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INITIAL AND COMPLETE POST-BUCKLING ANALYSES OF SLENDER ELASTIC RODS SUBJECTED TO UNIFORM THERMAL LOADS RESTING ON ELASTIC FOUNDATION

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Abstract. *This paper proposes analytical and numerical solutions respectively for the initial and complete post-buckling behavior of slender elastic rods, with non-movable hinged ends, resting on elastic foundation and subjected to thermal loads. The governing equations are derived from geometrical compatibility, equilibrium of forces and moments, constitutive and strain-displacement relations. A set of seven first order non-linear ordinary differential equations is obtained and boundary conditions are specified at the rod ends. The governing equations are then made non-dimensional thus reducing the problem to two parameters: the foundation stiffness and the rod slenderness ratio. A classical Perturbation Method is applied for solving the differential equations assuming that the structure is subjected to small strains and deflections. This gives a set o linear equations that can be solved sequentially. The results are presented and discussed for one slenderness ratio and a range of foundation stiffness corresponding to the third buckling mode. It is verified that for a specific range of foundation stiffness, the compressive load increases with the increase of the temperature gradient, when an opposite behavior was expected. The exact post-buckling result shows excellent agreement with the results presented here, given that in practical situations the deflections are usually moderate.*

Keywords: *Post-Buckling, Elastic Foundation, Perturbation Method*

1. INTRODUCTION

Beam stability is an important subject because in many engineering applications it is a typical failure mode mechanism. Thermal and mechanical rod buckling in elastic foundations has been studied by many authors such as Den Hartog (1952), Timoshenko and Gere (1961), Hetenyi (1966), Sundararajan (1974), Kalaba (1978), Gauss and Antman (1984), Eisenberger et al (1986), Hui (1988), Panayotounakos (1989), Xie and Vaziri (1992), Lee et al (1992, 1996), Hunt et al (1993), Raju and Rao (1993), Rao and Raju (1994, 2002), Lee and Waas (1996), Jekot (1996), Lin and Librescu (1998), Coffin and Bloom (1999), Li and Cheng (2000), Li et al (2002), Li and Zhou (2003), Vaz and Solano (2003, 2004), Li and Batra (2005), Li and Song (2006), Vaz et al (2007).

When a beam is heated up and its ends are constrained from moving axially an axial compressive stress develops and if this stress reaches a critical value, a serious stability problem may take place. The initial post-buckling and the complete post-buckling of slender elastic rods supported by a linear elastic foundation, subjected to a uniform temperature gradient are investigated here. The rod has double-hinged immovable ends and the classical perturbation method is employed for solving analytically and sequentially the governing equations for small strains. Furthermore the results are compared to the complete post-buckling, which is obtained numerically by the shooting method.

2. THE GOVERNING EQUATIONS

In figure 1 it is shown the initial and the buckled forms of the rod. X, Y are the Cartesian coordinates, ΔT is the uniform temperature gradient, K is the elastic foundation modulus and P is the compressive force that arises from the constrained ends. It is also shown the rod initial length L and buckled length L^* and their respective infinitesimal lengths dS and dS^*.

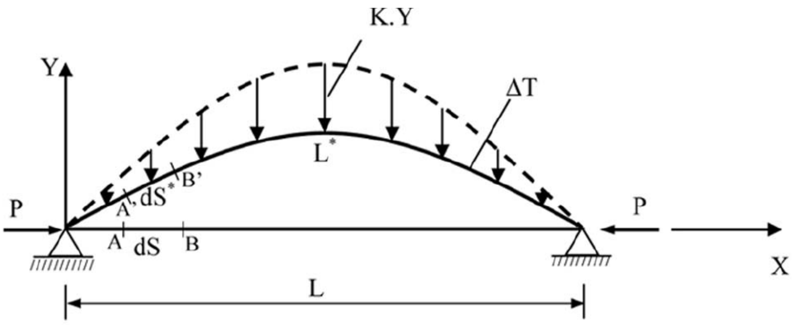


Figure 1. Rod initial and buckled forms on an elastic foundation.

Figure 2 shows the infinitesimal element $dS*$ of the buckled rod, where θ is the angle that the rod makes with the horizontal axis X , M is the bending moment and V is the force parallel to the vertical axis Y .

