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LIMIT ANALYSIS OF STRAIGHT AND CURVED TUBES

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Abstract. *The main objective of this work is related to the determination of incipient plastic collapse of tubes submitted to proportional concentrated loading. This goal is achieved through the implementation of a non-linear programming for limit analysis, as an alternative to frequently encountered linear programming approach. The algorithm is developed as a sequence of Newtonian iterations on the set of optimality equalities, followed by a sort of line search consisting of step relaxations combined with stress scaling. The numerical procedure is also prepared to be solved through the utilization of simple finite element techniques. In this work two types of elements are utilized, straight and constant radius curved tubes submitted to concentrated proportional loading. Also an example of problem is solved using this approach.*

Keywords*: limit analysis, optimization, incipient plastic collapse*

1. INTRODUCTION

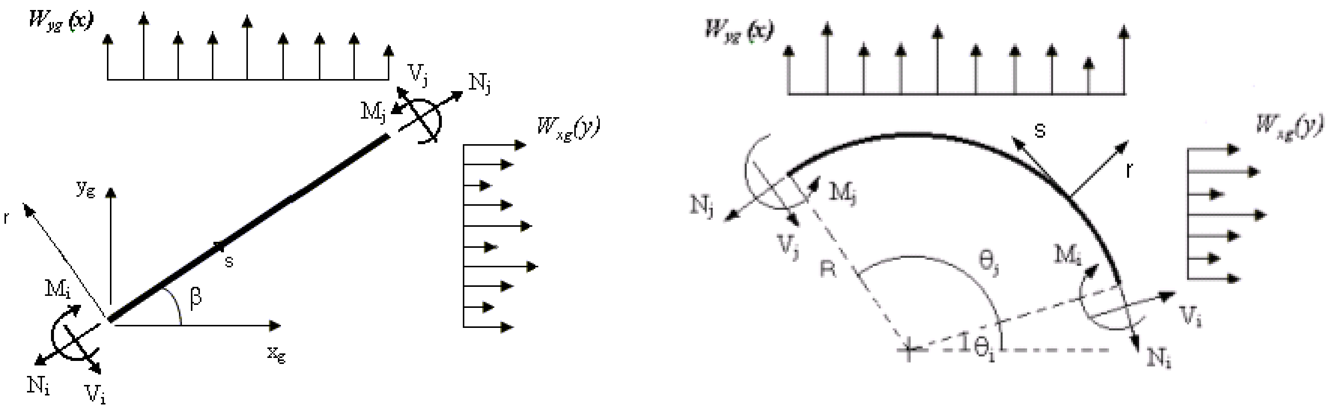
The determination of the limit load that a structure can support before the generation of incipient plastic collapse is achieved by limit analysis theory. This theory distinguishes from incremental plasticity approach by don't following strains evolution but determining limit loads where plastic strain in the structure keep growing without any further increase of load. In this work the limit analysis theory is applied to straight and curved tubes submitted to proportional concentrated loads. A non-linear yielding function is implemented as an alternative to frequently encountered approach of linearization of non-linear yielding function. This choice has the advantage of working with few non-linear constrains in opposition to linearization approach that works with a considerable greater number of linear constrains. Borges *et al.* (1996) proposed a Newtonian algorithm to solve a set of optimality equalities obtained from the limit analysis problem. The theoretical basis is prepared to be solved through the utilization of simple finite element techniques. An example was solved by the utilization of a limit analysis algorithm written in Visual Basic (VB) language.

2. THEORY

In this section the principal aspects of the theory of limit analysis as the expressions of equilibrium, constitutive relations and yielding criterion are shown.

2.1. Equilibrium

Figure 1 shows the straight and curved elements with, respectively, length of L and $R(\theta_j - \theta_i)$:



(a) (b)

Figure 1: Elements: (a) Straight and (b) curved.

