TOPOLOGICAL SENSITIVITY ANALYSIS APPLIED TO THE MASS MINIMIZATION PROBLEM UNDER MATERIAL FAILURE CONSTRAINTS

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Abstract. *The Topological Sensitivity Method provides a function whose value at a certain position* $\mathbf{x}$ *on the domain represents the sensitivity of a cost function when an infinitesimal hole or an inclusion is introduced at this point. This work deals with Topologial Sensitivity Analysis applied to the problem of mass minimization with local failure constraint. To this aim, equilibrium condition and material failure constraint are incorporated into the objective function through a Lagrangian and Augmented Lagrangean approach. The calculation of the Topological Sensitivity of the proposed cost function involves two stages. The first one refers to the Shape Sensitivity Analysis due to variations in the hole diameter which provides an expression based on stress and strains of the equilibrium and adjoint solution. The second one, not addressed here, is the Asymptotic Analysis of the resulting argument, when the hole size goes to zero.*

1. INTRODUCTION

The Topological Derivative (TD) is a relatively new concept that was firstly introduced by Eschenauer et al. (1994) and Eschenauer & Schumacher (1993). Given a state equation defined on a domain Ω and a cost function associated to it, this derivative provides the sensitivity of this cost function when an infinitesimal hole or inclusion is introduced in Ω. Since then, many works had been published related to further developments in the derivative concept itself and to different applications in specific problems (see, e.g, Eschenauer & Schumacher (1997), Sokolowski & Zochowski (1999), Céa (2000), among many others). In Novotny (2003) and Novotny et. al (2003) the existence of a link between the mentioned original definition of the TD and classical Shape Sensitivity Analysis was formally demonstrated, which allowed to use established concepts of the Continuum Mechanics for its calculation.

As consequence of its nature, the information contained in the TD has been naturally used in topological optimization procedures. In structural cases, however, these applications are mostly restricted to the classical compliance problem, following what is again observed in the application field of classical approaches like intermediate materials (Bendsøe and Sigmund (2003)).

A search in the literature immediately shows that the problem of structural topology optimization considering local constraints has received much less attention than the classic compliance problem. The main reasons for this are the difficulties introduced by the constraints due to their number and mostly by their nature. A first formalism addressing the problem of mass minimization with local failure constraints in continuum structures is attributed to Duysinx & Bendsøe (1998), Duysinx & Sigmund (1998). In Pereira et al. (2004) (and in later improvements shown in Fancello & Pereira (2003), Fancello (2006)) this problem is revisited proposing an Augmented Lagrangian approach in order to deal with stress constraints. Despite the satisfactory result shown in these works, it is quite clear that no conclusive results are available yet and the field is open to new developments.

The objective of this paper is to introduce the Topological Sensitivity formalism into the problem of mass minimization with failure (stress) constraints. To this aim, and for completeness of this paper, a brief description of the Topological Derivative and the Topological Sensitivity Method is initially presented in Section 2. In Section 3, the objective function is formulated and, finally, the outlines of the topological sensitivity for this problem are stated. This sensitivity is basically based in a two-step procedure: first, classical shape sensitivity analysis and, second, a localization of this sensitivity based on asymptotic analysis.

2. TOPOLOGICAL DERIVATIVE AND TOPOLOGICAL SHAPE SENSITIVITY METHOD

Consider an open domain $\Omega \subset R^N, (N = 2, 3)$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ (Dirichlet and Neumann boundaries) perturbed by the introduction of a hole F_ϵ of radius ϵ centered in an arbitrary point $\mathbf{x}$ in such a way that the new (open) domain results $\Omega_\epsilon = \Omega - \bar{F}_\epsilon$. Consider now two cost functions $\psi(\Omega)$ and $\psi(\Omega_\epsilon)$ defined, respectively, in the original and perturbed domains. Then, the Topological Derivative is given by the following limit (see Garreau et al., 2001):

$$D_T(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{\psi(\Omega_\epsilon) - \psi(\Omega)}{f(\epsilon)}, \tag{1}$$

where $f(\epsilon)$ is a negative and monotonic function, such that $f(\epsilon) \to 0$ with $\epsilon \to 0$. This expression can also be viewed as the second term of the Taylor series

$$\psi(\Omega_\epsilon) = \psi(\Omega) + f(\epsilon) D_T(\mathbf{x}) + \mathcal{O}(f(\epsilon)). \tag{2}$$