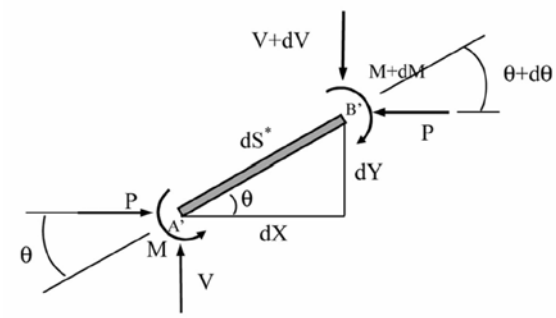


Figure 2. Infinitesimal element of the buckled rod.

Assuming that the rod is linear elastic, it is possible to obtain from figures 1 and 2 the seven governing equations. These equations were derived from the geometrical compatibility, equilibrium of forces and moments, constitutive and strain-displacement relations. With the purpose of making the comparison of results easier and facilitating their comprehension the governing equations were made non-dimensional by employing the following relations: $S^* = s^* L$, $X = x L$, $Y = y L$, $S = s L$, $\Omega = \kappa / L$, $K = k\,EI / L^4$, $P = p\,EI / L^2$ and $V = v\,EI / L^2$. Where Ω is the rod curvature, E is the Young's modulus, I is the cross-sectional second moment of inertia and ε is the specific linear strain, defined as the ratio between the elongations in strained and initial configurations. Therefore the governing equations will be:

$$\frac{dx}{ds^*} = \cos\theta \tag{1}$$

$$\frac{dy}{ds^*} = \sin\theta \tag{2}$$

$$\frac{d\theta}{ds^*} = \kappa \tag{3}$$

$$\frac{dv}{ds^*} = -k\,y \tag{4}$$

$$\frac{d\kappa}{ds^*} = -p\sin\theta + v\cos\theta \tag{5}$$

$$\frac{dp}{ds^*} = 0 \tag{6}$$

$$\frac{ds}{ds^*} = \frac{1}{(1+\varepsilon)} \tag{7}$$

The temperature gradient induces an elongation of the rod which is prevented by its hinged ends and a compressive force arises. Employing these relations: $\lambda = L\sqrt{A/I}$, $L^* = l^* L$ and $\Delta T = \Delta t / (\lambda^2 \alpha)$, where λ is the slenderness ratio. The strain that results from this thermo-mechanical equation is:

$$\varepsilon = \frac{\Delta t}{\lambda^2} - \frac{p}{\lambda^2}\cos\theta - \frac{v}{\lambda^2}\sin\theta \tag{8}$$

Where A is the cross-sectional area and α is the linear thermal expansion coefficient. The boundary conditions are:

$$x(0) = y(0) = \kappa(0) = s(0) = x(l^*) - 1 = y(l^*) = \kappa(l^*) = s(l^*) - 1 = 0 \tag{9}$$

When small displacements are admitted it is possible to solve this complex boundary value problem analytically, for the initial post-buckling, utilizing a classical perturbation method. The full methodology is presented in Vaz et al (2007) and it will be summarized here.

2.1. Initial Post-Buckling Solution Using the Perturbation Method

The variables of the problem are expanded in terms of a perturbation parameter ξ and due to the symmetry of the problem the odd and even functions can be separated. The expanded variables are:

$$x(s^*) = s^* + \xi^2 \; x_1(s^*) + \cdots \tag{10}$$
$$y(s^*) = \xi \; y_0(s^*) + \xi^3 \; y_1(s^*) + \cdots \tag{11}$$
$$\theta(s^*) = \xi \; \theta_0(s^*) + \xi^3 \; \theta_1(s^*) + \cdots \tag{12}$$
$$v(s^*) = \xi \; v_0(s^*) + \xi^3 \; v_1(s^*) + \cdots \tag{13}$$
$$\kappa(s^*) = \xi \; \kappa_0(s^*) + \xi^3 \; \kappa_1(s^*) + \cdots \tag{14}$$
$$p = a_0 + \xi^2 \; a_1 + \cdots \tag{15}$$
$$\Delta t = b_0 + \xi^2 \; b_1 + \cdots \tag{16}$$
$$\varepsilon = \varepsilon_0 + \xi^2 \; \varepsilon_1\left(s^*\right) + \cdots \tag{17}$$

