LARGE DEFLECTIONS OF A LINEAR VISCOELASTIC CANTILEVER BEAM SUBJECTED TO A CONCENTRATED END LOAD

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Abstract. *The large deflection of a cantilever beam made of linear viscoelastic material subjected to a constant concentrated load at the free end is investigated in this paper. The mathematical formulation and the method of solution are presented. The set of four first order non-linear ordinary integro-differential-partial governing equations is derived from geometrical compatibility, equilibrium of forces and moments and linear viscoelastic constitutive relations. The set of equations is then reduced to a non-dimensional form showing that the problem is governed by non-dimensional force and relaxation modulus parameters. The numerical solution is obtained using an iterative one parameter shooting-method with a fourth-order Runge-Kutta algorithm. A case study is presented and the results are validated with a finite element model performed in the software Abaqus.*

Keywords: *large deflections, beams, viscoelastic analysis.*

1. INTRODUCTION

Nowadays a large variety of structural components built with polymeric materials is frequently designed considering the material with linear or non-linear elastic behavior. Polymeric materials show intrinsic viscoelastic (time-dependent) behavior and this may cause a significantly different mechanical response. In the offshore industry, for instance, some specific applications of the viscoelastic large deflections beam theory may be cited. Caire *et al*. (2005) employed the viscoelastic theory to characterize the response of polyurethane bend stiffeners used to protect flexible risers and umbilical cables top connection. Not only bend stiffeners, but also flexible pipes and umbilical cables are manufactured with polymeric materials and usually designed considering linear elastic response. A better understanding of the mechanical response of these structures may be obtained if constitutive equations are correctly assessed.

The viscoelastic behavior must be expressed by a constitutive equation that includes time as a variable in addition to the stress and strain variables. Findley *et al*. (1976), Tschoegl (1989) and Wineman and Rajagopal (2000) presented a broad review of the macro-mechanical behavior of linear and non-linear viscoelastic materials. The subject of large deflection of beams with linear elastic and non-linear elastic materials subjected to a variety of external loads has received considerable attention and is already well known. Among the many works regarding this subject it may be mentioned: Iyengar and Rao (1955), Wang *et al.* (1961), Kemper (1968), Lewis and Monasa (1981-2), Lee (2002) and Dado and Al-Sadder (2005).

In this paper the large deflection of a linear viscoelastic cantilever beam subjected to a concentrated load at the free end is investigated. The mathematical formulation assumes the Euler-Bernoulli theory for pure bending where the standard assumptions apply for each time t. The constitutive law is represented by a hereditary integral equation where the response becomes dependent on the loading history and an exponential stress relaxation curve is used to characterize the material viscoelastic behavior. The resulting set of non-linear integro-differential-partial equations are numerically solved using an iterative fourth-order Runge-Kutta method. A finite element model is also performed using the software Abaqus v6.5 and results are then compared.

2. PROBLEM STATEMENT

A constant concentrated load P is instantaneously applied to the free end of a cantilever beam of length L and symmetrical cross-section, as shown in Fig. 1.

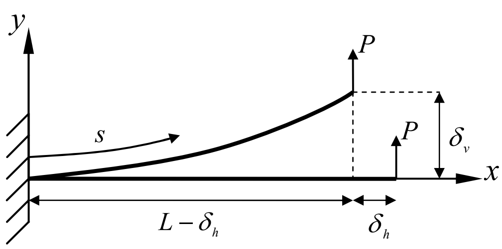

Figure 1. Cantilever beam subjected to a concentrate end load

The shape of the elastica is the curve defined by the (x, y) coordinate system whose origin is at the clamped end. A point on the deflected central axis is identified by the x and y coordinates, the arc length s, and the angle of rotation θ. The vertical and horizontal tip displacements are denoted by δ_v and δ_h respectively. For this model the following assumptions are considered and apply at each time t: (a) it is a quasi-static viscoelastic problem (beam dynamical behavior is not evaluated), (b) the self-weight is disregarded, (c) strains remain small, (d) the axial extensibility is neglected, (e) the effect of shear is negligible, which means that the Euler-Bernoulli beam theory may be applied, i.e., plane sections before bending remain plane after bending and (f) the tip point load is assumed to be vertical for every time t which means that it is a non-follower force. The beam is made of a linear viscoelastic material where the stress relaxation curve may be expressed by,

$$G(t) = G_\infty + (G_0 - G_\infty).e^{-t/\tau_R} \tag{1}$$

where $G_0 = G(0)$, $G_\infty = \lim G(t)$ as $t \to \infty$, and τ_R is a characteristic relaxation time. Tensile and compressive material properties are similar which implies that the neutral axis is at the centroid of the cross-section for all times t. Equation (1) expresses a three-parameter solid linear viscoelastic constitutive behavior.