It is trivial to show that doing the equilibrium of a part of straight element of length s, the distribution of axial load $N(s)$ and bending moment $M(s)$ can be obtained:

$$N(s) = N_i - \int_0^s \omega_s(s)ds \tag{1}$$

$$M(s) = \left(1 - \frac{s}{L}\right)M_i + \frac{s}{L}M_j + s\left[\frac{1}{L}\int_0^L s\omega_r(s)ds - \int_0^L \omega_r(s)ds\right] + \int_0^s s\omega_r(s)ds \tag{2}$$

Where, N_i, V_i , M_i and N_j, V_j , M_j are respectively the axial, shear and bending moment components at nodes i and j. β is the angle of straight element. θ_i and θ_j are respectively angles at nodes i and j. The local coordinates are s (longitudinal) and r (transversal); and the global coordinates are x_g and y_g. The expressions of longitudinal and transversal loads, in local coordinates, are respectively $\omega_s(s) = W_{xg}\cos(\beta) + W_{yg}\sin(\beta)$ and $\omega_r(s) = -W_{xg}\sin(\beta) + W_{yg}\cos(\beta)$, where W_{xg} and W_{yg} are respectively the horizontal and vertical loads, in global coordinates.

In an analogous way the distribution of axial load $N(s)$ and bending moment $M(s)$ through an arc of angle θ of the curved element can be obtained:

$$N(s) = -\left(\frac{\sin(\theta - \theta_i) + \sin(\theta_j - \theta)}{\sin(\Delta\theta)}\right)N_i - \left(\frac{\sin(\theta - \theta_i)}{R\sin(\Delta\theta)}\right)\left(M_i - M_j\right) +$$

$$+ \frac{\sin(\theta - \theta_i)}{R\sin(\Delta\theta)}\left(\int_{x_i}^{x_j} xW_{yg}(x)dx + \int_{y_i}^{y_j} yW_{xg}(y)dy\right) + \cos(\theta)\int_{x_i}^{x} W_{yg}(x)dx - \sin(\theta)\int_{y_i}^{y} W_{xg}(y)dy \tag{3}$$

$$M(s) = -\left(\frac{R}{\sin(\Delta\theta)}\right)\left(\sin(\theta_j - \theta) - \sin(\Delta\theta) + \sin(\theta - \theta_i)\right)N_i + \left(1 - \frac{\sin(\theta - \theta_i)}{\sin(\Delta\theta)}\right)M_i + \left(\frac{\sin(\theta - \theta_i)}{\sin(\Delta\theta)}\right)M_j -$$

$$- \frac{\sin(\theta - \theta_i)}{\sin(\Delta\theta)}\left(\int_{x_i}^{x_j} xW_{yg}(x)dx + \int_{y_i}^{y_j} yW_{xg}(y)dy\right) - \int_{x_i}^{x} xW_{yg}(x)dx - \int_{y_i}^{y} yW_{xg}(y)dy \tag{4}$$

Expressions (1) and (2), as well as, expressions (3) and (4) can be rewritten in matrix form:

$$Q_s^e(s) = Y(s)Q^e + Q_w(s) \tag{5}$$

Where matrix $Q_w(s)$ represents the effect of distributed load. $Q_s^e(s)$ is the internal load vector and Q^e is the parameter vector:

$$Q_s^e(s) = \begin{pmatrix} N(s) \\ M(s) \end{pmatrix} \tag{6}$$

$$Q^e = \begin{pmatrix} N_i \\ M_i \\ M_j \end{pmatrix} \tag{7}$$

The $Y(s)$ matrices for straight and curved element are shown:

$$Y(s)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1-\frac{s}{L} & \frac{s}{L}\end{pmatrix},\ \text{for straight element} \tag{8}$$

$$Y(s)=\frac{1}{\sin(\Delta\theta)}\begin{pmatrix}\sin(\theta-\theta_i)+\sin(\theta_j-\theta) & \frac{\sin(\theta-\theta_i)}{R} & -\frac{\sin(\theta-\theta_i)}{R}\\ -R(\sin(\theta-\theta_i)-\sin(\Delta\theta)+\sin(\theta_j-\theta)) & \sin(\Delta\theta)-\sin(\theta-\theta_i) & \sin(\theta-\theta_j)\end{pmatrix},\ \text{for curved element} \tag{9}$$