It is necessary to remark that, although the above definition is quite general, the limit given by Eq. (1) cannot be performed easily. Domains Ω_ϵ and Ω do not have the same topology and consequently it is not possibly to define an homeomorphic map between them. Novotny (2003) proposed an alternative method to compute the Topological Derivative based on the concept of Shape Sensitivity Analysis. This new procedure considers the sensitivity of the cost function due to a modification on the radius ϵ of an existing hole. This perturbation, of size $\tau \in R$ provides a new domain $\Omega_\tau = \Omega - \bar{F}_\tau$ where $\bar{F}_{\epsilon+\tau}$ is a hole of radius $\epsilon + \tau$. The smooth and inversive mapping between Ω_τ and Ω_ϵ is represented by

$$\mathbf{x}_\tau = \mathbf{x} + \tau \mathbf{v}(\mathbf{x}) \tag{3}$$

where $\mathbf{v}(\mathbf{x})$ is the shape change velocity field. Zolézio (1981), proved that only the component in the direction normal to the boundary is relevant for shape sensitivity expressions and only this term contributes for a change on the domain. A radial velocity field may be defined by

$$\mathbf{v}(\mathbf{x}) = \begin{cases} 0, & \text{se } \mathbf{x} \in \partial\Omega_\epsilon \\ -\mathbf{n}, & \text{se } \mathbf{x} \in \partial F_{\epsilon+\tau} \end{cases} \tag{4}$$

that represents an uniform expansion of the hole F_ϵ, as show Fig. (1).

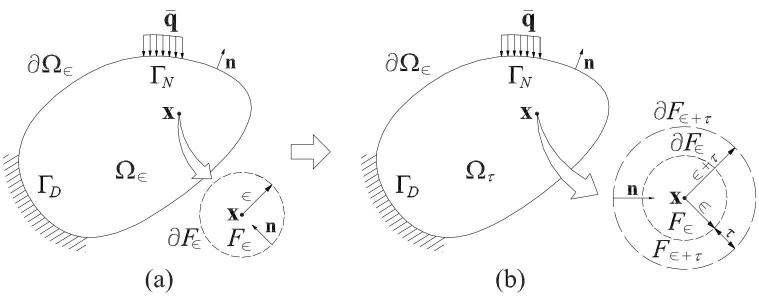

Figure 1. Original and perturbed domain through expansion in the radius of the infinitesimal hole.

The sensitivity of the function $\psi(\Omega_\tau)$ in relation to the parameter τ, for $\tau = 0$, is given by

$$\left.\frac{d}{d\tau}\psi(\Omega_\tau)\right|_{\tau=0} = \lim_{\tau \to 0} \frac{\psi(\Omega_\tau) - \psi(\Omega_\epsilon)}{\tau}. \tag{5}$$

In Novotny et al. (2003) it is formally shown that (1) and (5) are related by the limit

$$D_T(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{1}{f'(\epsilon)} \left.\frac{d}{d\tau}\psi(\Omega_\tau)\right|_{\tau=0}. \tag{6}$$

This expression allows the calculation of the Topological Derivative to be performed in a two-step procedure: The first one is the systematic application of the well known mathematical framework provided by shape sensitivity analysis. The second one is an asymptotic analysis of the shape sensitivity expressions in order to make the final limit operation possible. Although the first step is sistematically performed, the last one requires particular attention.

