POST-BUCKLING ANALYSIS OF INCLINED ELASTIC RODS SUBJECTED TO TERMINAL AXIAL LOADS AND SELF-WEIGHT

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Abstract. *The behavior of inclined slender elastic rods subjected to axial forces and distributed load is discussed in this paper. Mathematical models and numerical solutions are developed for small and large displacements. A double-hinged boundary condition is assumed and the analysis is carried out for different values of non-dimensional weight and angle of inclination. The mathematical formulation results from considering geometrical compatibility, equilibrium of forces and moments and constitutive relations. For large displacements a set of six first order non-linear ordinary differential equations with boundary conditions prescribed at both ends is obtained. This two-point boundary value problem is numerically integrated using a three parameter shooting method. When small displacements are assumed the problem simplifies and a power series solution may be conveniently employed. The results for both simulations are presented, compared and discussed.*

Keywords*: Post-Buckling, Elastic Rods, Inclined Rods.*

1. INTRODUCTION

The subject of buckling, post-buckling and large deflection analyses of slender elastic rods has developed since the early classical contributions from Bernoulli, Euler and Lagrange in the 18th century, see Love (1944). This class of problem may exhibit interesting phenomena such as limit load, bifurcation, jump and hysteresis, given the non-linear nature of geometrical, physical or load assumptions.

Buckling and post-buckling of weightless rods have been addressed, for instance, by Gurfinkel (1965), Wang (1997), Tan and Witz (1995), Lee and Oh (2000) and Vaz and Silva (2002). The use of long submersed columns such as marine risers and drill-strings in the offshore oil&gas industry motivated the study of buckling and initial post-buckling of vertical rods subjected to variable axial forces resulting from self-weight. In Lubinski (1950), Huang and Dareing (1966, 1968 and 1969), Plunkett (1967), Wang (1983), Bernitsas and Kokkinis (1983a-b and 1984a-b), Kokkinis and Bernitsas (1985 and 1987), Vaz and Patel (1995), Patel and Vaz (1996), Jurjo et al (2001), Vaz and Mascaro (2005) the rod buckling, initial post-buckling and post-buckling solutions are developed.

Sampaio Jr and Hundhausen (1998) employed generalized hypergeometric functions to solve the small displacement problem of inclined beam-columns. In this situation the gravitational field not only imposes a variable axial force (problem is no longer symmetrical) but it also imparts to the rod a lateral distributed load. In this paper a large displacement formulation is developed, a numerical solution is obtained and results are compared. This paper has a potential practical application where a heavy slender structure is supported in two points, such as in the areas of drill string mechanics, global riser static analysis and flexible pipe jumper configuration, for instance.

2. THE MATHEMATICAL MODEL

Consider an inextensible slender uniform rod with length L and self-weight (per unit length) ρ supported by two points at an inclination β with respect to the horizontal axis, as shown in Figure 1a. The self-weight simultaneously modifies the distribution of longitudinal and lateral loads, respectively P and H . In addition pure bending formulation is assumed and the material is linear elastic so the bending stiffness is given by EI , where E is the Young's Modulus and I is the second moment of cross-sectional area. The problem can be load or displacement controlled so let Δ be the axial displacement as shown in Fig. 1a.

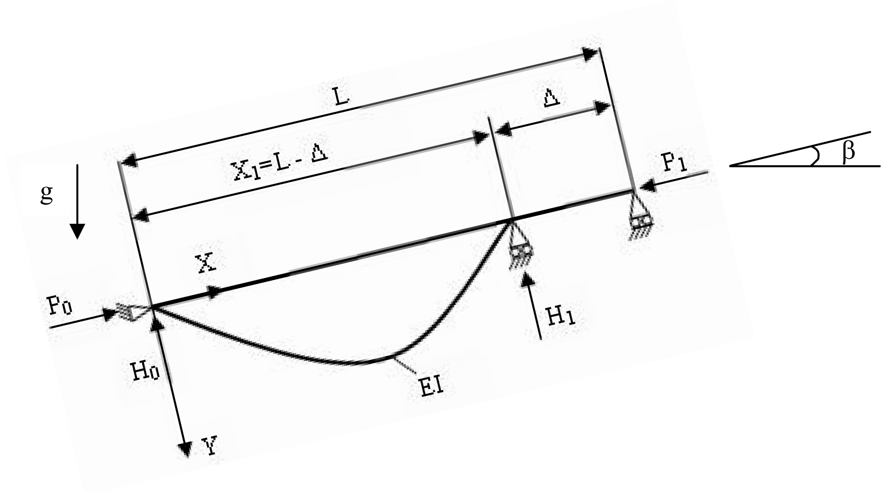

Figure 1a - Schematic of an Inclined Deflected Vertical Rod Subjected to Self-Weight.

