STRATEGY FOR GENERATION OF BIDIMENSIONAL TRIANGLE MESHES TRACING OFFSET CURVES

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Abstract. *This work presents a strategy for the generation of bidimensional triangle meshes in regions of arbitrary domains tracing offset curves. The process mesh generation is an important computacional topic to simulate engineering problems, due to the fact that the majority of the numerical analyses based on the domains discretization of these problems demands efficient and robust procedures for its represetation. The proposed strategy combines the Delaunay generation of triangles meshes with procedures of front advancing, using the trace of offset curves. The Delaunay criterion is used to generate triangles closer to the ideals, reducing the generation of bad quality triangles. The offset curves allows the gradual generation of points inside of the domain that are used as the base for the generation of the triangles. Numerical examples are presented to validate the proposed methodology. A study on the form quality of the generated triangles is also presented.*

Keywords*: Mesh Generation, Delaunay Triangulation, Offset Curves.*

1. INTRODUCTION

The meshes generation process is an important topic related to the numerical simulation of engineering problems using the Finite Element Method (FEM). In this simulation, most of the problems need efficient procedures for the meshes generation, very fundamental in the discretization of the domains.

In complex problems, the strategy adopted for the meshes generation is the decomposition of the problem in several sub-domains. The final mesh must be consistent and formed by the composition of the local meshes generated in each sub-domain. This strategy demands that the contour of each sub-domain be compatible among itself, respecting the border (interface) between the adjacent sub-domain. This boarding is typical for known algorithms such as advancing front. These algorithms consist in the triangulation of the domains inserting elements starting from the border to the interior, where an initial front is created by the discretization of the domain's border. The elements of the mesh are inserted one by one, updating the front and generating new triangles.

In this context, the generation of triangular meshes of finite elements using the advancing front technique is an active research area where some related works are found ((Jin and Tanner, 1993), (Miranda *et al*., 1999), (Cavalcante *et al*., 2000), (Cavalcante *et al*., 2001), (Miranda and Martha, 2002).

This way, the present work proposes an alternative strategy for the generation of bidimensional triangular meshes in arbitrary domains regions, starting at the offset curves tracing. The strategy combines the Delaunay triangular mesh generation with procedures of advancing front using the offset curves tracing. This tracing is based on a methodology proposed by Del Savio *et al.* (2004). The Delaunay criterion is used because it allows generation of triangles closest to the ideal ones, reducing the bad qualities of the generated triangles. The offset curves make possible the gradual generation of the internal points in the domain, serving as base for the generation of the triangles.

Section 2 presents the strategy, describing step by step the procedure for the generation of the final mesh. The generation of the offset curves is presented in Section 3, describing the creation process of the themselves. In Section 4, the triangulation technique based on the Delaunay criterion is presented. In Section 5, a smoothing technique is presented to improve the mesh quality. With the objective to validate the proposed strategy, Section 6 shows application examples of the considered algorithm. Quality tests are done with the objective to measure the quality of the generated elements. Section 7 presents the conclusions of the work.

2. ADOPTED STRATEGY

The strategy adopted in this work for the triangular meshes generation in arbitrary regions consists in the use of the advancing front technique starting from the creation of the offset curves based on the boundary points of the studied problem. Figure 1 shows the diagram with the steps involved in this strategy. These offset curves are used to set up internal points in the domain that were used in triangulation. The use of points on these offset curves supplies a gradual generation of the triangular elements of the mesh.

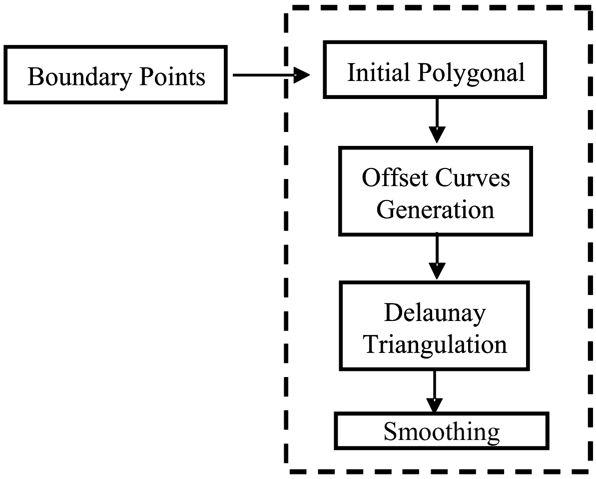

Figure 1. The adopted strategy steps.

Initially, the boundary points are supplied as input data of the algorithm. These points must be ordered, not mattering the direction (if clockwise or counter-clockwise). Figure 2a shows these initial points. A linear interpolation of these points is used, generating a boundary polygon (Fig. 2b) that serves as base for the creation of the offset curves.