3. MASS MINIMIZATION PROBLEM

Since the purpose of this work is analyzing the problem of mass minimization constrained by a material failure criterion within the context of expression (6), it is convenient to define a cost function ψ that accounts for both, objective and constraints. A possible way to do this is using an Augmented Lagrangian procedure. Consider then, the Problem P_1

defined on the original domain Ω

$$\begin{aligned} &\min_{\Omega} \mu = \int_{\Omega} \rho \, d\Omega \\ &\text{subject to:} \left\{ \begin{array}{r} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathcal{V} \\ g(\mathbf{u}) \leq 0, \end{array} \right. \end{aligned} \tag{7}$$

where ρ represents a relative density, μ accounts for the relative mass of the body in Ω, $g(\mathbf{u})$ defines the material failure function while a and l are the usual forms in linear elasticity:

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\tau} \sigma(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) \, d\Omega, \tag{8}$$

$$l(\mathbf{v}) = \int_{\Omega_\tau} \mathbf{b} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \mathbf{v} \, d\Gamma_N. \tag{9}$$

In this last equations, $\sigma(\mathbf{u})$ represents the stress tensor, $\mathbf{C}$ the elasticity tensor and $\varepsilon(\mathbf{u})$, the infinitesimal Green's strain tensor. Displacements fields $\mathbf{u}$ and $\mathbf{v}$ belong (just for simplicity reasons) to the same space $\mathcal{V}$,

$$\mathcal{V} = \left\{ \mathbf{v} \in H^1(\Omega_\tau) : \mathbf{v} = 0 \ \text{ on } \Gamma_D \right\}. \tag{10}$$

Adding the function $s \geq 0$ in the inequality constraint associated to the material failure criterion, the Problem P_1 can be rewritten as (Problem P_2)

$$\begin{aligned} &\min_{s,\,\Omega} \mu = \int_{\Omega} \rho \, d\Omega \\ &\text{subject to:} \left\{ \begin{array}{r} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathcal{V} \\ h(\mathbf{u}, s) = g(\mathbf{u}) + s = 0 \end{array} \right. \end{aligned} \tag{11}$$

The material failure constraint can be included into the cost function by using a Lagrangian functional $\psi(\mathbf{u}_\tau)$ and a Lagrangian multiplier α which defines the Problem P_3:

$$\begin{aligned} &\min_{s,\,\Omega} \psi(\mathbf{u}) = \int_{\Omega} \rho d\Omega + \int_{\Omega} \alpha \, h(\mathbf{u}, s) \ d\Omega \\ &\text{subject to:} \left\{ \begin{array}{r} a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathcal{V} \\ h(\mathbf{u}, s) = g(\mathbf{u}) + s = 0 \end{array} \right. \end{aligned} \tag{12}$$

It must be noted that in Problem P_3, if the constraint $h(\mathbf{u}_\tau, s)$ is satisfied, Problems P_1, P_2 and P_3 are equivalent. Problem P_3 can be solved by penalization (using the penalization parameter r), which defines Problem P_4 :

$$\begin{aligned} &\min_{s,\,\Omega} \psi(\mathbf{u}) = \int_{\Omega} \rho d\Omega + \int_{\Omega} \alpha \, h(\mathbf{u}, s) \, d\Omega + \frac{1}{2r} \int_{\Omega} h^2(\mathbf{u}, s) \ d\Omega \\ &\text{subject to:} \quad a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathcal{V} \end{aligned} \tag{13}$$

The minimization with respect to the variable $s > 0$ can be obtained analytically, following classical arguments in the Augmented Lagrangian technique. It is straightforward to show that the function s that minimizes the functionary is given by (Pereira, 2001):

$$s = \max\left\{0 \, ; \, -(g(\mathbf{u}) + r\alpha)\right\}, \tag{14}$$

that, from the Eq. (12), provides

$$h(\mathbf{u}) := \max\left\{g(\mathbf{u}) \ ; \ -r\alpha\right\}, \tag{15}$$

Substituting (15) in (13) the expression of Problem P_5 is obtained:

$$\begin{aligned} &\min_{\Omega} \psi(\mathbf{u}) = \int_{\Omega} \rho d\Omega + \int_{\Omega} \alpha \, h(\mathbf{u}) \, d\Omega + \frac{1}{2r} \int_{\Omega} h^2(\mathbf{u}) \ d\Omega \\ &\text{subject to:} \quad a(\mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathcal{V} \end{aligned} \tag{16}$$