The solution is obtained by substituting equations (10-17) in (1-7). Collecting terms of same order and after analytical manipulation of equations yield:

$$a_0 = b_0 = \left(\frac{n\pi}{l^*}\right)^2 + k\left(\frac{l^*}{n\pi}\right)^2 \qquad a_1 = \frac{1}{8}\left[\left(\frac{n\pi}{l^*}\right)^4 - 3k\right] \tag{18}$$

$$\xi^2 = 4\frac{l^*\text{-}1}{l^*}\left(\frac{l^*}{n\pi}\right)^2 \qquad b_1 = a_1 + \frac{1}{4}\left(\frac{n\pi}{l^*}\right)^2\left(\lambda^2 - a_0\right) \tag{19}$$

$$y_0(s^*) = \sin\left(\frac{n\pi s^*}{l^*}\right) \qquad y_1(s^*) = \frac{3\left(\frac{n\pi}{l^*}\right)^2\left[-3\left(\frac{n\pi}{l^*}\right)^4 + k\right]}{64\left[9\left(\frac{n\pi}{l^*}\right)^4 - k\right]}\sin\left(\frac{3n\pi s^*}{l^*}\right) \tag{20}$$

$$p = n^2\pi^2 + \frac{k}{n^2\pi^2} + \frac{\left(l^* - 1\right)}{2}\left(\frac{k}{n^2\pi^2} - 3n^2\pi^2\right) \tag{21}$$

$$\Delta t = n^2\pi^2 + \frac{k}{n^2\pi^2} + \left(l^* - 1\right)\left(\lambda^2 - \frac{k}{2n^2\pi^2} - \frac{5n^2\pi^2}{2}\right) \tag{22}$$

Where n, a positive integer number, is the rod buckling mode associated with a prescribed value of k. Equation (18) shows that the rod buckles at same load for mechanical or thermo-mechanical processes. The transition elastic foundation stiffness and its corresponding load are calculated from $a_0(n) = a_0(n+1)$ and $l^* = 1$:

$$k_t = \pi^4 n^2 (n+1)^2 \qquad p_t = \pi^2\left(2n^2 + 2n + 1\right) \qquad n = 1, 2, \ldots \tag{23}$$

Equation (21) indicates that the compressive load always decreases when the temperature gradient is increased except for $3\pi^4 \le k < 4\pi^4$, corresponding to mode n = 1. Equation (22) shows initially stable regions for

$\lambda^2 \leq \dfrac{k}{2n^2\pi^2} + \dfrac{5n^2\pi^2}{2}$, see figure 3. Note that for lower values of foundation stiffness a thick beam theory is required to properly describe the buckling behavior near the critical values of slenderness ratios.

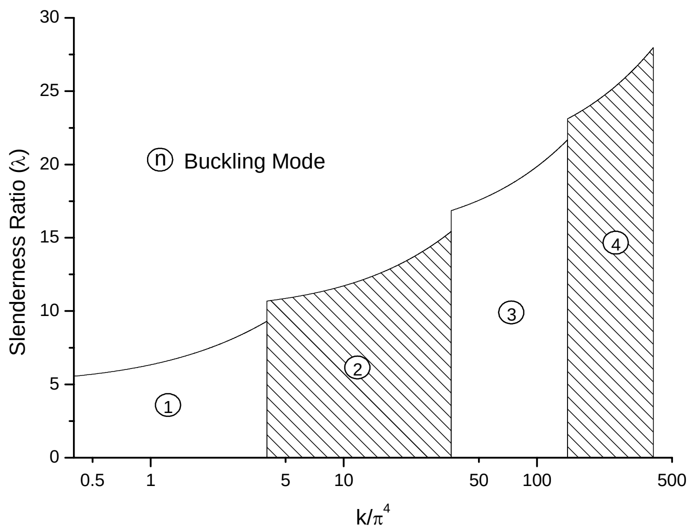


Figure 3. Slenderness Ratio as a function of the Foundation Stiffness.