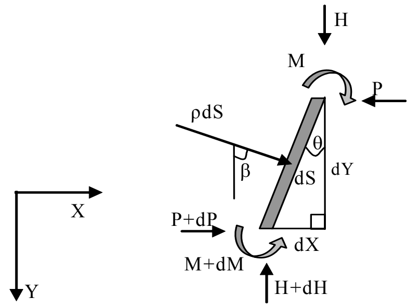

Figure 1b – Infinitesimal Element of Rod.

The governing equations result from geometrical compatibility, equilibrium of forces and moments and constitutive relations. A system of six first order non-linear ordinary differential equations describes the large displacement behavior of initially inclined rods subjected to self-weight.

2.1 Geometrical Equations

Geometrical restrictions are obtained from trigonometrical relations applied to an infinitesimal rod element dS (see Fig. 1b):

$$\frac{dX}{dS} = cos\,\theta \tag{1a}$$

$$\frac{dY}{dS} = sin\,\theta \tag{1b}$$

where S is the rod arc-length ($0 \leq S \leq L$), (X,Y) are the Cartesian coordinates of the deflected rod and θ is the angle between the tangent and the X-axis. Furthermore the curvature K may be defined by:

$$K = \frac{d\theta}{dS} \tag{1c}$$

2.2 Equilibrium of Forces and Moments

A schematic of the internal forces and moments in the rod infinitesimal element is shown in Fig. 1b. The equilibrium of longitudinal and lateral forces and bending moments, respectively yield:

$$\frac{dP}{dS} = -\rho \sin\beta \tag{2a}$$

$$\frac{dH}{dS} = \rho \cos\beta \tag{2b}$$

$$\frac{dM}{dS} = -P \sin\theta + H \cos\theta \tag{2c}$$

where M is the bending moment.

2.3 Constitutive Relations

Assuming linear elastic, homogeneous and isotropic materials, and considering the state of pure bending results in:

$$M = EI\,K \tag{3}$$

Therefore, substituting Eq. (3) into (2c) results:

$$\frac{dK}{dS} = \frac{1}{EI}\left(H \cos\theta - P \sin\theta\right) \tag{4}$$

2.4 Boundary Conditions

A set of six boundary conditions must be defined and for the double-hinged rod they may be specified as:

$$X(0) = Y(0) = K(0) = X(L) - X_1 = Y(L) = K(L) = 0 \tag{5}$$

where X_1 is the top end X-coordinate ($X_1 = L - \Delta$). The influence of the boundary conditions on the rod response is significant and it can be easily approached with same methodology presented here.

2.5 The Governing Equations

It is obviously convenient to reduce the set of differential Eqs. (1a-c), (2a-b) and (4) to a non-dimensional form using the following change of variables: $S = sL$, $Y = yL$, $X = xL$, $K = \kappa/L$, $\rho = \overline{\rho}\, EI/L^3$, $P = p\, EI/L^2$ and $H = h\, EI/L^2$, where $0 \le s \le 1$. Hence:

$$\frac{dx}{ds} = \cos\theta \tag{6a}$$

$$\frac{dy}{ds} = \sin\theta \tag{6b}$$

$$\frac{d\theta}{ds} = \kappa \tag{6c}$$

$$\frac{dp}{ds} = -\bar{\rho}\ sin\,\beta \tag{6d}$$

$$\frac{dh}{ds} = \bar{\rho}\ cos\,\beta \tag{6e}$$

$$\frac{d\kappa}{ds} = -p\ sin\,\theta + h\ cos\,\theta \tag{6f}$$

where (x, y) constitute the deflected rod non-dimensional Cartesian coordinates, s the non-dimensional arc-length, κ the non-dimensional curvature, θ the angle formed by the curve tangent and the longitudinal axis, p and h respectively the non-dimensional longitudinal and lateral loads and $\bar{\rho}$ the non-dimensional weight. Furthermore the boundary conditions given by Eq. (5) may be also made non-dimensional:

$$x(0) = y(0) = \kappa(0) = x(1) - x_1 = y(1) = \kappa(1) = 0 \tag{7}$$

where $x_1 = 1 - \delta$ ($\delta = \Delta/L$). Eq. (7) represents non-movable and movable hinged conditions respectively at the lower and upper ends.