Finally, if Problem P_5 is defined on the perturbed domain Ω_τ , an analogous expression is obtained:

$$\begin{aligned} &\min_{\Omega_\tau} \psi(\mathbf{u}_\tau) = \int_{\Omega_\tau} \rho d\Omega_\tau + \int_{\Omega_\tau} \alpha h_\tau(\mathbf{u}_\tau) \, d\Omega_\tau + \frac{1}{2r} \int_{\Omega_\tau} h_\tau^2(\mathbf{u}_\tau) \, d\Omega_\tau \\ &\text{subject to:} \quad a_\tau(\mathbf{u}_\tau, \mathbf{v}_\tau) - l_\tau(\mathbf{v}_\tau) = 0, \quad \forall\, \mathbf{v}_\tau \in \mathcal{V}_\tau, \end{aligned} \tag{17}$$

In this functional, the Lagrange multiplier function α is supposed to be known a-priori. The Augmented Lagrangian technique allows for an iterative updating of this value such that the constraints are satisfied at the end of the process.

4. TOPOLOGICAL SENSITIVITY ANALYSIS

The topological sensitivity analysis of Eq.(17) is the subject of this section. The information provided by the resulting function can be used later in a descent procedure based in expression (2).

Considering the configuration Ω_τ, we define the Lagrangian function associated with Problem P_5, that incorporates the equilibrium equation

$$\begin{aligned}\pounds_\tau\left(\mathbf{u}_\tau,\lambda_\tau,\tau\right) = &\int_{\Omega_\tau}\rho d\Omega_\tau + \int_{\Omega_\tau}\alpha h_\tau\left(\mathbf{u}_\tau\right)d\Omega_\tau + \frac{1}{2r}\int_{\Omega_\tau}h_\tau^2\left(\mathbf{u}_\tau\right)d\Omega_\tau \\ &+ a_\tau\left(\mathbf{u}_\tau,\lambda_\tau\right) - l_\tau\left(\lambda_\tau\right),\end{aligned} \tag{18}$$

where the function λ_τ is the corresponding Lagrange multiplier function. The total derivative of $\pounds_\tau$ is given by

$$\frac{d\pounds_\tau}{d\tau} = \frac{\partial\pounds_\tau}{\partial\tau} + \left\langle\frac{\partial\pounds_\tau}{\partial\lambda_\tau},\eta_\tau\right\rangle + \left\langle\frac{\partial\pounds_\tau}{\partial\mathbf{u}_\tau},\xi_\tau\right\rangle \tag{19}$$

Enforcing optimality conditions (null value) for the partial derivatives of the second and the third terms of Eq. (19), we recover the equilibrium and adjoint equations, respectively given by

$$\left\langle\frac{\partial\pounds_\tau}{\partial\lambda_\tau},\eta_\tau\right\rangle = a_\tau\left(\mathbf{u}_\tau,\eta_\tau\right) - l_\tau\left(\eta_\tau\right) = 0 \quad \forall\,\eta_\tau\in\mathcal{V}_\tau \tag{20}$$

$$\begin{aligned}\left\langle\frac{\partial\pounds_\tau}{\partial\mathbf{u}_\tau},\xi_\tau\right\rangle = &\,a_\tau\left(\lambda_\tau,\xi_\tau\right) + \left\langle\frac{\partial}{\partial\mathbf{u}_\tau}\int_{\Omega_\tau}\alpha h_\tau\left(\mathbf{u}_\tau\right)d\Omega_\tau\bigg|_{\tau=0},\xi_\tau\right\rangle \\ &+ \frac{1}{2r}\left\langle\frac{\partial}{\partial\mathbf{u}_\tau}\int_{\Omega_\tau}h_\tau^2\left(\mathbf{u}_\tau\right)d\Omega_\tau,\xi_\tau\right\rangle = 0 \quad \forall\,\xi_\tau\in\mathcal{V}_\tau,\end{aligned} \tag{21}$$