From equilibrium, in local coordinates generates:

$$R_L^e=B_L^TQ^e+W^e \tag{10}$$

Where, R_L^e is the internal nodal load vector, W^e is the distribution vector and B_L^T is the equilibrium matrix:

$$B_L^T=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{L} & -\frac{1}{L}\\ 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & \frac{1}{L} & -\frac{1}{L}\\ 0 & 0 & 1\end{pmatrix},\ \text{for straight element} \tag{11}$$

$$B_L^T=\begin{pmatrix}1 & 0 & 0\\ \frac{1-\cos(\Delta\theta)}{\sin(\Delta\theta)} & \frac{1}{R\sin(\Delta\theta)} & -\frac{1}{R\sin(\Delta\theta)}\\ 0 & 1 & 0\\ 1 & \frac{1}{R} & -\frac{1}{R}\\ \frac{-(1-\cos(\Delta\theta))}{\sin(\Delta\theta)} & \frac{\cos(\Delta\theta)}{R\sin(\Delta\theta)} & -\frac{\cos(\Delta\theta)}{R\sin(\Delta\theta)}\\ 0 & 0 & 1\end{pmatrix},\ \text{for curved element} \tag{12}$$

In global coordinates, from equilibrium, generates:

$$R^e=B^TQ^e+W_g^e \tag{13}$$

Where, R^e is the internal nodal load vector, in global coordinates, W_g^e is the distribution vector and B^T is the equilibrium matrix, in global coordinates. Transformation matrices have to be used to transform from local to global coordinates. See the expressions at Kenedi *et al.* (2006a) and Kenedi *et al.* (2006b).

2.2. Constitutive relations and Yielding criterion

The laws of physical behavior are characterized by constitutive relations between pertinent variables for the description of phenomenon, in this case the plastic collapse. The laws are presented for elastic ideally plastic materials, in isothermal and quasi-static processes. Defining *q* as generalized strain vector and *Q* as generalized stress vector:

$$q=\begin{bmatrix}\varepsilon_0\\ \mathrm{K}\end{bmatrix} \tag{14}$$

$$Q=\begin{bmatrix}\mathrm{N}\\ \mathrm{M}\end{bmatrix} \tag{15}$$

For problems of small strains it is possible to use additive decomposition for the calculation of total strain, in other words, the total strain q can be divided in elastic q^e and plastic q^p parts:

$$q = q^e + q^p \tag{16}$$

where,

$$q^e = D_e^{e^{-1}} Q \tag{17}$$

$$D_e^{e^{-1}} = \begin{bmatrix} \frac{1}{EA} & 0 \\ 0 & \frac{1}{EI} \end{bmatrix} \tag{18}$$

Where, A is the cross section area, E is the Young Modulus, I is the second moment of area and D_e^e is the elastic matrix. Expressions (16) and (17) can be written in rate form:

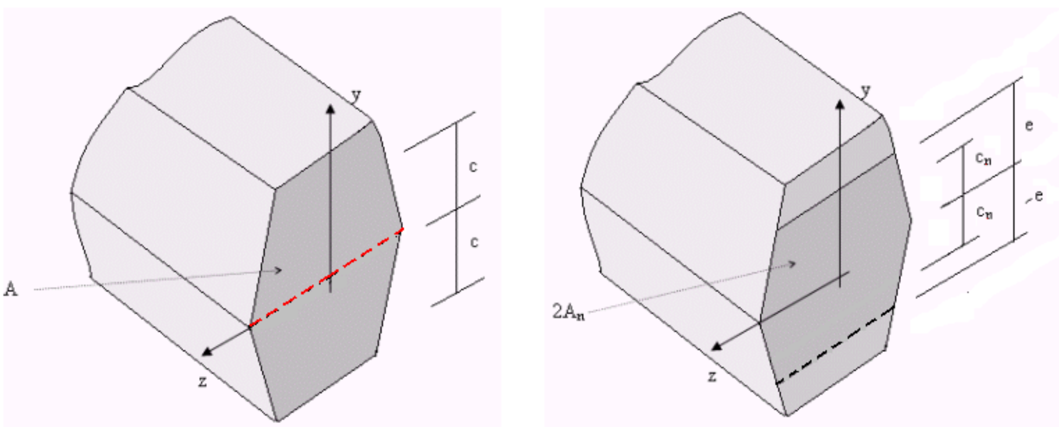
$$\dot{q} = \dot{q}^e + \dot{q}^p \tag{19}$$