2.2. Complete Post-Buckling Solution

A geometrically non-linear post-buckling solution was also developed employing the shooting method. A guess for the unspecified boundary conditions employed the previously calculated values for the initial post-buckling solution. For these values the shooting method converged in the first iteration for values of Δt < 800 in most cases. This fact indicated that the initial post-buckling solution via perturbation method was giving approximate values for a large range of temperature gradients. The results of these two methods are presented below.

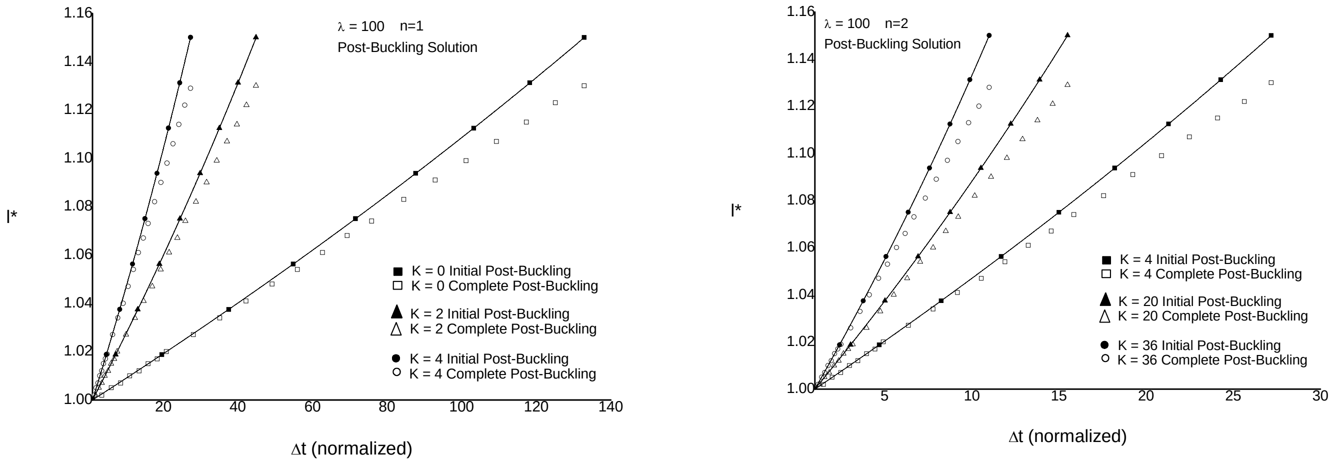
2.3. Analysis of Results

Two parameters control the buckling and the initial post-buckling problems: the elastic foundation stiffness k and the rod slenderness ratio λ. For a given combination of (k, λ) the solution is obtained for values of strained lengths l^*. A parametric study is carried out for $\lambda = 100$, $0 \leq k \leq 36\pi^4$. The results for the complete post-buckling solution are also plotted and show excellent agreement with the analytical solution developed here.

Figures 4 to 8 respectively present the strained length, normalized compressive load, maximum deflection, maximum angle and maximum curvature as a function of the normalized temperature gradient for modes 1 and 2. The concept of normalized loads and temperatures introduced in this work is explained next. The critical buckling load p_{cr} and temperature Δt_{cr} are obtained from Eq. (15) by setting $l^* = 1$. They increase linearly with the foundation stiffness and reach quite large values when a broad range of foundation stiffness is considered. It is then convenient to normalize the load and temperature variables, by simply dividing them by their respective critical values. Consequently the normalized temperatures and loads start from 1.0 (the bifurcation condition) for any foundation stiffness. Obviously the absolute temperatures and loads involved for higher values of foundation stiffness are increasingly higher.