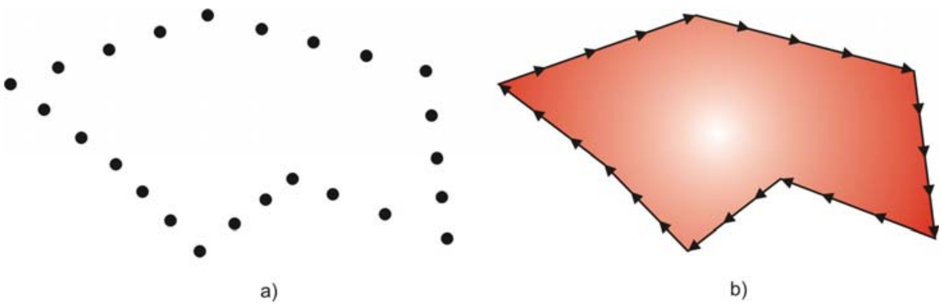

Figure 2. Initial stage of the strategy: the organized boundary points.

Starting from the boundary polygon, apply a technique for the offset curves generation taking as distance between these curves (offset) the length of the lesser edge of the boundary polygon. Offset Curves are generated as many as necessary, obeying a preset stopping criterion. In this work, it was adopted as stopped criterion 10% of the initial polygon's perimeter. Figure 3a illustrates this stage of the proposed strategy. The offset curves supply internal polygons which vertices coordinates are used as internal points to guide the triangulation.

The first boundary points supplied and the internal points calculated using offset curves generates a set of points that are used as base for the domain triangulation. In the proposed strategy, the Delaunay criterion is used for the triangulation. This criterion reduces the creation of bad triangles, trying to get, whenever possible, triangles with forms next to the ideal ones. Figure 3b illustrates this considered procedure.

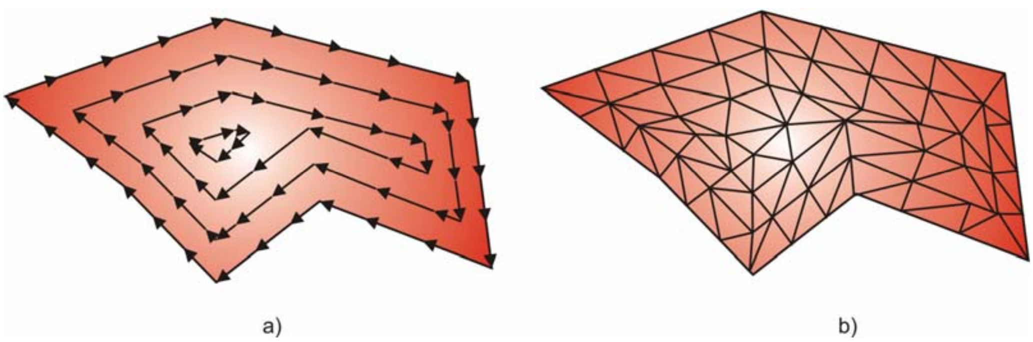

Figure 3. a) Creation of the offset curves; b) Delaunay triangulation.

In the methodology used for Delaunay triangulation, initially a rectangle containing the entire considered domain is defined (Fig. 4a). An algorithm for the Delaunay triangulation is applied considering such rectangle (Fig. 4b). The edges of the boundary polygon are treated like the internal restrictions of this rectangle, therefore, they are kept during the triangulation. The spurious regions which are external to the original domain, are removed, generating the final mesh (Fig. 4c).

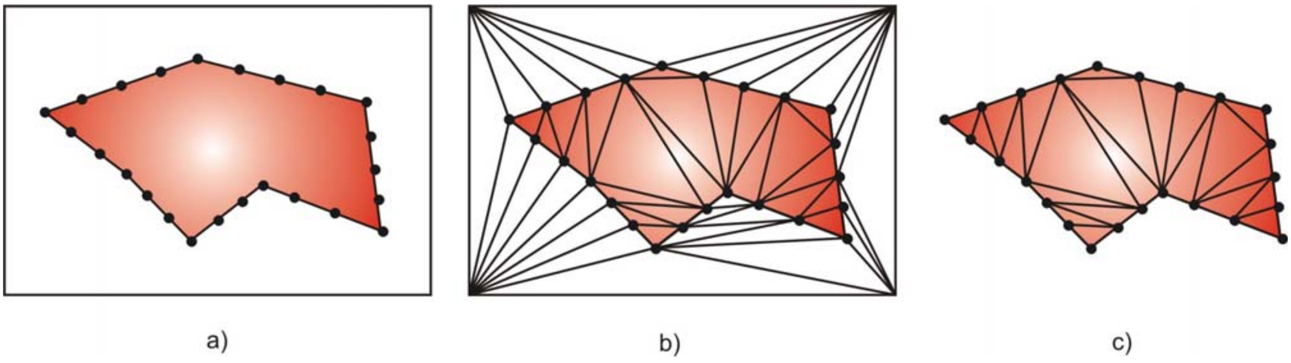

Figure 4. Delaunay Triangulation: a) Initial rectangle containing the domain; b) Initial triangulation of the domain; c) Triangulation without spurious regions.

3. OFFSET CURVES GENERATION

The offset curves generation used in this work is based on the work of Del Savio *et al.* (2004). The creation process of these curves is made with the objective to generate curves similar to the boundary. These similar curves supply the necessary internal points for the use of the Delaunay criterion for domain triangulation.