$$\dot{q}^e = D_e^{e^{-1}} \dot{Q} \tag{20}$$

Where $\dot{q}$ is the total strain rate, $\dot{q}^e$ is the elastic strain rate and $\dot{q}^p$ is the plastic strain rate. The flow rule, which relates generalized stress with plastic strain rate is shown:

$$\dot{q}^p = \begin{cases} 0 & if \quad f(Q) < 0 \\ 0 & if \; f(Q) = 0 \; and \; \dot{f}(Q) < 0 \\ \dot{\lambda} \nabla_Q f(Q), \dot{\lambda} \geq 0 & if \; f(Q) = 0 \; and \; \dot{f}(Q) = 0 \end{cases} \tag{21}$$

Where $f(Q)$ is a yielding criterion and $\dot{\lambda}$ is a plastic multiplier vector. In sequence it is shown the achievement of a yielding criterion adapted to beam elements $f(M, N)$. Figure 2 shows a symmetrical cross section in relation to y and z axis. At fig.2a, where only bending moment is present, the neutral axis (dashed axis) is coincident with the z axis. The distance between z axis and centroid of each semi-area is c. At fig.2b, a tensile load is added to a negative bending moment, the neutral axis moves to a new position, - e from z axis. The new distance between z axis and centroid of each semi-area is c_n.



(a) (b)

Figure 2: Beam with symmetrical sections with y and z axis (a) submitted to bending load and (b) submitted to tensile load in addition to negative bending moment.

The tensile load and bending moment are:

$$N = \int_A \sigma \, dA \tag{22}$$

$$M = \int_A y\sigma \, dA \tag{23}$$

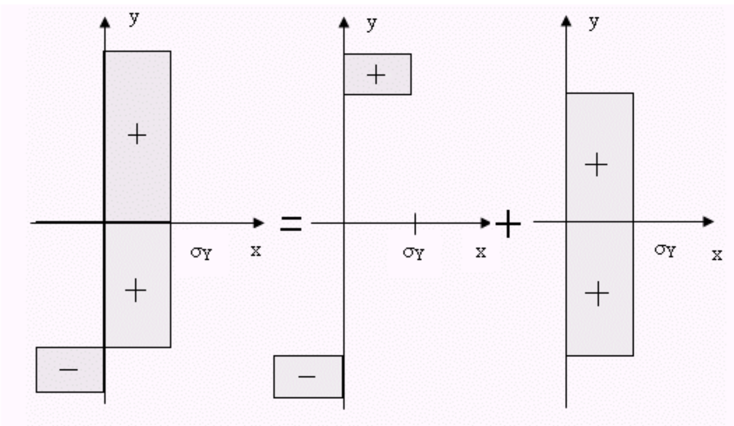
The tensile load and bending moment that plastifies the whole cross section are, respectively, N_0 and M_0:

$$N_0 = \sigma_y A \tag{24}$$

$$M_0 = \sigma_y A\,c \tag{25}$$

Where σ_y is the yield strength and the distance c is defined as: $c = \dfrac{\int_A y \, dA}{A}$

The stress distribution at a beam cross section, of a elastic-ideally plastic material, loaded by tension and negative bending moment can be decomposed as show at fig.3:



(a) (b) (c)

Figure 3: Distributions of stress in a beam of symmetrical cross section totally plastified: (a) Resultant of stress distribution, (b) a bending moment loading and (c) a tensile loading.