3. THE SOLUTION FOR LARGE DISPLACEMENTS

As the set of six first order non-linear ordinary differential equations and its boundary conditions characterize a two-point boundary value problem, a technique may be employed to transform it into an initial value problem and allow a direct integration scheme. Three boundary conditions are given at one end $x(0) = y(0) = \kappa(0) = 0$ whereas three other conditions are given at the other end, i.e. $x(1) - x_1 = y(1) = \kappa(1) = 0$, so $h(0)$, $\theta(0)$ and $p(0)$ must be found. A shooting method, available in Mathcad, is employed to compute the initial missing values. This procedure may be summarized with the following main steps: (a) the set of differential equations is defined (i.e., equations (6a-f)); (b) the initial missing values are guessed (i.e., values for $h(0)$, $\theta(0)$ and $p(0)$ are guessed); (c) the boundary value endpoints are specified (i.e., $x(1) - x_1 = y(1) = \kappa(1) = 0$); (d) a load function which returns the initial condition is established; (e) a score function to measure the distance between terminal and desired conditions is employed; (f) the equivalent initial conditions are calculated (i.e., the "exact" values for $h(0)$, $\theta(0)$ and $p(0)$ are calculated). From this point, a Runge-Kutta high order solution algorithm is applied to solve the set of non-linear ordinary differential equations.

4. THE SOLUTION FOR SMALL DISPLACEMENTS

When small displacements are assumed $dx \cong ds$, $sin(\theta) \cong \theta$ and $cos(\theta) \cong 1$, so a simplified equation is obtained:

$$\frac{d^3 y}{dx^3} + \left[p_0 - \bar{\rho}\, sin(\beta)\, x\right]\frac{dy}{dx} = \bar{\rho}\, cos(\beta)\, x + h_0 \tag{8}$$

where p_0 and h_0 are respectively the longitudinal and lateral loads at $x = 0$ and $\beta, \bar{\rho}$ and p_0 are known. A solution for Eq. (8) may be obtained via Maclaurin series, yielding:

$$y(x) = S(x) + C_1 T(x) + C_2 U(x) + h_0 V(x) + C_0 \tag{9}$$

where C_0, C_1, C_2, h_0 are constants and $S(x), T(x), U(x), V(x)$ are series functions given by:

$$
\begin{aligned}
S(x) &= \frac{-\rho \cos\beta\, x^4}{362880}(-15120 + 10\, p_0\, \rho \sin(\beta)\, x^5 + 504\, p_0\, x^2 + \ldots) \\
T(x) &= \frac{x}{362880}(81\, p_0^2\, \rho \sin(\beta)\, x^7 + 15120\, \rho \sin(\beta)\, x^3 - 2016\, p_0\, \rho \sin(\beta)\, x^5 \ldots) \\
U(x) &= \frac{x^2}{120960}(2016\, \rho \sin(\beta)\, x^3 - 144\, p_0\, \rho \sin(\beta)\, x^5 + 4\, p_0^2\, \rho \sin(\beta)\, x^7 + \ldots) \\
V(x) &= -\frac{x^3}{362880}(-1512\, \rho \sin(\beta)\, x^3 + 72\, p_0\, \rho \sin(\beta)\, x^5 + x^6\, p_0^3 + \ldots)
\end{aligned} \tag{10}
$$

Applying the boundary conditions results in the following linear problem:

$$
\begin{bmatrix} 1 & T(0) & U(0) & V(0) \\ 1 & T(1) & U(1) & V(1) \\ 0 & Q(0) & R(0) & W(0) \\ 0 & Q(1) & R(1) & W(1) \end{bmatrix} * \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ h_0 \end{bmatrix} = \begin{bmatrix} -S(0) \\ -S(1) \\ -P(0) \\ -P(1) \end{bmatrix} \tag{11}
$$

where $P(x), Q(x), R(x), W(x)$ are the second differentiation of $S(x), T(x), U(x), V(x)$ respectively.