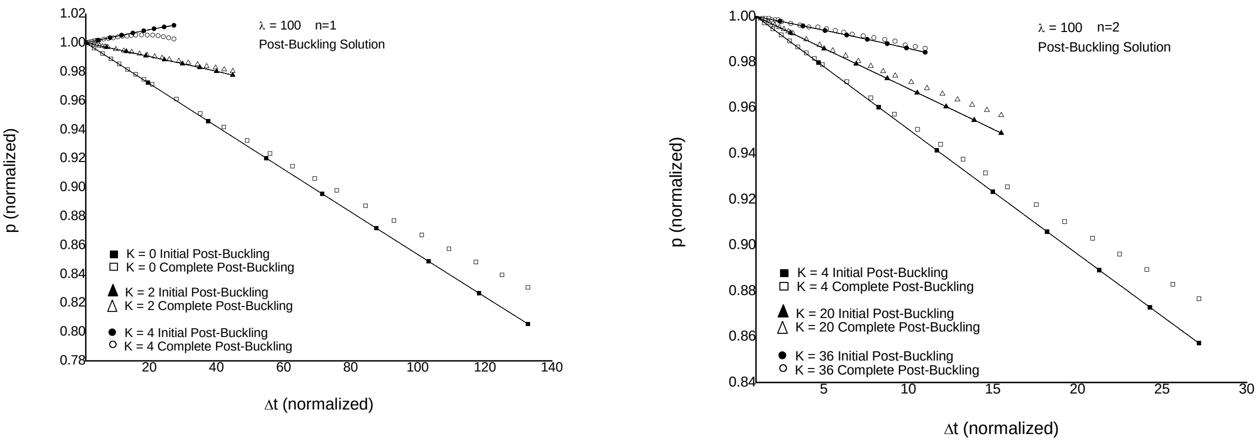
The strained length as a function of the normalized temperature gradient is plotted in Fig. 4 for the foundation stiffness $k_t = 0; 2\pi^4; 4\pi^4; 20\pi^4; 36\pi^4$. Large foundation stiffness requires relatively less temperature input for same deformation. The behavior of the normalized load as a function of the normalized temperature gradient for several values of foundation stiffness is presented in Fig. 5, respectively for n = 1 and 2. The normalized load always decreases with the normalized temperature except for n = 1 and $3\pi^4 \leq k < 4\pi^4$. Figure 6 shows the maximum lateral deflection as a function of the normalized temperature gradient. As expected, the lateral displacement is smaller for higher buckling modes. The rod maximum angle increases with the normalized temperature gradient, as depicted in Fig. 5. Figure 6 shows that the maximum curvature also increases with the normalized temperature gradient.

Figure 9 presents the deflected configurations for modes 1 and 2 comparing the results for initial and complete post-buckling subjected to the following thermal loads: $\Delta t = 225.00; 767.65; 1328.9; 282.48; 820.23$ and 1376.2. Observe that the loads and temperatures are not normalized. The graphs were plotted for intermediate values of foundation stiffness ($k = 2\pi^4; 20\pi^4$). The rod lateral deflections are larger the lower is the mode. For a given strained length, say $l^* = 1.020$, progressively higher temperatures are required for higher foundation stiffness (or mode), however note that a relatively smaller temperature increment is required. It can be seen that as the temperature increases, the complete post-buckling gives smaller lengths than the initial post-buckling solution.



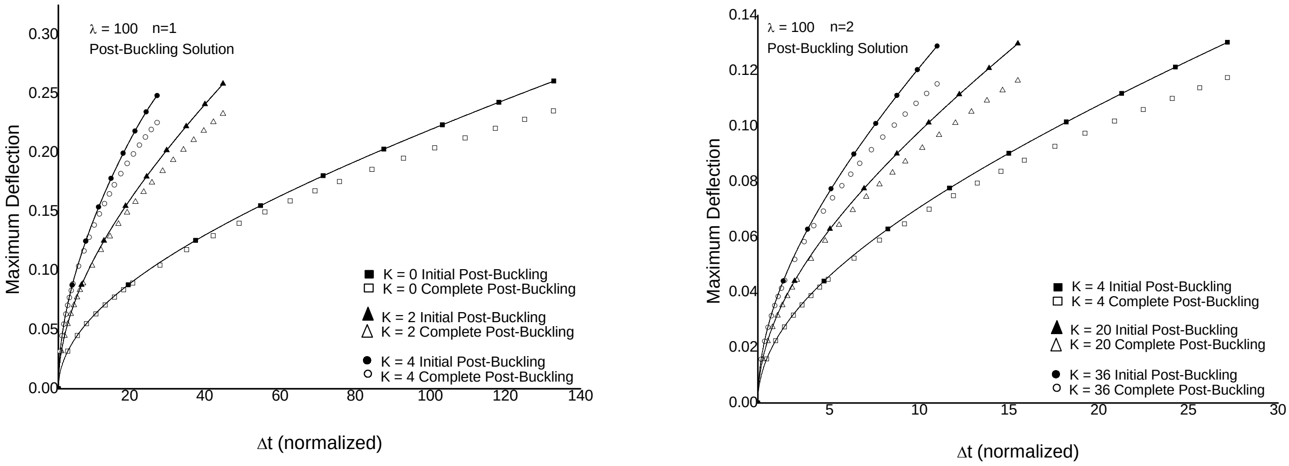
(a) (b)