In this work, the creation of the offset curves is based on the boundary polygon. The base for the determination of these curves is purely geometric. Normal and bisector vectors of the edges of the boundary polygon are used, guaranteeing a good representation of the geometry of this polygon.

Figure 5 illustrates the creation process of these offset curves. Initially, all bisector of the boundary polygon are calculated, as shown in the Fig. 5a. These bisectors are stored following the same sorting of the boundary polygon points. The first point of the offset curve is obtained using one of the calculated bisector and tracing an imaginary straight line segment throughout this bisector with size D, as can be seen in the Fig. 5b where D is the size of the smallest edge of the boundary polygon.

The next point is obtained from the determination of the intersection between the line that is parallel to the edge of the boundary polygon that passes through the previous point calculated and the next bisector, as can be seen in the Fig. 5c. This procedure is repeated until all bisectors are covered (Fig. 5d and Fig. 5e). This process can result in the appearance of spurious regions. These regions are identified and removed. A simple data structure is used to simplify the identification of these regions in an efficient way. The calculated points are linearly interpolated generating an offset curve similar to the boundary polygon. The process described is repeated for the determination of other offset curves, as seen in Fig. 5f.

Figure 6 shows examples of generated offset curves using the proposed strategy. Figure 6a shows a square domain and its respective offset curves, demonstrating their similarity to the boundary. Figure 6b shows a region with a contraction. It can be observed that the technique used does not generate curves in regions where this contraction is critical.

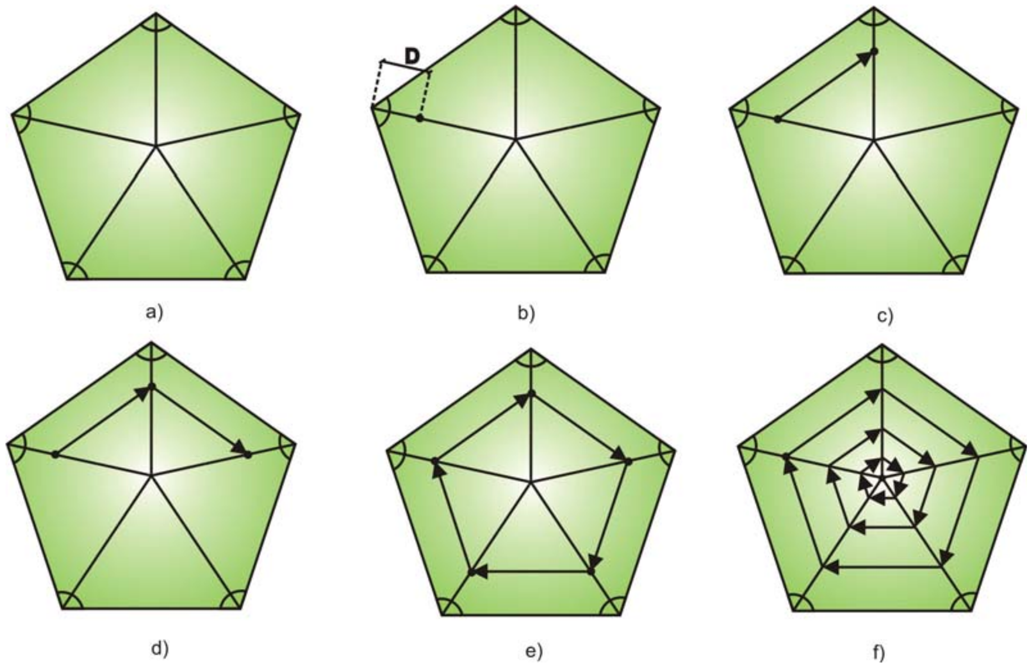

Figure 5. The creation of offset curves: a) Calculating the bisector; b) Determining the first point of the curve; c) Determining the second point of the curve; d) Determining the following points; e) First offset curve; f) Offset curves.

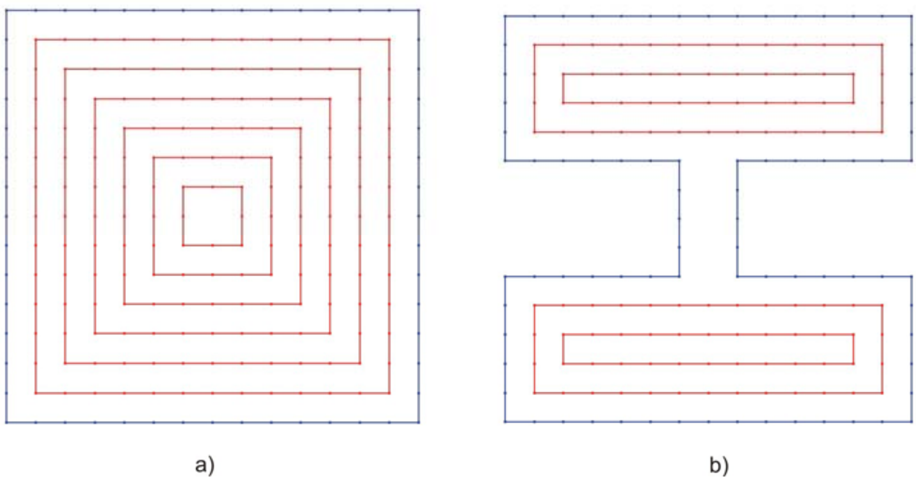

Figure 6. Examples of offset curves: a) Square domain b) Domain with contraction.

