{A} \psi_s(\xi)\ \ \phi_j(x)\ \ \psi_k(\xi)\ \ \phi_i(x) \right\}\ \ dxdP, \tag{38}$$

$$F_{ki} = \int_\Omega \int_D \frac{hP_e}{A}\left(T_f - \overline{T}\right)\ \ \psi_k(\xi(\theta))\ \ \phi_i(x)\ \ dxdP, \tag{39}$$

$$\mathbf{F}_k = \sum_{i=1}^{n} F_{ki} \mathbf{e}_i \text{ and } \left[\mathbf{K}_{ks}\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} K_{kisj}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right). \tag{40}$$

one may reformulate the SFEM as: determine $\mathbf{U} \in R^{n+q}$, that solves the following system of equations

$$\begin{bmatrix} \left[\mathbf{K}_{11}\right] & \left[\mathbf{K}_{12}\right] & \left[\mathbf{K}_{13}\right] & ... & \left[\mathbf{K}_{1q}\right] \\ & \left[\mathbf{K}_{22}\right] & \left[\mathbf{K}_{23}\right] & ... & \left[\mathbf{K}_{2q}\right] \\ & & \left[\mathbf{K}_{33}\right] & ... & \left[\mathbf{K}_{3q}\right] \\ Sym & & & ... & ... \\ & & & & \left[\mathbf{K}_{qq}\right] \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ ... \\ U_q \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \\ ... \\ \mathbf{F}_q \end{Bmatrix} \tag{44}$$

### 2.6. Numerical results

Here, one considers the fin problem defined in figure 1 where $L$=0.5m, $ko$=398.0$W/m^2°C$ and the convection coefficient $h$=100.0$W/m°C$. The prescribed temperature $\overline{T}$ =100 0°C and the fluid temperature $T_f$ =25°C. The cross section of the fin is circular with a radius of $R$=0.005$m$. The covariance function of the thermal conductivity is given by Eq. (21), with $b$=1.0 and $\sigma_b$ =100.0. Figure 4 depicts the exact covariance function

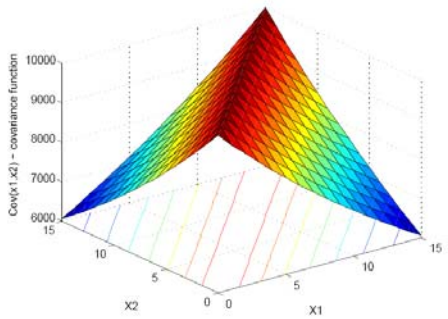


Figure 4 – exact covariance function

Figure 5 depicts the Karhunen-Loeve spectral approximation for $m$=3

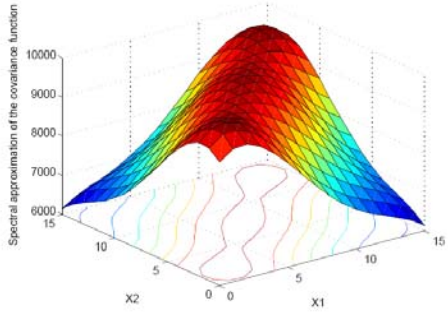


Figure 5 - Karhunen-Loeve spectral approximation for $m$=3

Figure 6 depicts the difference between the exact and the Karhunen-Loeve spectral approximation ($m$=3)

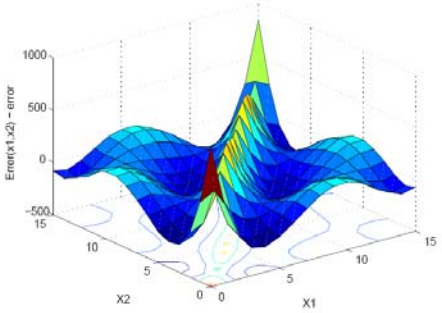


Figure 6. Error in the approximation of the covariance function

Here, the fin problem is solved by a Monte Carlo simulation using 100 samplings together with the Askey-Wiener for $m$=3 in the KL approximation and $p$=2 for the order of the generalized polynomial chaos (Askey-chaos). The adequacy of the Monte Carlo simulation was tested by determining the average thermal conductivity, whose exact value is $ko$=398.0$W/m^2°C$. Figure 7 depicts the temperatures distributions obtained by the Monte Carlo simulation (in blue) and the Askey-chaos solution form $m$=3 and $p$=2 (in read).

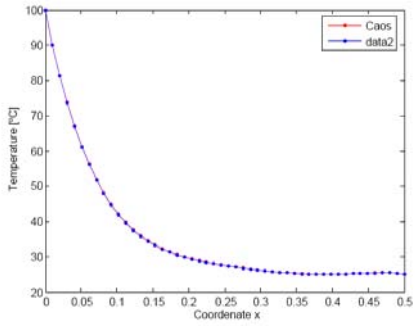


Figure 7. – Temperatures distributions obtained by Monte Carlo (blue) and Askey-chaos (read)

Figure 8 depicts the variance distributions obtained by the Monte Carlo simulation (in blue) and the Askey-chaos solution form $m$=3 and $p$=2 (in read).

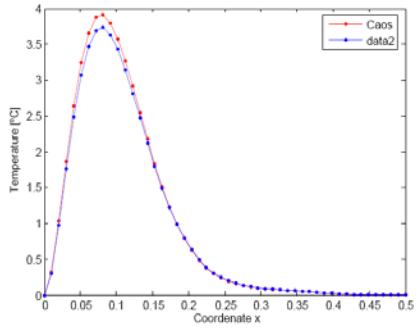


Figure 7. - Variance distributions obtained by Monte Carlo (blue) and Askey-chaos (read)

## 3. CONCLUSION

The spectral approach have shown to converge very fast to an approximated solution to the stochastic heat problem.

## 4. ACKNOWLEDGEMENTS

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## 5. RESPONSIBILITY NOTICE

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