Figure 4. Strained Length as a Function of the Normalized Temperature Gradient.



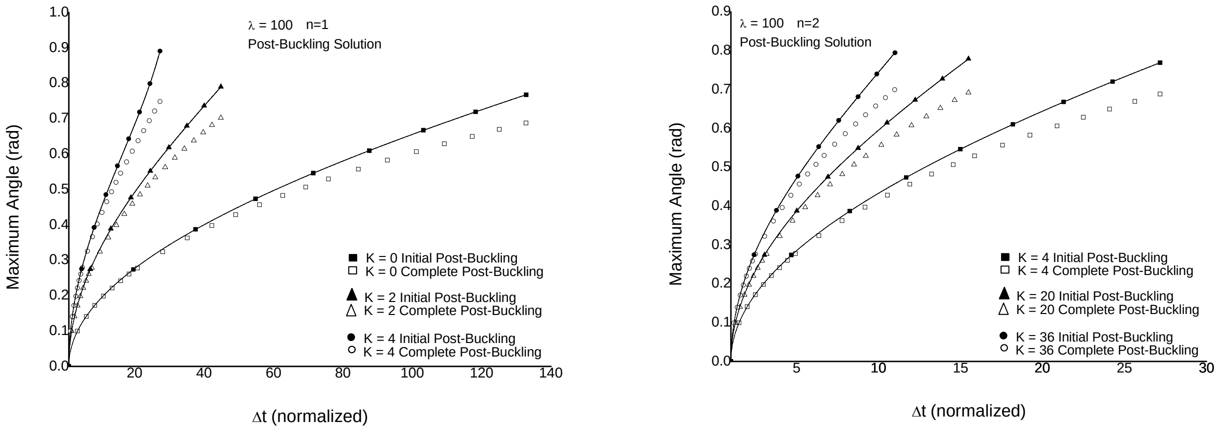
(a) (b)

Figure 5. Normalized Load as a Function of the Normalized Temperature Gradient.



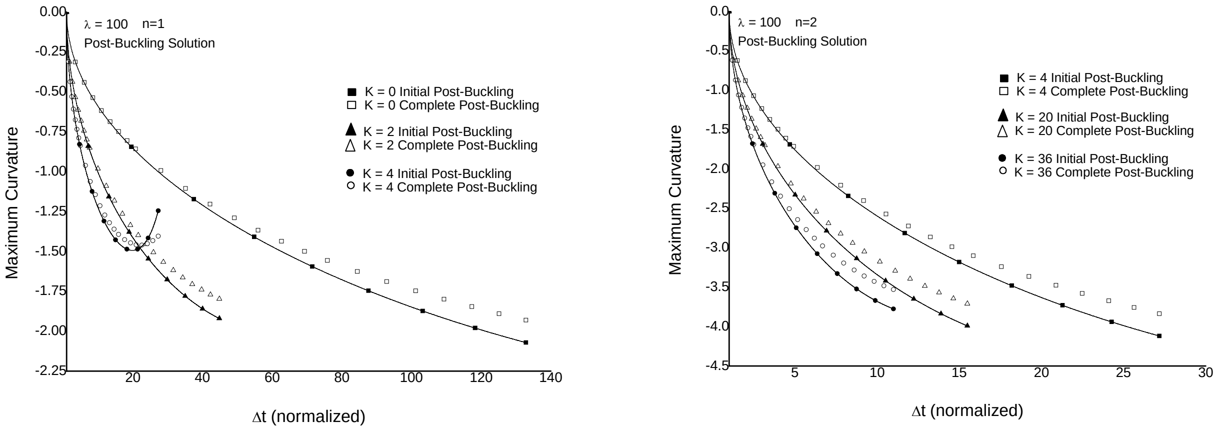
(a) (b)

Figure 6. Maximum Deflection as a Function of the Normalized Temperature Gradient.