4. DELAUNAY TRIANGULATION

The triangulation of a region supplies triangles from a set of points and following determined rules to connect these points, establishing a subdivision of the space that they occupy. For majority of the applications that needs triangulation, these triangles need to be closer to equilateral. That is the case of applications focused on finite elements. One triangulation technique is known as Delaunay triangulation. The Delaunay triangulation is considered to be simple to implement, not to mention that it normally generates good quality triangles. The implementation of the Delaunay triangulation used in this work is based on the one proposed by of De Floriani and Puppo (1992).

4.1. Empty circle property

One of the properties that define a Delaunay triangulation is known as the property of the empty circle. It says that the circle that passes on the three vertices of each triangle does not contain in its interior any another point (Fig. 7a). In the case to have restrictions, the circle that circumscribes each triangle does not contain in its interior any another point that is visible by the three vertices (Fig. 7b).

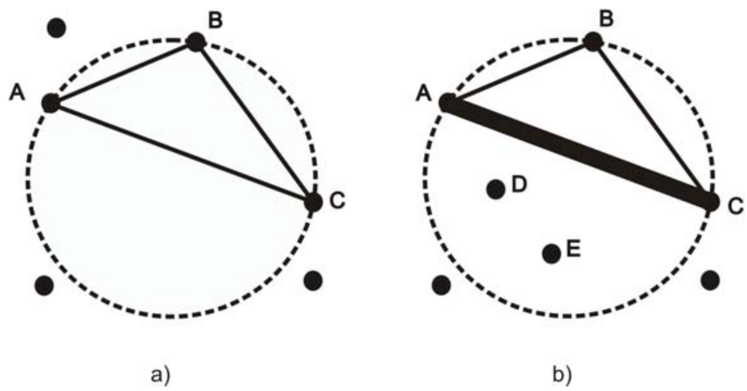

Figure 7. Property of the empty circle: a) Delaunay triangulation; b) Delaunay triangulation with restriction.

4.2. Maximum and minimum angle property

Another important property of the Delaunay triangulation is the maximum and minimum angle. This property says that in a convex quadrilateral formed by two triangles, the smallest of its six angles is bigger than the smallest of the six angles formed by the other possible triangulation of the same quadrilateral. In a Delaunay triangulation with restrictions, a convex quadrilateral formed by two triangles, if the diagonal line is not be a restriction, the smallest of its six angles is bigger than the smallest of the six angles formed by the other possible triangulation in the same quadrilateral. Figure 8 shows a quadrilateral ABCD with possible diagonal lines AC and BD. The maximum and minimum angle criterion indicates diagonal line BD because angle B is bigger than the smallest angle formed for the other diagonal line (C).

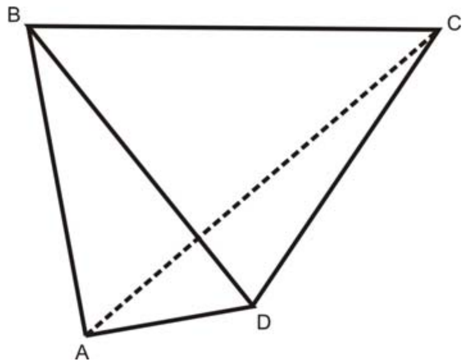

Figure 8. Maximum and minimum angle criterion.

4.3. De Foriani and Puppo algorithm

The Delaunay triangulation algorithm with restriction is based on the basic properties of the empty circle (Section 4.1) and the maximum and minimum angles (Section 4.2). The triangulation receives a set of points and nonintersecting line segments as input data. With these set of points and line segments (which extreme points belong to the aforementioned set of points), the algorithm makes a Delaunay triangulation with restriction by gradually inserting points and line segments.

The problem of making a stepwise Delaunay triangulation with restriction is the initial triangulation and the insertion of points and line segments. Details of this algorithm can be seen in the work of De Floriani and Puppo (1992).

5. LAPLACIAN SMOOTHING

A smoothing technique is used to improve mesh quality by relocating nodes within a patch. A general formulation for this technique is given through Eq. (1), which is a generic form of a weighted Laplacian function (Foley *et al.*, 1989):

$$X_O^{n+1} = X_O^n + \phi \frac{\sum_{i=1}^{m} \omega_{iO} \left(X_i^n - X_O^n \right)}{\sum_{i=1}^{m} \omega_{iO}} \tag{1}$$

In Eq. (1), m is the number of nodes connected to node O, X_O^{n+1} is the position of node O at smoothing iteration $n + 1$, ω_{iO} is the weighted function between nodes i and O, and ϕ is a relaxation parameter which is normally set in the interval (0, 1]. In this work, a value of ϕ = 0.5 and ω_{iO} = 1 were adopted, resulting in a simple average of nodes. The smoothing procedure is repeated 5 times.