3. THEORETICAL FORMULATION

The mathematical formulation derives from considering geometrical compatibility, equilibrium of forces and moment and linear viscoelastic constitutive relations. Hence a system of four first order non-linear ordinary integro-differential-partial equations is set to describe the problem.

3.1. Geometrical relations

Applying trigonometrical relations to an infinitesimal beam element the following equations are found,

$$\frac{\partial x(s,t)}{\partial s} = \cos(\theta(s,t)) \tag{2}$$

$$\frac{\partial y(s,t)}{\partial s} = \sin(\theta(s,t)) \tag{3}$$

In addition curvature may be defined by,

$$\frac{\partial \theta(s,t)}{\partial s} = k(s,t) \tag{4}$$

where s is the rod arc-length measured from the fixed end, $(x(s,t), y(s,t))$ are the Cartesian coordinates of the deflected rod that varies with time, $\theta(s,t)$ is the slope with respect to the x-axis of any point along the arc length and $k(s,t)$ is the curvature.

3.2. Equilibrium of forces and moment

Considering quasi-static equilibrium for the beam at each time t, the reaction forces may be determined from the free-body diagram for the entire beam. Solving the equilibrium equations, the bending moment relation at s for a given time t for the deflected beam under the concentrated end load is given by,

$$M(s,t) = P.\left(\int_0^L \cos(\theta(s,t)).ds - \int_0^s \cos(\theta(s,t)).ds \right) \tag{5}$$

Differentiating equation (5) with respect to position the shear force equation is obtained,

$$\frac{\partial M(s,t)}{\partial s} = -P.\cos(\theta(s,t)) \tag{6}$$

3.3. Constitutive relations

Supposing the linearity of viscoelastic response (i.e., the material behavior is independent on the stress or strain levels), homogeneous and isotropic material and considering pure bending where each material element is in a uniaxial stress state, the following constitutive equation may be employed according to Wineman and Rajagopal (2000),

$$\sigma(s,t) = G(0).\varepsilon(s,t) + \int_0^t \varepsilon(s,\tau).\frac{\partial G(t-\tau)}{\partial(t-\tau)} d\tau \tag{7}$$

where t is the current time, τ is a representative previous time, $\sigma(s,t)$ is the normal stress, $\varepsilon(s,t)$ is the axial strain and $G(t)$ is the relaxation function. Assuming that plane sections remain plane after bending implies that the axial strain at a given time t varies linearly with the distance from the neutral axis y and is described by,

$$\varepsilon(y,s,t) = y.k(s,t) \tag{8}$$

The equilibrium of bending moments for the cross-section area of the beam yields,

$$M(s,t) = \int \sigma(s,t) y.dA \tag{9}$$

Substituing equation (8) in (7) and the resultant equation in (9), the bending moment-curvature relation for the linear viscoelastic case is obtained,

$$M(s,t) = I.G(0).k(s,t) + I\int_0^t k(s,\tau).\frac{\partial G(t-\tau)}{\partial(t-\tau)} d\tau \tag{10}$$

where $I = \int_A y^2 dA$, is the second moment of area for the beam cross section. Algebraically manipulating equation (10) to explicitly obtain the curvature, differentiating with respect to position and introducing equation (6), yields,

$$\frac{\partial k(s,t)}{\partial s} = -\frac{1}{G(0)}\left[\frac{P}{I}.\cos(\theta(s,t)) + \int_0^t \frac{\partial k(s,\tau)}{\partial s}\frac{\partial G(t-\tau)}{\partial(t-\tau)} d\tau\right] \tag{11}$$

The geometrical relations (2), (3), (4) and (11) form the system of four non-linear integro-differential partial equations that represents the linear viscoelastic beam bending problem. It is worthy emphasizing that when the problem is evaluated for the initial time $t = 0$ the integral in Eq. (11) vanishes and the solution is the same as for linear elastic materials with the elasticity modulus given by $G(0)$, see Barten (1944-5) and Bisshopp and Drucker (1945).