Thus, for $\mathbf{u}_\tau$ and λ_τ satisfying conditions above, the total derivative of the Lagrangian function with respect to τ is given by

$$\begin{aligned}\frac{d\pounds_\tau}{d\tau}\bigg|_{\tau=0} = &\frac{\partial}{\partial\tau}\int_{\Omega_\tau}\rho d\Omega_\tau\bigg|_{\tau=0} + \frac{\partial}{\partial\tau}\int_{\Omega_\tau}\alpha h_\tau\left(\mathbf{u}_\tau\right)d\Omega_\tau\bigg|_{\tau=0} + \frac{1}{2r}\frac{\partial}{\partial\tau}\int_{\Omega_\tau}h_\tau^2\left(\mathbf{u}_\tau\right)d\Omega_\tau\bigg|_{\tau=0} \\ &+ \frac{\partial}{\partial\tau}a_\tau\left(\mathbf{u}_\tau,\lambda_\tau\right)\bigg|_{\tau=0} - \frac{\partial}{\partial\tau}l_\tau\left(\lambda_\tau\right)\bigg|_{\tau=0}\end{aligned} \tag{22}$$

Using the Reynolds Transport Theorem, the first and the second terms of this equation can be written, respectively, as

$$\frac{\partial}{\partial\tau}\int_{\Omega_\tau}\rho\,d\Omega_\tau\bigg|_{\tau=0} = \int_{\Omega_\epsilon}\rho\,div\,\mathbf{v}\,d\Omega_\epsilon \tag{23}$$

$$\frac{\partial}{\partial\tau}\int_{\Omega_\tau}\alpha h_\tau\left(\mathbf{u}_\tau\right)d\Omega_\tau\bigg|_{\tau=0} = \int_{\Omega_\epsilon}\alpha\left(h_\epsilon\left(\mathbf{u}_\epsilon\right)div\,\mathbf{v}+\frac{\partial h_\tau\left(\mathbf{u}_\tau\right)}{\partial\tau}\bigg|_{\tau=0}\right)d\Omega_\epsilon \tag{24}$$

where div is a derivation operation in relation to the reference coordinate system Ω_ϵ.

A wide group of isotropic material failure functions can be written in terms of the first invariant I_1 of the Cauchy stress tensor and the second and third stress invariants J_2 and J_3, of the deviatoric stress tensor. By using the chain rule and after some algebra it is possible to write that

$$\frac{\partial}{\partial\tau}\int_{\Omega_\tau}\alpha h_\tau\left(\mathbf{u}_\tau\right)d\Omega_\tau\bigg|_{\tau=0} = \int_{\Omega_\epsilon}\alpha\left[h_\epsilon\left(\mathbf{u}_\epsilon\right)\mathbf{I} - \left(\nabla\mathbf{u}_\epsilon\right)^T\mathbf{C}\mathbf{A}\left(\mathbf{u}_\epsilon\right)\right]\cdot\nabla\mathbf{v}\,d\Omega_\epsilon \tag{25}$$

where

$$\mathbf{A}\left(\mathbf{u}_\epsilon\right) = \left[\frac{\partial h_\epsilon\left(\mathbf{u}_\epsilon\right)}{\partial I_1}\mathbf{I} + \frac{\partial h_\epsilon\left(\mathbf{u}_\epsilon\right)}{\partial J_2}\mathbf{B}^T\mathbf{S}\left(\mathbf{u}_\epsilon\right) + \frac{\partial h_\epsilon\left(\mathbf{u}_\epsilon\right)}{\partial J_3}\mathbf{B}^T\mathbf{S}\left(\mathbf{u}_\epsilon\right)\mathbf{S}\left(\mathbf{u}_\epsilon\right)\right] \tag{26}$$

The deviatoric stress tensor $\mathbf{S}$ is obtained from the following projection operation,

$$\mathbf{S}\left(\mathbf{u}_\epsilon\right) = \mathbf{B}\sigma\left(\mathbf{u}_\epsilon\right) \tag{27}$$

where the fourth order operator $\mathbf{B}$ is given by

$$\mathbf{B} = \left(\mathbf{II} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right) \tag{28}$$