6. EXAMPLES

In this section, comparisons are made between the proposed algorithm and the algorithm proposed by Miranda (Miranda *et al.*, 1999) based on quadtree in relation to the quality of the generated meshes. Two examples are presented: first, an “I” section and second, an octagonal section. In both cases, the number of generated elements is around 300, so that it is possible to easily visualize the differences in quality of elements between both algorithms.

The quality of generated meshes was measured with the normalized metric γ/γ^* (Miranda *et al.*, 1999). This metric has a valid interval between 1.0 and infinity, and the value for the equilateral triangle is 1.0. It is desirable to have elements with values close to 1.0. In this case, the adopted quality measure is a normalized ratio between the root mean square of the lengths' edges of a triangle (S_{rms}) and the area A of the triangle:

$$\gamma = \frac{S_{rms}^2}{A} = \frac{\sum_{i=1}^{3} S_i^2}{3A} \tag{2}$$

Where Si is the length of an edge.

$$S_{rms} = \sqrt{1/3 \sum_{i=1}^{3} S_i^2} \tag{3}$$

This metric presents a good quality expressivity and is computationally efficient. γ^* is the metric for equilateral triangle. For each element of the generated mesh, the quality measure γ/γ^* is evaluated. If the value of this metric is above a pre-defined limit value, the element is classified as a poorly shaped element. The limit value is empirically defined based on experiments and observations. In this work, the adopted value is 1.3.

The first example (an “I” section) is presented in the Fig. 9a. This figure also shows the boundary points that are the input data. Figure 9b shows the offset curves based on boundary edges as described in previous sections. After all offset curves generated, an involving rectangle is found, and that is the boundary box to Delaunay triangulation. Applying the Delaunay triangulation, a mesh is obtained as shown in Fig. 9c. Removing the spurious regions (outside triangles of the input boundary), an intermediary mesh is obtained in Fig. 9d. Using the smoothing process (Eq. 1), the final mesh is achieved as shown in Fig. 10a. In this figure, the quality of the final mesh is compared with a mesh generated by Miranda algorithm, Fig. 10b, employing a contour process of visualization in elements with the normalized metric. Note some similarities about distribution of elements and color appearance between the two meshes. These similarities are evident in Fig. 11, where the normalized metrics are plotted for all elements using both algorithms. Most of elements present normalized metric bellow 1.1, which means good quality shape of the elements.

The second example, an octagonal section, is presented in Fig. 12a. Figures 12b, 12c and 12d follow the same procedure described on the last paragraph. The Fig. 13 shows the final mesh using the proposed algorithm and Miranda's one. In appearance, there are some differences between the meshes; however, most of elements present normalized metric bellow 1.1. The Fig. 14 provides evidence of this affirmation.

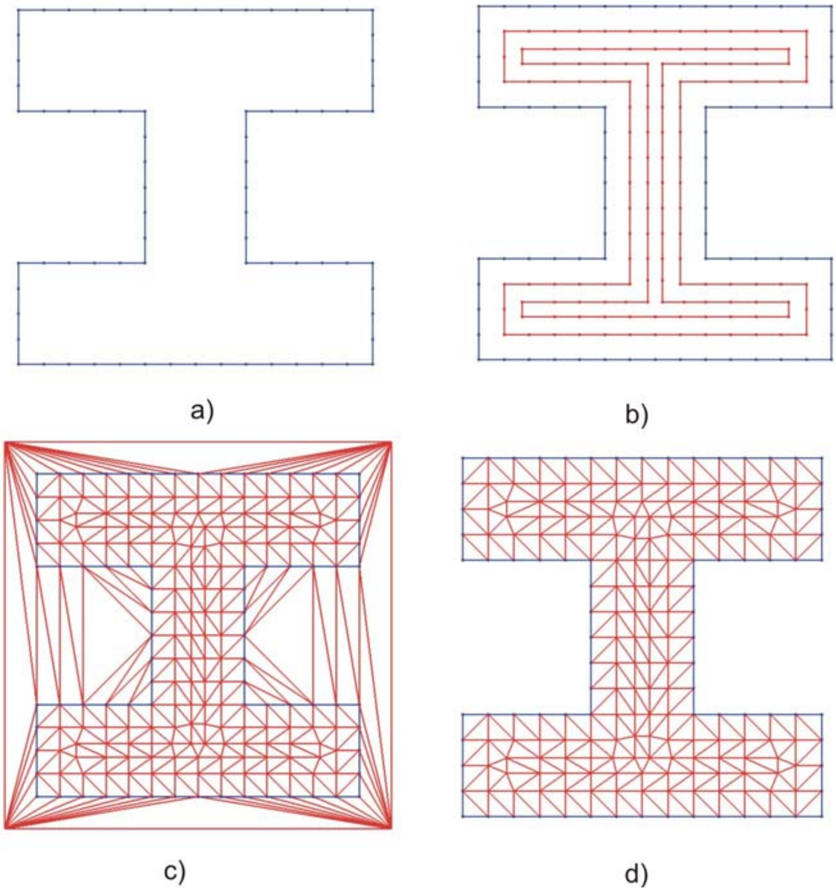

Figure 9. Process to obtain mesh using the proposed algorithm for example 1: a) bound edges and points; b) offset curves; c) Delaunay triangulation; d) intermediary mesh after removing spurious regions.