5. ANALYSIS OF RESULTS

A comparative study is carried out for power series (small displacements) and numerical (large displacements) solutions for several values of inclination ($\beta = 0, 22.5, 45, 67.5, 90$ deg) and non-dimensional rod self-weight ($\overline{\rho} = 35, 100$). Figs 2 and 3 respectively show the large displacement configuration for $\overline{\rho}$ = 35 and 100. In both figures the geometrical configuration is plotted for $\delta = 0, 0.2, 0.4, 0.6, 0.8, 1.0$. The heavier and more inclined rod (i.e., $\overline{\rho}$ = 100 and $\beta = 90$ deg) exhibits a more pronounced lower bulge. For horizontal rods ($\beta = 0$) the solution is, as expected, symmetrical but as the inclination increases the lateral displacement becomes more asymmetric.

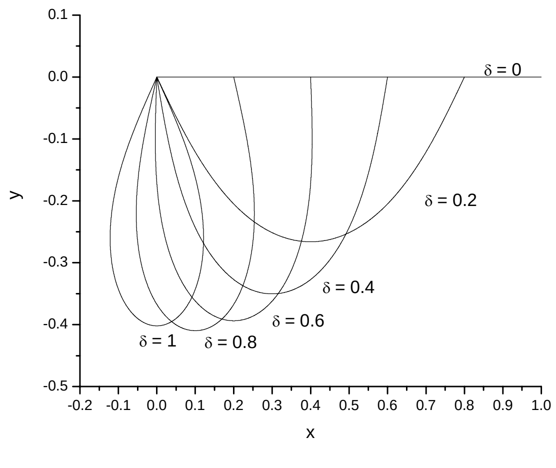

Figure. 2a - Configuration for $\overline{\rho} = 35$, $\beta = 0^0$

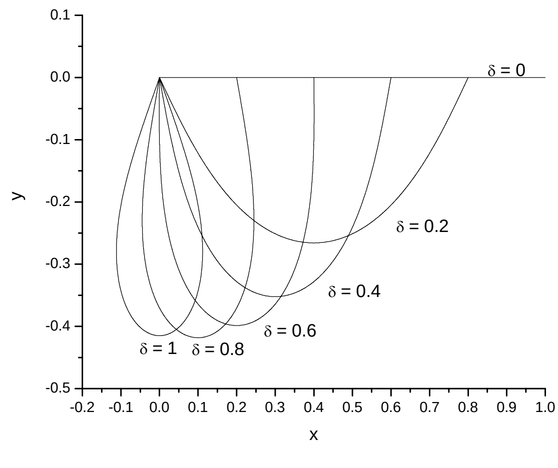