From figures 2 and 3 and expressions (22) and (23):

$$N = \sigma_y 2A_n \tag{26}$$

$$M = \sigma_y A\,c - \sigma_y 2A_n c_n \tag{27}$$

Notice from figs. 2b and 3c that area $2A_n$ is only submitted to tensile load. Dividing expressions (26) and (27) respectively by expression (24) and (25) generates:

$$\frac{M}{M_0} + \frac{c_n}{c}\frac{N}{N_0} = 1 \qquad \text{where, } \frac{c_n}{c} = \frac{N}{N_0} \text{ for rectangular cross section beam.}$$

Generating the yielding criterion:

$$f(M,N) = \frac{\langle M \rangle}{M_0} + \left(\frac{N}{N_0}\right)^2 - 1 \quad \text{where, } \langle M \rangle \text{ is +M for positive and } \langle M \rangle \text{ is -M for negative bending moments.} \tag{28}$$

Figure 4 shows the graphical representation of expression (28):

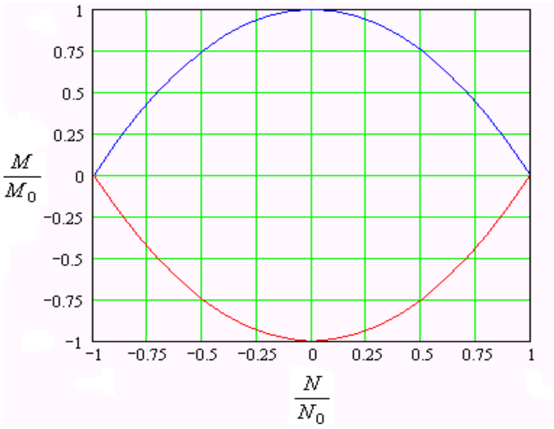


Figure 4: Yielding criterion locus.

Júnior, (2006) showed that yielding surface of beams of rectangular and tubular cross sections are practically the same, so in this work the expression (28) will be used to represent the yielding surface of tubes, and can be rewritten in matrix form:

$$f(M,N)=\frac{1}{2}CQ_s^e(s)\cdot Q_s^e(s)\pm A\cdot Q_x^e(s)-1,\ \text{where,}\ A=\begin{pmatrix}0\\ \dfrac{1}{M_0}\end{pmatrix}\ \text{and}\ C=\begin{pmatrix}\dfrac{2}{N_0{}^2} & 0\\ 0 & 0\end{pmatrix} \tag{29}$$

Substituting (5) in (29) the yielding criterion is, now, written in function of nodal loads:

$$f(M,N)=\frac{1}{2}\mathbf{C}Q^e\cdot Q^e\pm\mathbf{A}\cdot Q^e+\mathbf{R},$$

$$\text{where}\quad \mathbf{C}=Y(s)^T\,C\,Y(s),\quad \mathbf{A}=Y(s)^T\left(\alpha\,CQ_w(s)+A\right)\ \text{and}\ \mathbf{R}=\left(\frac{\alpha}{2}CQ_w(s)+A\right)\cdot Q_w(s)-1 \tag{30}$$

and α is the collapse factor.

3. THEORY IMPLEMENTATION

The analysis of discrete problem can be done by the utilization of Static Formulation (Borges *et al.*, 1996):

$$\alpha=\begin{array}{c}\sup\alpha^*\\ \alpha^*\in\Re\\ Q\in\Re^q\end{array}\left|\begin{array}{c}B^TQ-\alpha^*R=0\\ f(Q)\le 0\end{array}\right. \tag{31}$$

The application of optimality conditions (expression 32-35) are equivalent to expression (31) of Static Formulation (Borges *et al.* , 1996):

$$B\dot{U}-\nabla_Q f(\alpha,Q)\dot{\lambda}=0 \tag{32}$$

$$B^TQ-\alpha R=0 \tag{33}$$

$$R\cdot\dot{U}+\nabla_\alpha f(\alpha,Q)\dot{\lambda}=1 \tag{34}$$

$$f(\alpha,Q)_k^e\le 0 \qquad \dot{\lambda}_k^e\ge 0 \qquad f(\alpha,Q)_k^e\dot{\lambda}_k^e=0 \qquad e=1\ldots nel,\ k=1\ldots nk \tag{35}$$

Where B is the deformation operator that is the transpose of equilibrium matrix B^T used in expression (13), $\dot{U}$ is the velocity vector and R is the external load vector, also used in expression (13). k is the number nodes and n is the number of plastic modes per node.