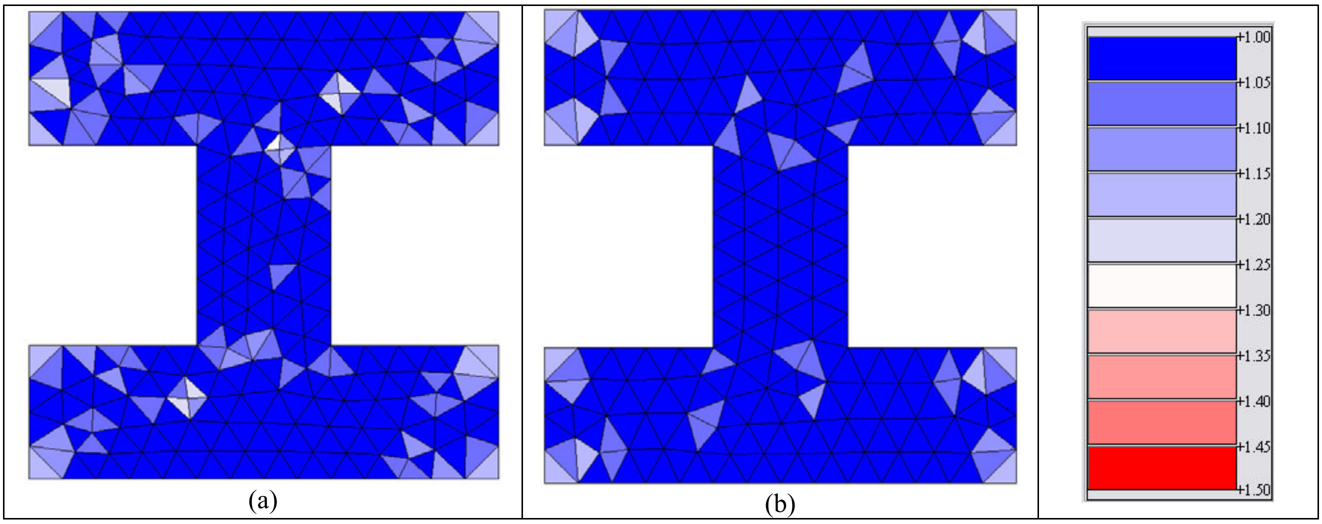

Figure 10. Contour visualization of the normalized metric on example 1: a) final mesh using the proposed algorithm; b) mesh obtained by Miranda Algorithm.

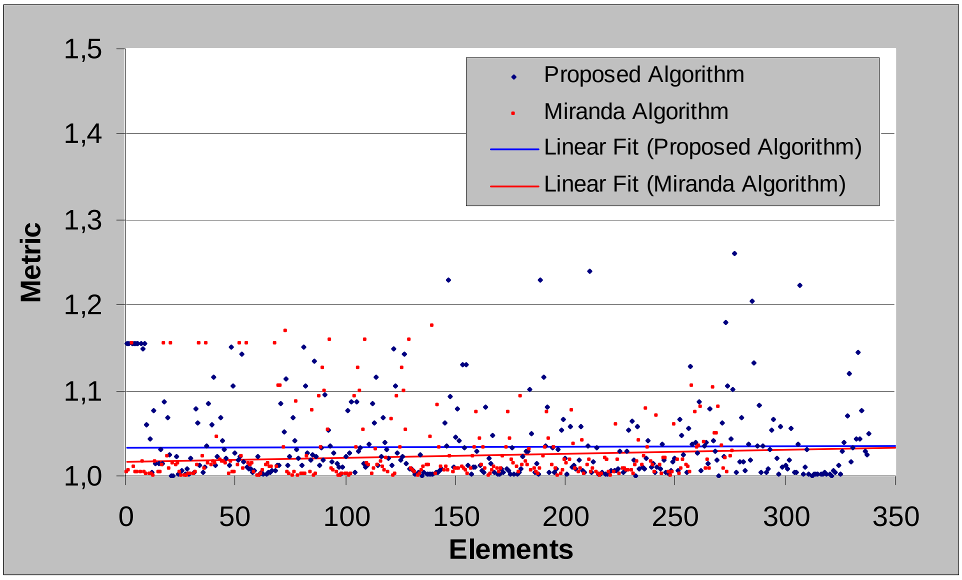

Figure 11. For example 1, normalized metrics are plotted for all elements.