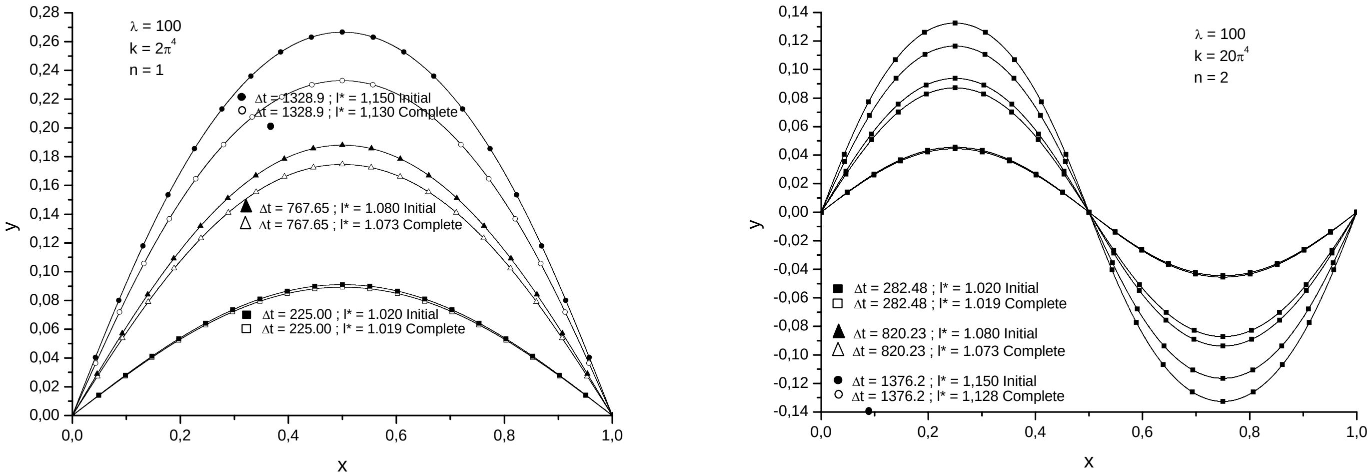
(a) (b)

Figure 7. Maximum Angle as a Function of the Normalized Temperature Gradient.



(a) (b)

Figure 8. Maximum Curvature as a Function of the Normalized Temperature Gradient.



(a) (b)

Figure 9. Deformed Configurations for $\lambda = 100$ and $k = 2\pi^4$ and $\lambda = 100$ and $k = 20\pi^4$.

3. CONCLUSIONS

This paper presents formulation and solution for the initial post-buckling and complete post-buckling analyses of slender elastic rods supported on linear elastic foundations and subjected to uniform temperature gradients. The material thermal strain-temperature relationship is considered linear and the rod ends are assumed hinged and immovable. The governing equations are made non-dimensional and the problem is shown to be controlled by two parameters: the elastic foundation stiffness and the slenderness ratio. A perturbation method is employed to expand the non-linear equations into a set of sequentially solvable analytical equations which describe the initial post-buckling regime. The critical buckling load and temperature gradient as well as its respective buckling modes are calculated. It is shown that there are limiting values of slenderness ratios where the rod is intrinsically initially stable. The critical slenderness ratio increases with the foundation stiffness. The results for the initial post-buckling analysis are obtained for a range of foundation stiffness corresponding to the fourth mode for a typical value of slenderness ratio. The results for the complete post-buckling solution are in excellent agreement with the solution developed here. The results are presented for the deformed length, compressive load, maximum deflection, angle and curvature as a function of the normalized temperature gradient, defined as the temperature gradient divided by its respective buckling temperature. The results indicate that the thermo-induced compressive load always decreases with the temperature gradient, except for a specific range of non-dimensional foundation stiffness in the first mode.

4. ACKNOWLEDGEMENTS

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