The optimality conditions can be rewritten as shown:

$$g(x)=0, \quad f(Q)\le 0 \quad \text{and} \quad \dot{\lambda}\ge 0 \tag{36}$$

where,

$$g(x)=\begin{pmatrix} B\dot{U}-\nabla_Q f(\alpha,Q,s)\dot{\lambda} \\ B^T Q-\alpha R \\ R\dot{U}+\nabla_\alpha f(\alpha,Q,s)\dot{\lambda}-1 \\ G(\alpha,Q,s)\dot{\lambda} \end{pmatrix}, \quad x=\begin{pmatrix} Q \\ \dot{U} \\ \alpha \\ \dot{\lambda} \end{pmatrix} \text{ and } \quad G(\alpha,Q)=diag\left(f_k^e(\alpha,Q)\right) \quad e=1\ldots nel,\ k=1\ldots nk \tag{37}$$

3.1 Algorithm

A two stage procedure is utilized to determine a new value of x vector in function of its present value. At first stage an increment of x vector, called d_x, is determined using a Newton-like iteration for the non linear system of equations of expression (37). At second stage the stress increment d_x is reduced by a step length factor s and the resulting stress is scaled by a factor p.

Applying a Newton-like approach:

$$\nabla g(x)d_x=-g(x) \tag{38}$$

where,

$$\nabla g(x)=\begin{pmatrix} -H_{QQ} & B & -H_{Q\alpha} & -\nabla_Q f \\ B^T & 0 & -R & 0 \\ H_{Q\alpha} & R^T & H_{\alpha\alpha} & \nabla_\alpha f^T \\ \Lambda\nabla_Q^T f & 0 & \nabla_\alpha f & G \end{pmatrix} \tag{39}$$

$$\Lambda=diag(\dot{\lambda}_m) \quad m=1\ldots nk\times nel \qquad H_j=\sum \dot{\lambda}_j\nabla^2 f_j$$

At first stage the increment of x vector is updated:

$$dQ^0=\left(D^{ep}B\hat{\dot{U}}+TW_Q^{-1}W_\alpha-H_{QQ}^{-1}H_{Q\alpha}\right)d\alpha^0 \tag{40}$$

$$\dot{U}^0=\hat{U}\,d\alpha^0 \tag{41}$$

$$d\alpha^0=\frac{1}{\left(\sum R_0\cdot\hat{U}\right)+r_\alpha} \tag{42}$$

$$\dot{\lambda}^0=W_Q^{-1}\left(T^T B\hat{\dot{U}}-W_\alpha\right)d\alpha^0 \tag{43}$$

Where, $D^{ep}=H_{QQ}^{-1}-TW_Q^{-1}T^T$, $T=H_{QQ}^{-1}\nabla_Q f$, $W_Q=\nabla_Q^T f\,T-\Lambda^{-1}G$ and $W_\alpha=T^T H_{Q\alpha}-\Lambda^{-1}\nabla_\alpha f$
$\hat{U}$ is the displacement vector from the solution of $K\hat{U}=\widetilde{R}$, where $K=B^T D^{ep} B$ is the variable elastic-plastic stiffness matrix that is obtained by the assembling the contribution of every element K_i. Expression (30) is used to calculate each of m plastic functions.

At second stage step relaxation and stress scaling; and updating are done:

$$Q_i^e = p\left(Q_{i-1}^e + s\,dQ^0\right) \tag{44}$$

$$\alpha_i = p\left(\alpha_{i-1} + s\,d\alpha^0\right) \tag{45}$$

Also

$$\dot{\lambda}_m = \max\left(\dot{\lambda}_j^0 \quad, \gamma_\lambda \left\|\max\left(\dot{\lambda}^0\right)\right\|_\infty\right) \tag{46}$$

The value of p for each iteration is achieved by picking the p that is nearer to unity of the m calculated using $f_j\left(p^j Q^0\right) - \gamma_f f_j(Q) = 0$. Borges *et al.* (1996) has a very detailed description of this algorithm, including the initialization and convergence criterion.