Following analogous arguments, the derivative of the last three terms of the Eq. (22) are respectively given by

$$\frac{\partial}{\partial\tau}\int_{\Omega_\tau} h_\tau^2(\mathbf{u}_\tau)\,d\Omega_\tau\bigg|_{\tau=0} = \int_{\Omega_\epsilon}\left(h_\epsilon^2(\mathbf{u}_\epsilon)\mathbf{I} - 2h_\epsilon(\mathbf{u}_\epsilon)(\nabla\mathbf{u}_\epsilon)^T\mathbf{CA}(\mathbf{u}_\epsilon)\right)\cdot\nabla\mathbf{v}\,d\Omega_\tau \tag{29}$$

$$\frac{\partial}{\partial\tau}a_\tau(\mathbf{u}_\tau,\lambda_\tau)\bigg|_{\tau=0} = \int_{\Omega_\epsilon}\left\{[\sigma(\mathbf{u}_\epsilon)\cdot\varepsilon(\lambda_\epsilon)]\mathbf{I} - (\nabla\lambda_\epsilon)^T\sigma(\mathbf{u}_\epsilon) - (\nabla\mathbf{u}_\epsilon)^T\sigma(\lambda_\epsilon)\right\}\cdot\nabla\mathbf{v}d\Omega_\epsilon \tag{30}$$

$$\frac{\partial}{\partial\tau}l_\tau(\mathbf{u}_\tau)\bigg|_{\tau=0} = \int_{\Omega_\epsilon}(\mathbf{b}\cdot\mathbf{u}_\epsilon)\mathbf{I}\cdot\nabla\mathbf{v}d\Omega_\epsilon. \tag{31}$$

Substituting (23), (25), and (29) to (31) in (22) and rearranging terms, the total derivative of $\pounds_\tau$ can be rewritten as

$$\frac{d\pounds_\tau}{d\tau}\bigg|_{\tau=0} = \int_{\Omega_\epsilon}\mathbf{\Sigma}_\epsilon\cdot\nabla\mathbf{v}\,d\Omega_\epsilon, \tag{32}$$

where

$$\begin{aligned}\mathbf{\Sigma}_\epsilon = {}&\rho\mathbf{I} + \alpha\left[h_\epsilon(\mathbf{u}_\epsilon)\mathbf{I} - (\nabla\mathbf{u}_\epsilon)^T\mathbf{CA}(\mathbf{u}_\epsilon)\right] + \frac{1}{2r}\left[h_\epsilon^2(\mathbf{u}_\epsilon)\mathbf{I} - 2h_\epsilon(\mathbf{u}_\epsilon)(\nabla\mathbf{u}_\epsilon)^T\mathbf{CA}(\mathbf{u}_\epsilon)\right]\\ &+ [\sigma(\mathbf{u}_\epsilon)\cdot\varepsilon(\lambda_\epsilon)]\mathbf{I} - (\nabla\lambda_\epsilon)^T\sigma(\mathbf{u}_\epsilon) - (\nabla\mathbf{u}_\epsilon)^T\sigma(\lambda_\epsilon) + (\mathbf{b}\cdot\mathbf{u}_\epsilon)\mathbf{I}\end{aligned} \tag{33}$$

is denoted Eshelby energy-momentum tensor. This tensor represents the forces associated to the shape changes in the domain, provided by $\nabla\mathbf{v}$. Applying the Divergence Theorem, Eq. (32) becomes

$$\frac{\partial\pounds_\tau}{\partial\tau}\bigg|_{\tau=0} = \int_{\partial F_\epsilon}\mathbf{\Sigma}_\epsilon\mathbf{n}\cdot\mathbf{v}d\partial F_\epsilon + \int_{\partial\Omega_\epsilon}\mathbf{\Sigma}_\epsilon\mathbf{n}\cdot\mathbf{v}d\Omega_\epsilon - \int_{\Omega_\epsilon}div\mathbf{\Sigma}_\epsilon\cdot\mathbf{v}d\Omega_\epsilon \tag{34}$$