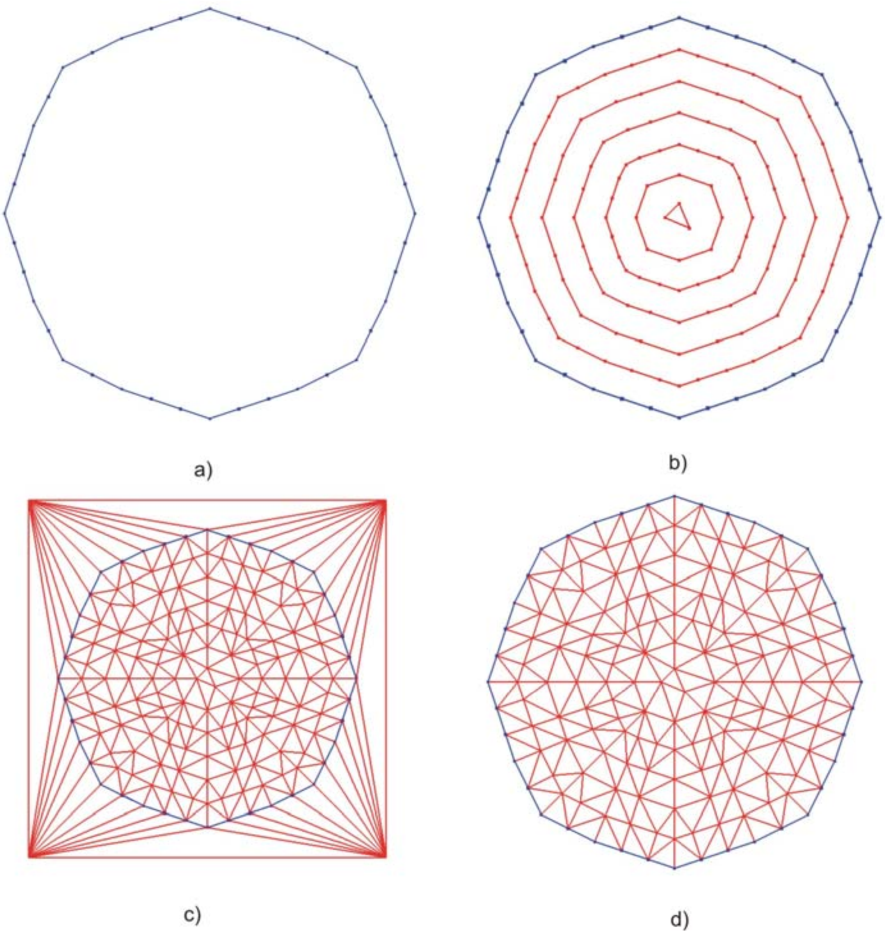

Figure 12. Process to obtain mesh using the proposed algorithm for example 2: a) bound edges and points; b) offset curves; c) Delaunay triangulation; d) intermediary mesh after removing spurious regions.

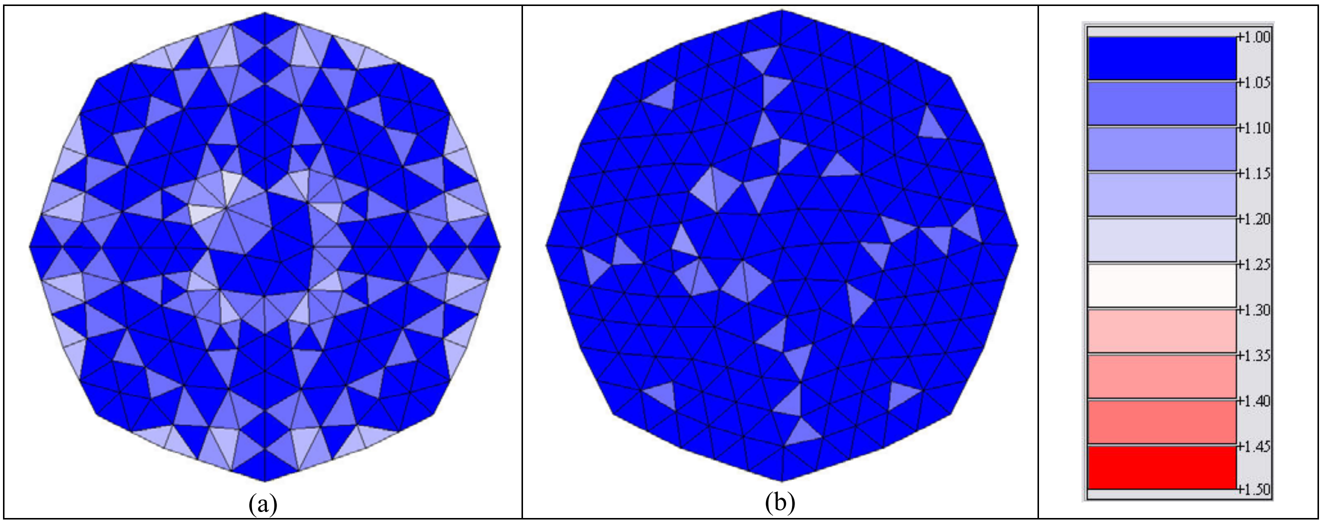

Figure 13. Contour visualization of the normalized metric on example 2: a) final mesh using proposed algorithm; b) mesh obtained by Miranda Algorithm.