Figure. 3a - Configuration for $\overline{\rho} = 100$, $\beta = 0^0$

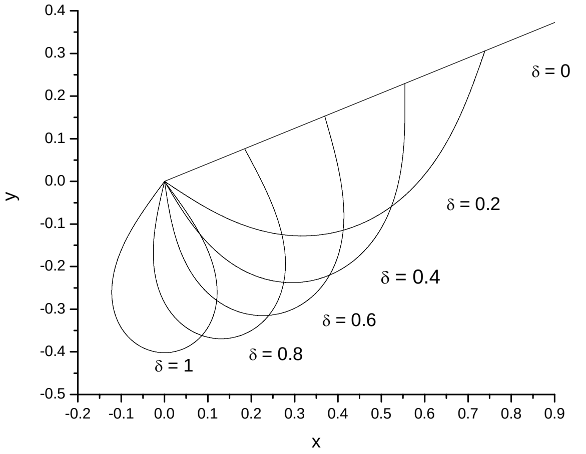

Figure. 2b - Configuration for $\overline{\rho} = 35$, $\beta = 22.5^0$

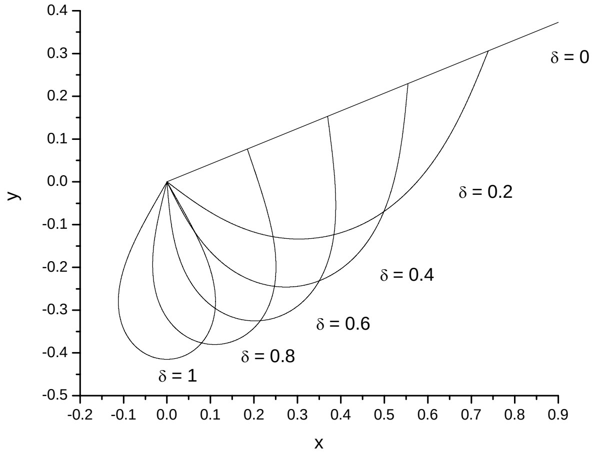

Figure. 3b - Configuration for $\overline{\rho} = 100$, $\beta = 22.5^0$

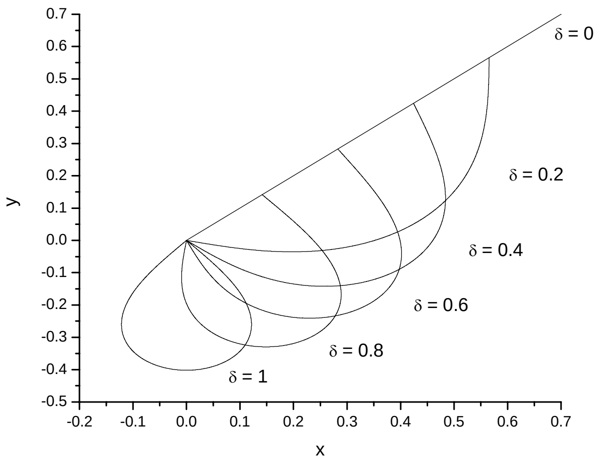

Figure. 2c - Configuration for $\overline{\rho} = 35$, $\beta = 45^0$

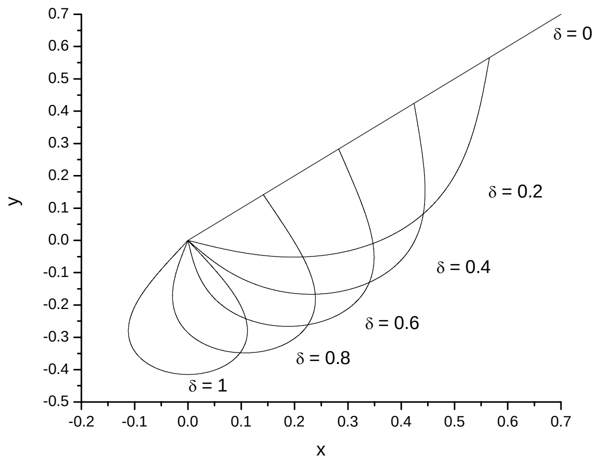

Figure. 3c - Configuration for $\overline{\rho} = 100$, $\beta = 45^0$

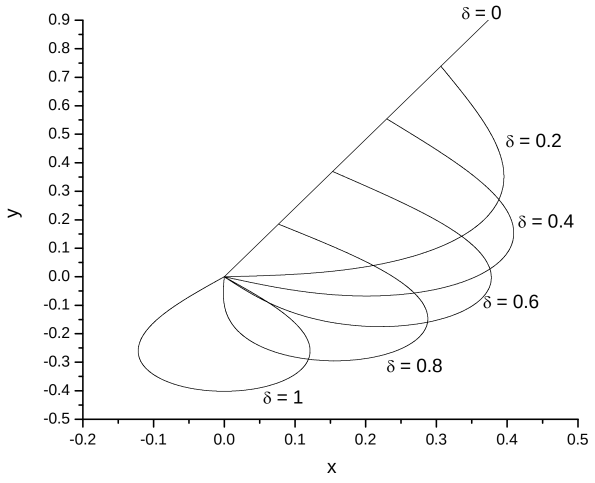

Figure. 2d - Configuration for $\overline{\rho} = 35$, $\beta = 67.5^0$

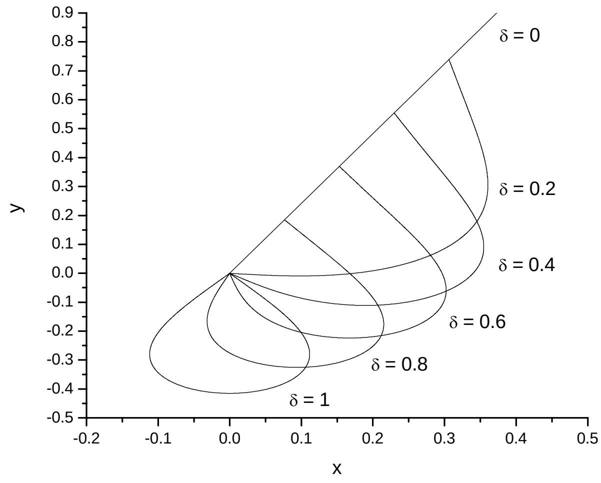

Figure. 3d - Configuration for $\overline{\rho} = 100$, $\beta = 67.5^0$

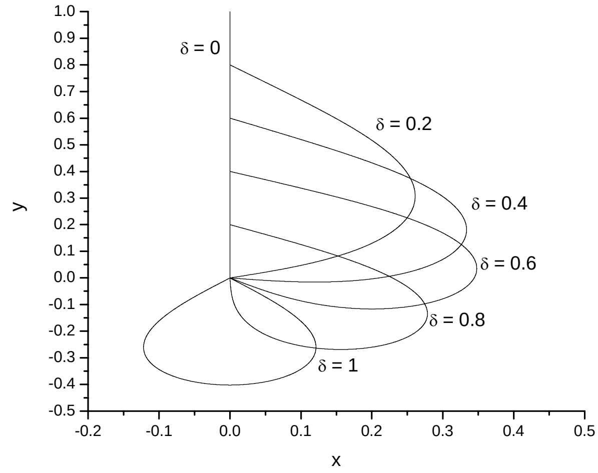

Figure. 2e - Configuration for $\overline{\rho} = 35$, $\beta = 90^0$