It is possible to show, after some algebra, that div $\mathbf{\Sigma}_\epsilon = 0$. Thus, being the velocity field null on the external boundary $\partial\Omega_\epsilon$ (see Eq. (4)) and considering the velocity definition (4), one has an expression for the shape derivative based on a boundary integration performed exclusively over the hole boundary:

$$\frac{\partial\pounds_\tau}{\partial\tau}\bigg|_{\tau=0} = -\int_{\partial F_\epsilon}\mathbf{\Sigma}_\epsilon\mathbf{n}\cdot\mathbf{n}\,d\partial F_\epsilon \tag{35}$$

It is now convenient to recall that the tensors $\varepsilon(\lambda_\epsilon)$ and $\sigma(\lambda_\epsilon)$ present in expression (33) are obtained from the solution of the adjoint equation (21). Using definitions (26) (27) and (28), the second and third terms of the right hand side in (21) can be rewritten as

$$\left\langle\frac{\partial}{\partial\mathbf{u}_\tau}\int_{\Omega_\tau}\alpha h_\tau(\mathbf{u}_\tau)\,d\Omega_\tau\bigg|_{\tau=0},\xi\right\rangle = \int_{\Omega_\epsilon}\alpha\mathbf{CA}(\mathbf{u}_\epsilon)\cdot\varepsilon(\xi)\;d\Omega_\epsilon \tag{36}$$

$$\left\langle\frac{\partial}{\partial\mathbf{u}_\tau}\int_{\Omega_\tau}h_\tau^2(\mathbf{u}_\tau)\,d\Omega_\tau,\xi\right\rangle\bigg|_{\tau=0} = 2\int_{\Omega_\epsilon}h_\epsilon(\mathbf{u}_\epsilon)\mathbf{CA}(\mathbf{u}_\epsilon)\cdot\varepsilon(\xi)\;d\Omega_\epsilon \tag{37}$$

Substituting these results in (21) and evaluating for $\tau = 0$, one has

$$\int_{\Omega_\epsilon}\left[\sigma(\lambda_\epsilon) - \left[\alpha + \frac{1}{r}h_\epsilon(\mathbf{u}_\epsilon)\right]\mathbf{CA}(\mathbf{u}_\epsilon)\right]\cdot\varepsilon(\xi)\,d\Omega_\epsilon = 0\quad\forall\,\xi\in\mathcal{V}_\epsilon, \tag{38}$$

which allows to conclude that

$$\sigma(\lambda_\epsilon) = -\left[\left(\alpha + \frac{1}{r}h_\epsilon(\mathbf{u}_\epsilon)\right)\mathbf{CA}(\mathbf{u}_\epsilon) + \sigma\left(\lambda_\epsilon^h\right)\right] \tag{39}$$

$$\varepsilon(\lambda_\epsilon) = -\left[\left(\alpha + \frac{1}{r}h_\epsilon(\mathbf{u}_\epsilon)\right)\mathbf{A}(\mathbf{u}_\epsilon) + \varepsilon\left(\lambda_\epsilon^h\right)\right] \tag{40}$$

In these last expressions, the tensor $\sigma\left(\lambda_\epsilon^h\right) = \mathbf{C}\varepsilon\left(\lambda_\epsilon^h\right)$ is a self-equilibrated stress field such that

$$\int_{\Omega_\epsilon}\sigma\left(\lambda_\epsilon^h\right)\cdot\varepsilon(\xi)\;d\Omega_\epsilon = 0\quad\forall\,\xi\in\mathcal{V}_\epsilon \tag{41}$$

Substituting (39) and (40) in (33), one finally obtain a new expression for the Elsheby tensor given by

$$\begin{aligned}[\mathbf{\Sigma}_\epsilon\mathbf{n}]\cdot\mathbf{n} = {}&\rho + \alpha h_\epsilon(\mathbf{u}_\epsilon) + \frac{1}{2r}h_\epsilon^2(\mathbf{u}_\epsilon) - \left[\alpha + \frac{1}{r}h_\epsilon(\mathbf{u}_\epsilon)\right]\;\sigma(\mathbf{u}_\epsilon)\cdot\mathbf{A}(\mathbf{u}_\epsilon)\\ &- \sigma(\mathbf{u}_\epsilon)\cdot\varepsilon\left(\lambda_\epsilon^h\right) + (\mathbf{b}\cdot\mathbf{u}_\epsilon)\end{aligned} \tag{42}$$