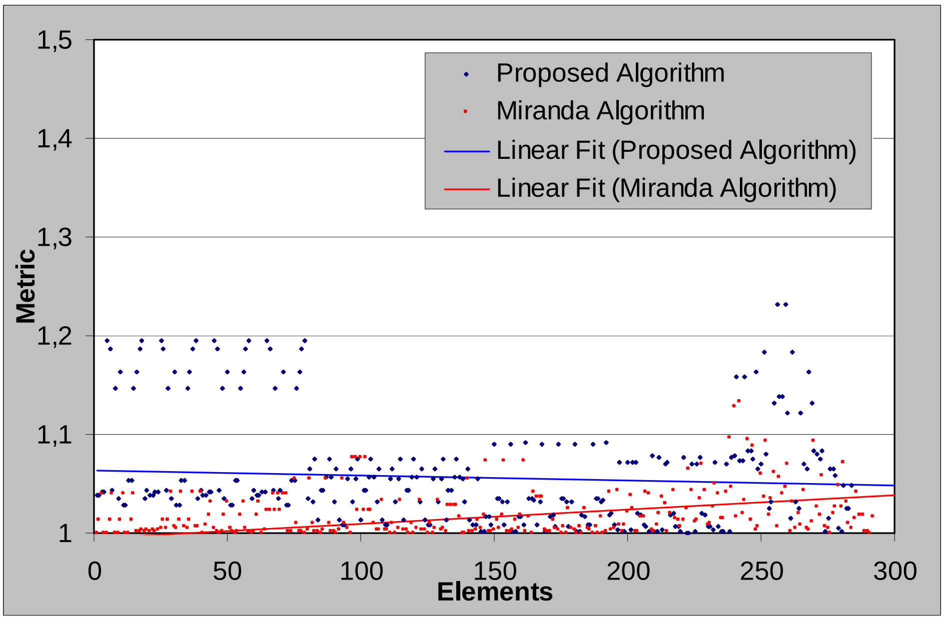

Figure 14. For example 2, normalized metrics are plotted for all elements.

Figures 15a and 15b shows the comparison of both algorithms presenting the percentage of elements with metric bellow 1.1.

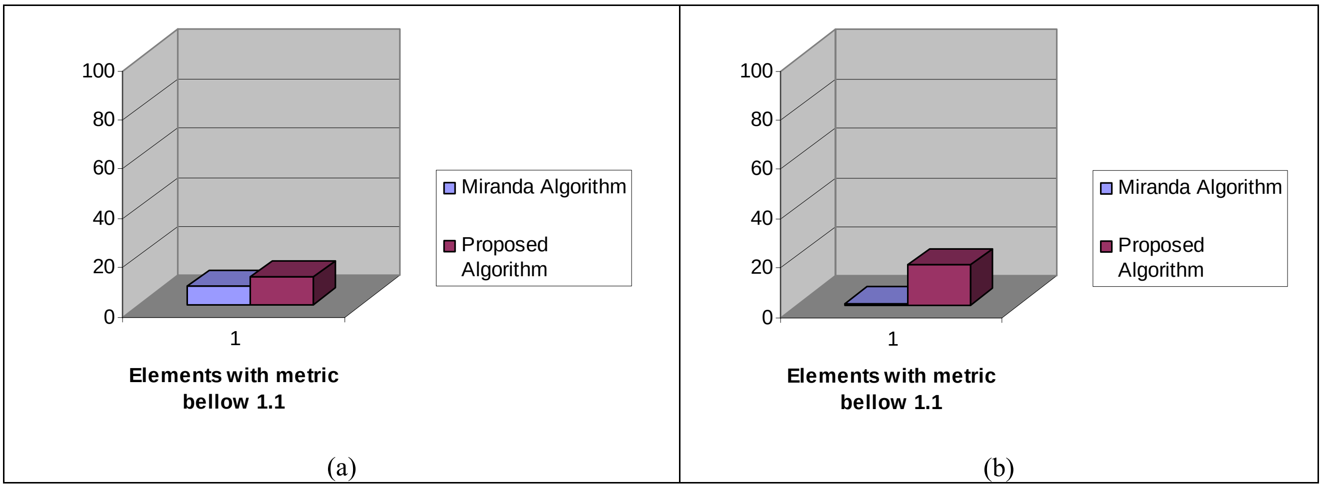

Figure 15. Percentage of elements with metric bellow 1.1: a) Example 1; b) Example 2.

7. CONCLUSION

This work describes an algorithm to generate two-dimensional triangle meshes in arbitrary regions using offset curves and the Delaunay triangulation. The process of offset curve uses the boundary edges and point to trace internal "parallel" curves. These curves and a boundary box are the input data to the Delaunay triangulation. To perform this triangulation, two properties must be followed: the empty circle, and the maximum and minimum angle. In this work, the algorithm implemented is based on De Floriani and Puppo (1992) algorithm. Finally, the final mesh is obtained by smoothing internal nodes.

Two models were presented to compare the quality of meshes generated by the proposed algorithm and by Miranda algorithm. It was shown that the proposed algorithm generates meshes with good quality. However, Miranda algorithm needs a background quadtree to help the generation of elements and, in some cases, it presents undesired behaviors. In contrast, the proposed algorithm creates an alternative way to control the size of elements by tracing offset curves, and it almost achieves the same quality of elements.

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