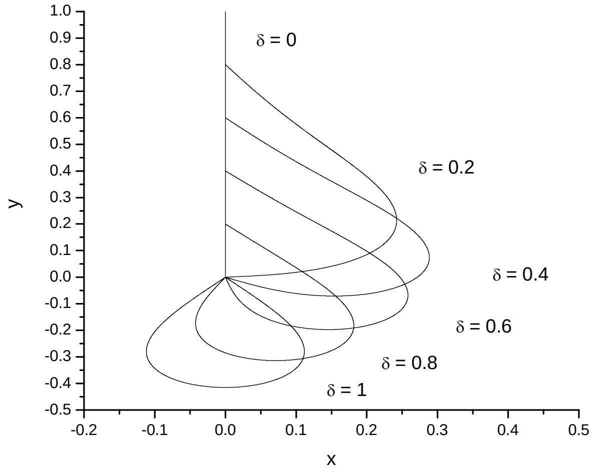

Figure. 3e - Configuration for $\overline{\rho} = 100$, $\beta = 90^0$

Figures 4 and 5 show the values of the non-dimensional variables h_0, δ, p_0, θ_0 and y_{max} (maximum lateral deflection) when small and large displacements are considered, respectively for $\overline{\rho} = 35$ and 100. The comparison between results for small and large displacement formulations indicate when geometrical non-linear effects take place and must be included for a correct response characterization.

In Figs 4a and 5a it is seen that h_0 is constant for $\beta = 0$ and varies more intensively for $\beta = 90$ deg . Observe that negative values for p_0 in Figs 4b and 5b indicate tensile forces. The change in behavior from a laterally and axially loaded rod to post-buckling phenomenon is evidenced in Figs 4c-d and 5c-d by comparing results for $\beta = 90$ deg and $\beta < 90$ deg .