4.1 VON MISES YIELD CRITERION

A possible particularization of the failure function is the von Mises stress criterion:

$$g_\epsilon(\mathbf{u}_\epsilon) = \sigma_{eq}(\mathbf{u}_\epsilon) - \bar{\sigma}, \tag{43}$$

where $\bar{\sigma}$ represents the admissible stress field and

$$\sigma_{eq}(\mathbf{u}_\epsilon) = \sqrt{3J_2} = \sqrt{\frac{3}{2}\mathbf{S}(\mathbf{u}_\epsilon)\cdot\mathbf{S}(\mathbf{u}_\epsilon)} \tag{44}$$

is the von Mises stress. For this criterion, the Eq. (15) reduces to

$$h_\epsilon(\mathbf{u}_\epsilon) = \max\{(\sigma_{eq}(\mathbf{u}_\epsilon) - \bar{\sigma})\ ;\ -r\alpha\} \tag{45}$$

It is worth remembering that the tensor $\mathbf{A}$ depends on the derivatives of h_ϵ with respect to the stress invariants. Consider the case when

$$(\sigma_{eq}(\mathbf{u}_\epsilon) - \bar{\sigma}) \geq -r\alpha, \quad \Rightarrow \quad h_\epsilon(\mathbf{u}_\epsilon) = (\sigma_{eq}(\mathbf{u}_\epsilon) - \bar{\sigma}) \tag{46}$$

(otherwise, $h_\epsilon(\mathbf{u}_\epsilon)$ is constant). Then, Eq. (26) reduces to

$$\mathbf{A}(\mathbf{u}_\epsilon) = \frac{3}{2\sigma_{eq}(\mathbf{u}_\epsilon)}\mathbf{B}^T\mathbf{S}(\mathbf{u}_\epsilon) \tag{47}$$

Substituting these results into the expression of the Eshelby tensor (42), and taking into account (6) and (35), the Topological Derivative due to the inclusion of an infinitesimal hole is given by the following limit operation:

$$D_T(\mathbf{x}) = \lim_{\epsilon\to 0}\frac{1}{f'(\epsilon)}\int_{\partial F_\epsilon}[-\rho + \alpha\bar{\sigma} - \frac{1}{2r}\left[\bar{\sigma}^2 - \sigma_{eq}^2(\mathbf{u}_\epsilon)\right] + \sigma(\mathbf{u}_\epsilon)\cdot\varepsilon\left(\lambda_\epsilon^h\right) - (\mathbf{b}\cdot\mathbf{u}_\epsilon)]d\partial F_\epsilon. \tag{48}$$

In order to perform this last limit, asymptotic expressions of functions $\mathbf{u}_\epsilon$ and $\lambda_\epsilon^h(\mathbf{u}_\epsilon, \lambda_\epsilon)$ are needed. The asymptotic behavior of $\mathbf{u}_\epsilon$ in a neighborhood of the hole is already given in Novotny (2003). The limit is completed based in the assumption that the asymptotic behavior of the self-equilibrated adjoint displacement field λ_ϵ^h follows that of the solution $\mathbf{u}_\epsilon$.

5. FINAL REMARKS

The problem of mass minimization considering material failure (local) constraints is here studied within the formalism provided by the Topological Derivative concept. A cost function that simultaneously accounts for the objective function and constraints (equilibrium and local failure) was constructed by means of a Augmented Lagrangian approach. The topological derivative of this cost function was partially derived, obtaining a final expression that depends on the asymptotic behavior of two displacement fields, solution and adjoint functions respectively. The limit operation will provide a derivative function depending only on the values of stress and strain calculated on the domain without the hole. The expression of this limit as well as validation numerical test are the subject of developments in progress.

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