3.3. Boundary conditions

A set of four boundary conditions must be specified for the problem as follows,

$$x(0,t) = y(0,t) = \theta(0,t) = k(L,t) = 0 \tag{12}$$

3.3. The governing equations

In order to obtain the most general results for the problem, the set of equations is reduced to a non-dimensional form using the following change of variables: $\bar{s} = s/L$ $(0 \le \bar{s} \le 1)$, $\bar{t} = t/\tau_R$, $\bar{\tau} = \tau/\tau_R$, $\bar{y} = y/L$, $\bar{x} = x/L$, $\bar{k} = kL$, $\bar{P} = PL^2/IG_0$ and $\bar{\alpha} = G_\infty/G_0$, yielding,

$$\begin{cases} \dfrac{\partial \bar{x}(\bar{s},\bar{t})}{\partial \bar{s}} = \cos(\bar{\theta}(\bar{s},\bar{t})) \\ \dfrac{\partial \bar{y}(\bar{s},\bar{t})}{\partial \bar{s}} = \sin(\bar{\theta}(\bar{s},\bar{t})) \\ \dfrac{\partial \bar{\theta}(\bar{s},\bar{t})}{\partial \bar{s}} = \bar{k}(\bar{s},\bar{t}) \\ \dfrac{\partial \bar{k}(\bar{s},\bar{t})}{\partial \bar{s}} = -\bar{P}.\cos\left(\theta(\bar{s},\bar{t})\right) + (1-\bar{\alpha}).\displaystyle\int_0^{\bar{t}} \dfrac{\partial \bar{k}(\bar{s},\bar{\tau})}{\partial \bar{s}}.e^{-(\bar{t}-\bar{\tau})} d\bar{\tau} \end{cases} \tag{13}$$

where $(\bar{x},\bar{y})$ constitute non-dimensional Cartesian coordinates, $\bar{s}$ the non-dimensional arc-length, $\bar{k}$ the non-dimensional curvature, $\bar{\theta}$ the angle formed by the curve tangent and the longitudinal axis, $\bar{P}$ the non-dimensional vertical load and $\bar{\alpha}$ the non-dimensional relaxation modulus.

4. NUMERICAL SOLUTION

The resultant system of four non-linear integro-differential-partial equations is solved for each time t using a one-parameter shooting method which converts the boundary value problem into an equivalent initial value problem. The mathematical package Mathematica v5.1 has been employed in the algorithm implementation. Fixed end curvatures are guessed for the current time $\bar{k}(0,\bar{t})$ and then the system of equations (13) is integrated using a fourth-order Runge-Kutta method with a trial-and-error approach until the value of $\bar{k}(0,\bar{t})$ that corresponds to the boundary condition $\bar{k}(1,\bar{t})=0$ is obtained in accordance with a stop criterion. The Simpson's one-third rule is employed to the integral in (13). This method is used to incrementally solve the system from the initial time ($\bar{t}=0$) up to a final specified time such as $\bar{t}=20$. Each solution obtained from a previous time is required to solve the problem in the current time as can be observed in the hereditary integral of equation (13) where the curvature at $\bar{t}$ is a function of the curvature at $\bar{\tau}$.

5. FINITE ELEMENT MODEL

Using the same assumptions as per in the analytical procedure, a finite element model using Abaqus v6.5 is developed in order to compare and validate the results. The model similarity is achieved using the following commands:

- two dimensional Euler-Bernoulli beam element B23 is used to generate the beam mesh;
- the Nlgeom parameter is included in the *Step command to indicate that geometric nonlinearity should be accounted for during the step as the problem undergoes large displacements;
- the option *Visco is used to obtain a transient static response in an analysis with viscoelastic material behavior.

The analysis is divided in two steps. In the first step a static concentrated load $\bar{P}$ is applied to the free end. In the second step the load is kept constant and the option *Visco is employed to obtain the viscoelastic response. As the software Abaqus needs dimensional parameters to perform the analysis, the beam geometry, material property and load condition must be given in accordance with the non-dimensional load $\bar{P}$ and the non-dimensional relaxation modulus $\bar{\alpha}$.

6. RESULTS AND DISCUSSION

A case study is presented in non-dimensional form for the non-dimensional vertical load $\bar{P}$ and hypothetical relaxation modulus $\bar{\alpha}$ parameter values given in Table 1.

Table 1. Case study parameters

Case	$\bar{P}$	$\bar{\alpha}$
1	¼	¼
2	1	¼
3	¼	½
4	1	½

The analytical and finite element results demonstrated total agreement which validate the mathematical formulation and the method of solution. The results of non-dimensional curvature, non-dimensional vertical and horizontal tip displacement versus non-dimensional time are presented in Fig. 2, 3 and 4, respectively. A limit (asymptotic) value is also shown in the graphs representing the elastic solution when the elasticity modulus is given by G_∞ (in non-dimensional plots the solution converges to an equivalent load of magnitude $\overline{P}/\alpha$ as $\bar{t} \to \infty$). The deflected configurations for each applied load ($\overline{P} = 1/4$ and $\overline{P} = 1$) are respectively shown for three different times ($\bar{t} = 2, 4, 20$) and $\overline{\alpha} = 1/4, 1/2$ in Figs. 5 and 6.