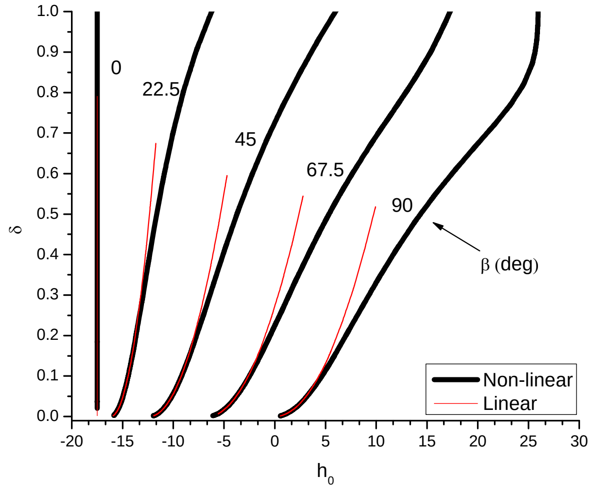

Figure 4a - δ versus h_0 for $\overline{\rho} = 35$

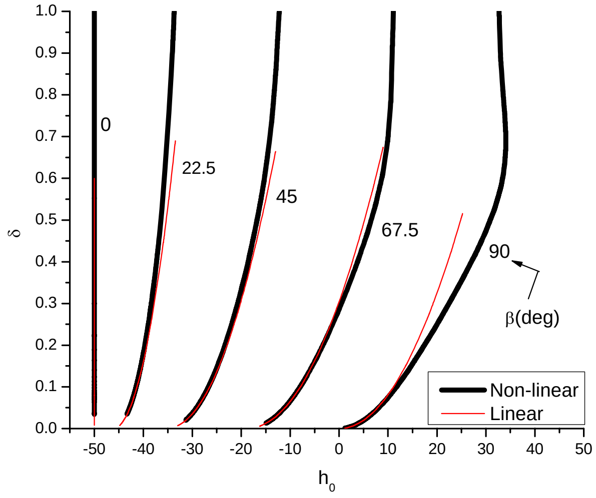

Figure 5a - δ versus h_0 for $\overline{\rho} = 100$

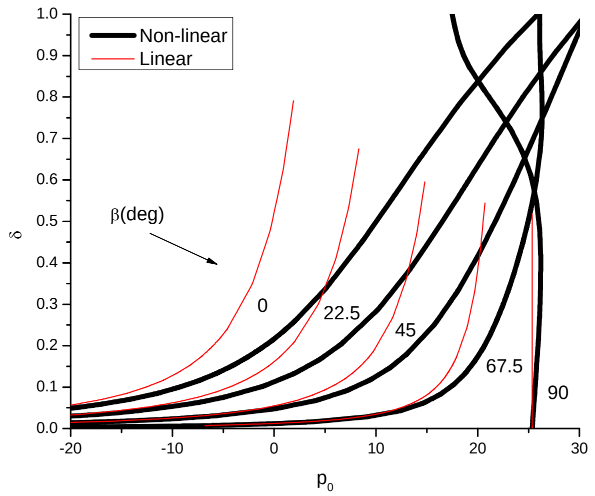

Figure 4b - δ versus p_0 for $\overline{\rho} = 35$

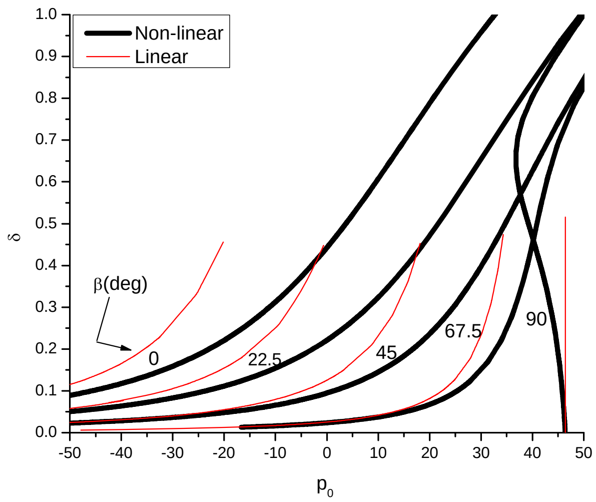

Figure 5b - δ versus p_0 for $\overline{\rho} = 100$

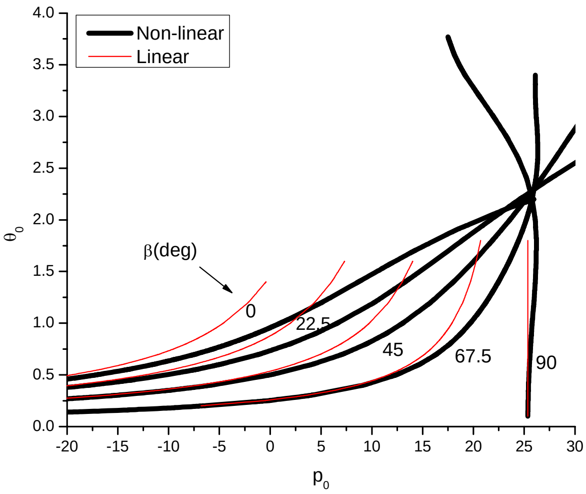

Figure 4c - θ_0 versus p_0 for $\overline{\rho} = 35$

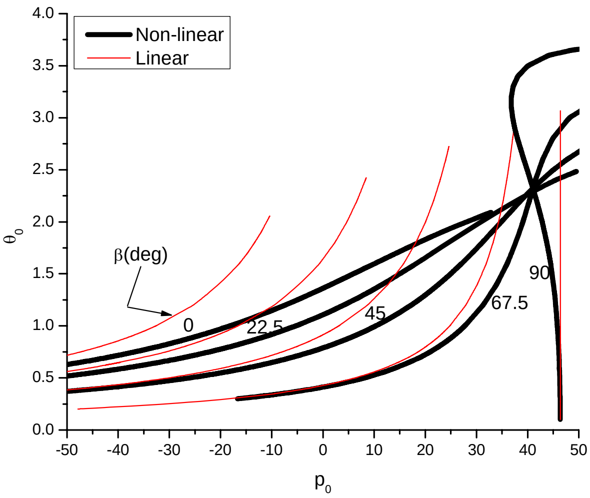

Figure 5c - θ_0 versus p_0 for $\overline{\rho} = 100$

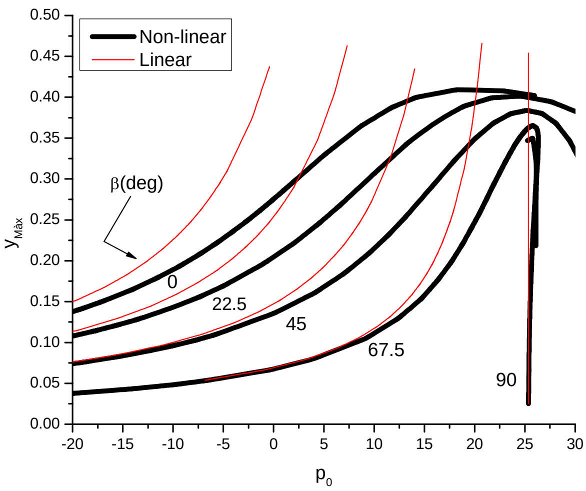

Figure 4d - y_{max} versus p_0 for $\overline{\rho} = 35$

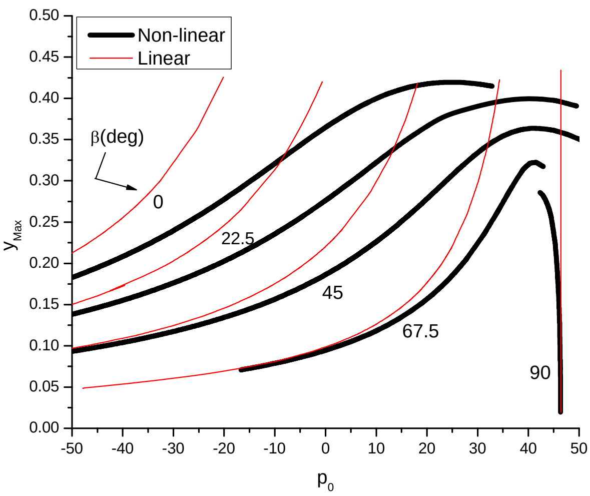

Figure 5d - y_{max} versus p_0 for $\overline{\rho} = 100$

6. CONCLUSIONS

This paper presents formulation and solution for inclined elastic rods subjected to terminal forces and a gravitational field. The rod is assumed hinged in both ends. An analytical (power series) solution is obtained when small deflections are considered. The large deflection non-linear analysis is obtained from solving a complex two-point boundary value problem governed by a set of six first order non-linear ordinary differential equations. As expected the numerical and analytical solutions are in good agreement when displacements are kept small once the geometrical non-linearities do not significantly influence the results. In addition the results evidence a change in the response behavior as the rod becomes vertical and a post-buckling instability phenomenon takes place. The boundary conditions affect the rod response and it can be readily calculated with the methodologies developed here.

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ACKNOWLEDGEMENT

The authors acknowledge the support of the National Council for Scientific and Technological Development (CNPq) for this work.