3.2 VB Program

To handle the large amount of mathematical operations needed to implement a limit analysis routine a VB program was developed for straight and constant radius curved elements. Figure 5 shows a couple of screens of the pre-processor of this program, such as the initial conditions, the materials properties, the cross-section options, the input of nodes and elements, the localization the center of curved elements and the specification of concentrated loading and constrains.

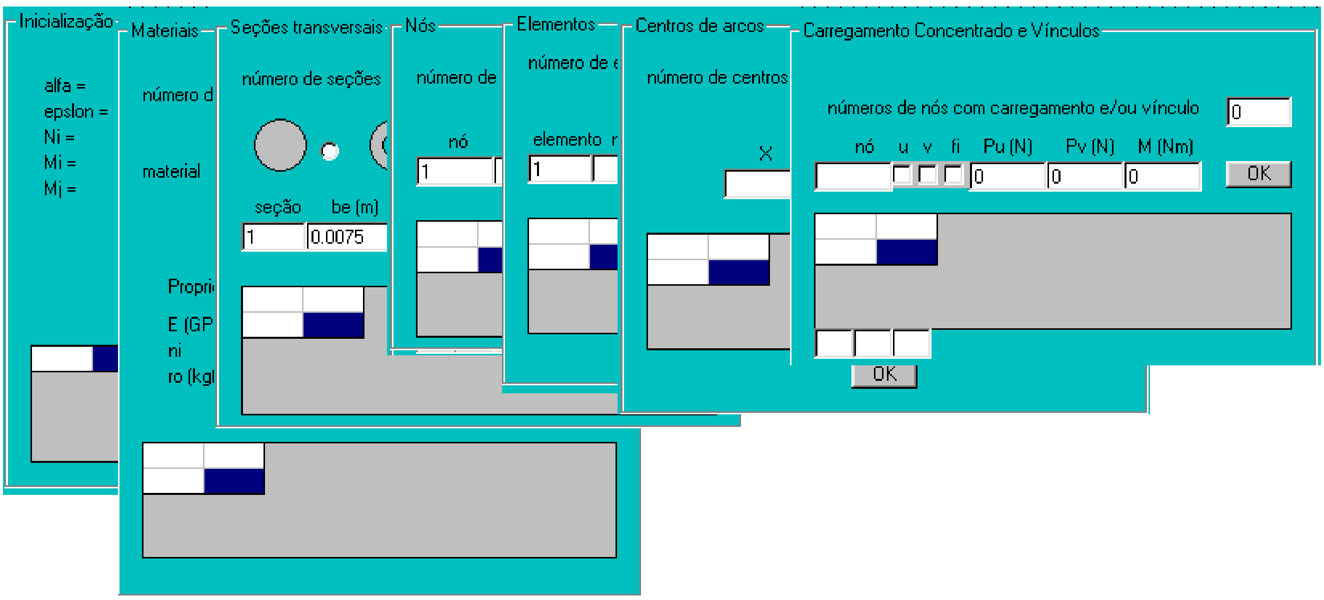


Figure 5 - Pre-processor screens of this program.

There are many advantages of using a software object orientated like VB such as, the natural organized way of programming that arise from the utilization of a great number of small codes inside each command button, the intuitive but powerful sets of commands available that is similar to well know Fortran language, the capacity of implementation of complex routines of algorithm and the capacity to handle a massive amount of calculations needed to be processed. This last characteristic is fundamental to solve a limit analysis algorithm since the calculation of elastic-plastic stiffness matrix has to be actualized, at each iteration, to determine the displacement vector by the utilization of Gauss approach. From fig.5 it is apparent another advantage of using VB that is the professional display of screens. So far the pos processor of this program is limited to generate files of .txt extension of the output variables such as Q^e, $\dot{U}$,α and $\dot{\lambda}_m$. With the development of this VB program a graphical output will be implemented to include a graphical representation of the final geometry of collapsed structure.