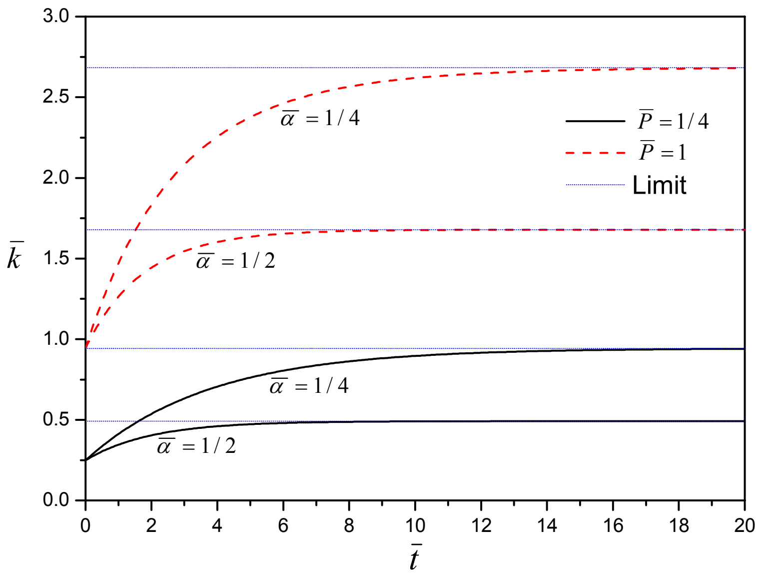

Figure 2. Non-dimensional curvature x non-dimensional time

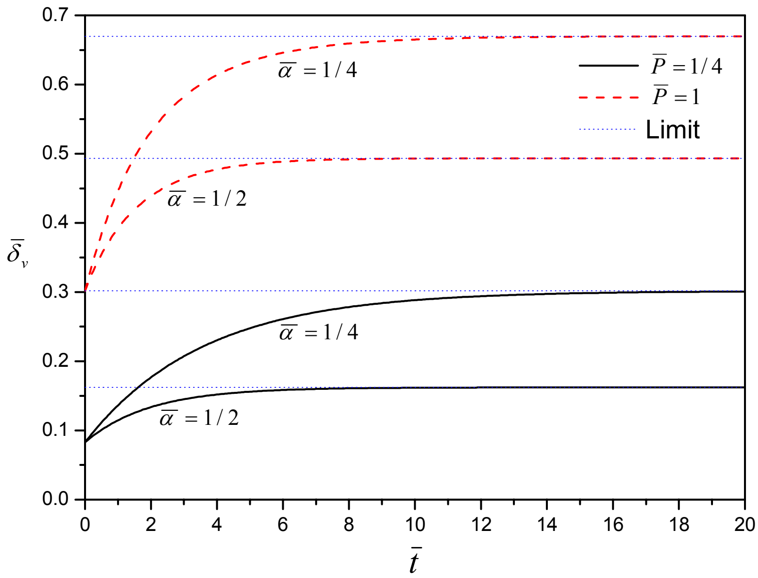

Figure 3. Non-dimensional vertical tip displacement x non-dimensional time

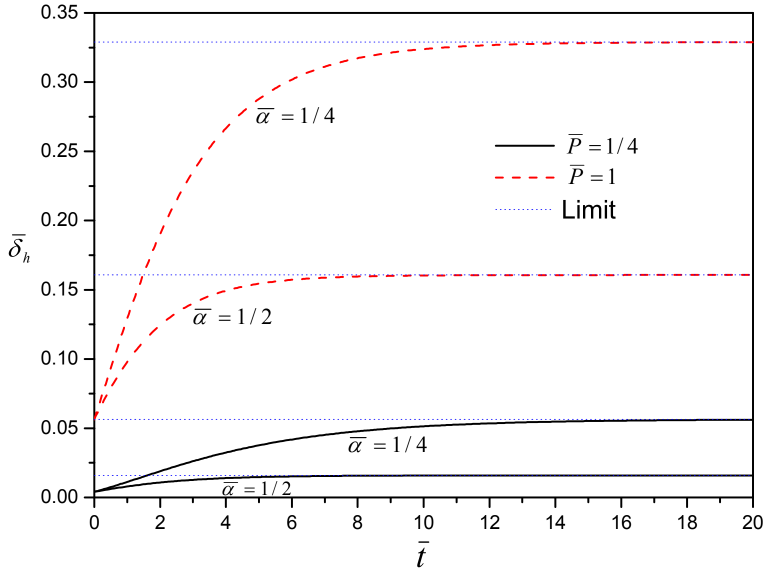

Figure 4. Non-dimensional horizontal tip displacement x non-dimensional time

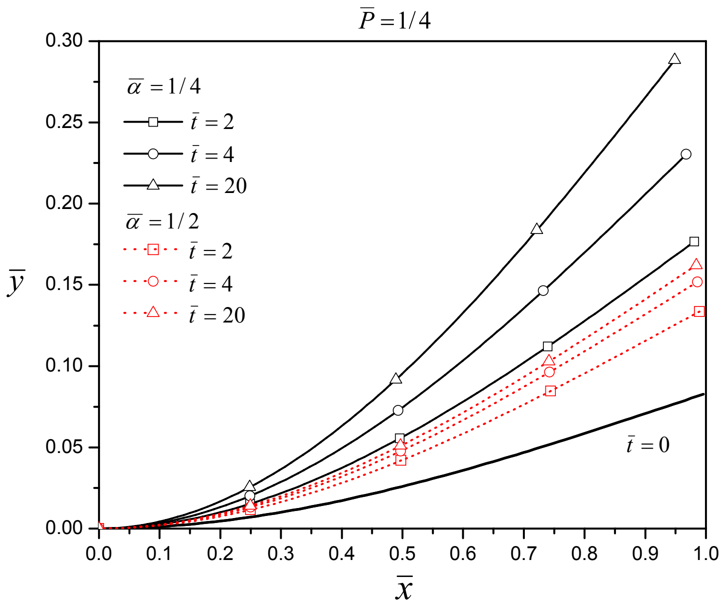

Figure 5. Deflection shapes for $\overline{P}=1/4$

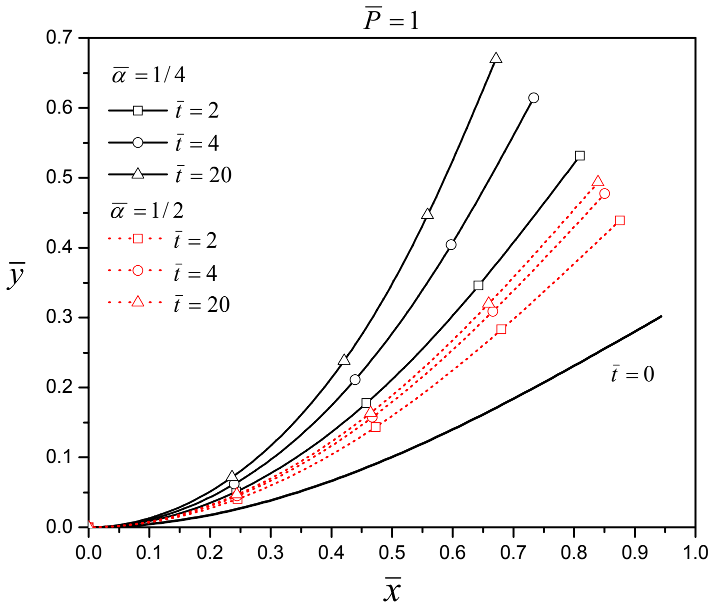

Figure 6. Deflection shapes for $\overline{P}=1$

It can be observed from the results, as expected, that the higher the non-dimensional relaxation modulus $\overline{\alpha}$, the smaller the curvature and tip displacements limit values. This is explained by the fact that as $\overline{\alpha}$ increases the final value of the relaxation function G_∞ decreases. It is also observed from Fig. 5 and 6 that the final deflected configurations are reached faster as the relaxation modulus increases.

7. CONCLUSIONS

This work presents the mathematical formulation and the numerical solution method for the large deflection analysis of a linear viscoelastic cantilever beam subjected to a concentrated load at the free end. The system of four non-linear integro-differential-partial equations obtained from geometrical compatibility, equilibrium of forces and moments and constitutive equations is solved using an iterative fourth-order Runge-Kutta method. A finite element model is also developed using the software Abaqus v6.5 with equivalent model assumptions. The same results are obtained from both models which validate the mathematical formulation and the numerical solution method.

The results presented point out the importance of considering the correct material characterization when polymeric structures are employed. The response of these structures, that show inherent viscoelastic behavior, may vary significantly from the elastic models usually adopted to represent the component material response.

8. ACKNOWLEDGEMENTS

The authors would like to thank the support of the National Council for Scientific and Technological Development (CNPq) and National Petroleum Agency (ANP) for this work.

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