3.3 Example

An example obtained from Lubliner, (1990) is implemented at VB program. Figure 6 shows the structure before and after the development of plastic hinges:

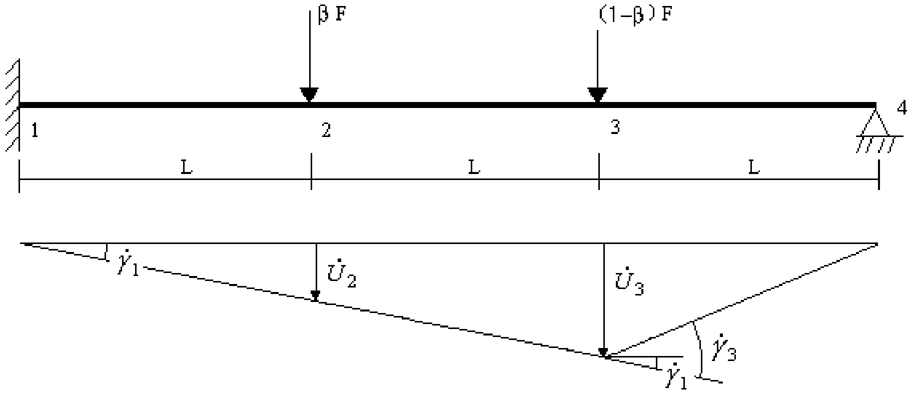


Figure 6 – A beam with two constrains at ends, one built in and other simple supported loaded by two concentrated loads at intermediary positions, before and after the development of plastic hinges at nodes 1 and 3.

The beam is divided in three elements with the same length, with four nodes. The velocities of nodes $\dot{U}_i$ and stain rate of plastic hinges $\dot{\gamma}_i$ are:

$$\dot{U}_1 = 0 \text{ and } \dot{\gamma}_1 = -\frac{\dot{U}}{L},\ \dot{U}_2 = -\dot{U},\ \dot{U}_3 = -2\dot{U} \text{ and } \dot{\gamma}_3 = -\frac{3\dot{U}}{L} \text{ and } \dot{U}_4 = 0 \tag{47}$$

Equating internal and external power and using expression (47) is possible to obtain:

$$\alpha = \frac{4M_0}{(2-\beta)FL}\ \text{, where } M_0 = S_y \frac{bh^2}{4} \text{ (for a rectangular cross section b x h)} \tag{48}$$

Substituting b = 0.0075 m, h = 0.003 m, β = 0.3, F =1 N, L = 1 m and σ_y = 250 MPa at expression (48) it is possible to determine the analytical alpha $\alpha_{analytical}$ = 9.9264. The VB program generated Q^e, $\dot{U}$,α and $\dot{\lambda}_m$ output in *.txt files. The Q^e file shows that *N* are always zero and *M* is -1 at node 1 and +1 at node 3, the $\dot{U}$ file shows that horizontal velocities at all nodes and vertical velocities at nodes 1 and 4 are null; and the vertical velocity of node 3 is the double of node 2. The α file shows the numeric alpha $\alpha_{numeric}$ = 9.9259, after four iterations (1^{st} = 9.2272, 2^{nd} = 9.6452, 3^{rd} = 9.9161 and 4^{th} = 9.9259) that is very close to $\alpha_{analytical}$ = 9.9264. Also $\dot{\lambda}_m$ file shows correctly, only in nodes 1 and 3 there are generation of plastic hinges.

4. CONCLUSIONS

The determination of incipient plastic collapse of tubes submitted to proportional concentrated loading was achieved through the utilization of limit analysis theory. An explanation about equilibrium, constitutive relations and yielding criterion for limit analysis theory was done. The discretization of problem was executed through the proposition of optimality conditions. A brief explanation of implementation of a Newton-like iteration for solution of non linear system of equations in two stages, increment and step relaxation combined with stress scaling was done. This approach was implemented in a VB program to handle the large amount of mathematical operations needed to implement a limit analysis routine. A problem was solved and the results compared with reference literature. With the continuation of this research the output capabilities of VB program will be improved and a successive application of limit analysis theory will be implemented to extend the utilization of limit analysis theory for large displacement and rotations.

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