INCORPORATION OF METROLOGICAL ASPECTS IN THE STRUCTURAL SIMULATION IN VISCOELASTIC MATERIALS

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Abstract. *This paper discusses the inclusion of metrological aspects in the structural process simulation, having as a case study the analysis in viscoelastic materials. The research focus is the identification, quantification and propagation of the measurement of uncertainty (involved in the problem modeling), through the computational model of structural simulation. The proposal methodology presents alternatives for feasibility in the use of the method of the Monte Carlo simulation in viscoelastic problems through the correspondence principle and Response Surface Method.*

Keywords*: metrology, uncertainty analysis, structural simulation, viscoelasticity.*

1. INTRODUCTION

This work proposes to discuss the inclusion of metrological aspects in the structural simulation process, having as a case study the analysis of components in viscoelastic materials. The work focus is the uncertainty analysis, whose main objective is to present the methodology used to identify and quantify the propagation of the influence of measurement uncertainty (presented in the process of input parameters determination) in the results supplied by computational simulation models of structural problems in viscoelastic materials (figure 1). This methodology will help answer important questions to guarantee the reliability and efficiency of structural projects, such as:

1. **What is the influence of certain levels of uncertainty of an input parameter in the output of the simulation model?**
2. **What level of uncertainty can be allowed in the measurements processes, used in the characterization of the input parameters of a simulation model, in order to obtain a determined uncertainty level in the model answer?**
3. **Which main source of uncertainty should be minimized in order to obtain an acceptable level of uncertainty in the model answer inside of the structural component requirements?**

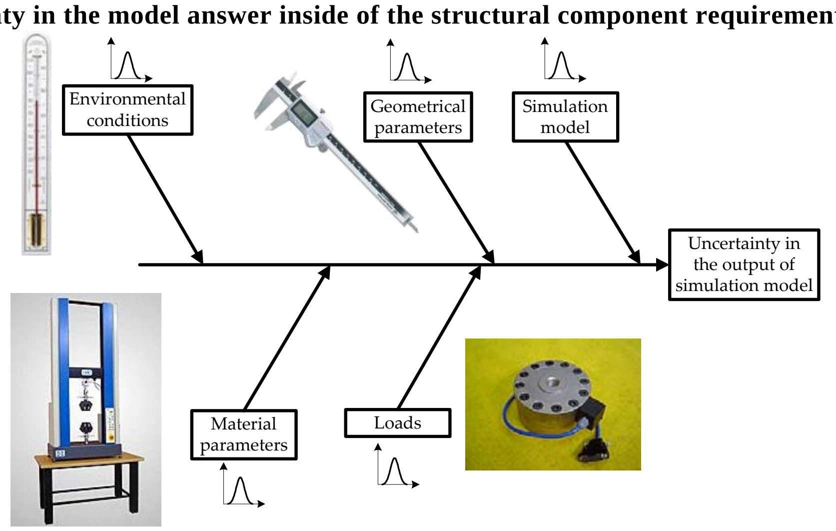


Figure 1. Propagation of measurement uncertainties through the structural simulation model.

The methodology proposed in the solution problem is based on four modules:

I. Material modeling - the study and implementation of mathematical model algorithms for viscoelastic materials in the time domain, using classic and fractionary rheological models.
II. Material characterization - the study and accomplishment of experimental creep tests for viscoelastic material characterization and the determination of material parameter uncertainty.
III. Structural simulation - the study of finite element method for viscoelastic problems and correspondence principle method.
IV. Analysis and propagation of uncertainties - the study of uncertainty propagation techniques and

application of metrological concepts for identification, quantification and propagation of measurement uncertainties involved in the structural modeling in viscoelastic material.

2. MATHEMATICAL MATERIAL MODELS FOR LINEARLY VISCOELASTIC RESPONSE

The constitutive law for a linearly viscoelastic material can be represented through hereditary integrals, where the uni-axial relationship between stress and strain can be written as:

$$\varepsilon(t) = J(0)\sigma(t) + \int_{0^+}^{t} J(t-\tau)\frac{d\sigma}{dt}d\tau \tag{1}$$

where ε - strain, σ - stress and $J(t)$ is defined as creep compliance (Christensen, 1982).

For classic rheological models, the creep compliance $J(t)$, is given by a sum of exponential functions well-known as Prony series (Flugge,1978)

$$J(t) = J_{\inf} - \sum_{i=1}^{n} J_i e^{-t/\tau_i} \tag{2}$$

where τ_i times of relaxation, J_i material parameters and J_{inf} creep compliance for $t=\infty$.

With the development of fractional calculus, Koeller (1984) developed a new rheological element, the "spring-pot". The constitutive equation of this element possesses derivatives of fractional order, which mixed the behavior between an elastic and viscous material. This element essentially substitutes the dashpot in the rheological classic models.

In agreement with Bagley (1986), the viscoelastic behavior of a great amount of polymeric materials can be represented with the use of a fractional model with only four parameters. The fractional model used in this work is the Fractional Zener model (Welch *et al*, 1999), where the creep compliance is given by the equation

$$J(t) = \frac{1}{E_0}\left\{1 + \left(\frac{E_0}{E_1}\right)\left(1 - E_\alpha\left[-(t/\tau)^\alpha\right]\right)\right\} \tag{3}$$

where E_0, E_1 and τ material parameters, as well as the fractional order derivative α. The nucleus of this function is given by the Mittag-Leffler function, $E_\alpha(*)$, which is defined by an infinite series (Enelund and Olson, 1999)

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)} \tag{4}$$

where $\Gamma(*)$ represents the Gamma function. The references (Gorenflo, 1997, Diethelm *et al*, 2000, Diethelm *et al*, 2002) present algorithms for the implementation of this function.

The finite element method (Zienkiewicks and Taylor, 1991) will be used in the simulation of structural problems through commercial software ANSYS (ANSYS, 2003).

2.1. The classic and numerical correspondence principle

The classic correspondence principle, CP, states that if a solution to a linear elasticity problem is known, the solution to the corresponding problem for a linearly viscoelastic material can be obtained by replacing each quantity which can depend on time by its Laplace transform multiplied by the transform variable (p or s), and the by transforming back to the time domain. There is the restriction that the interface between boundaries under prescribed load and boundaries under prescribed displacement may not change with time, although the loads and displacement can be time dependent (Findley, 1960). The case study, to be presented, it will illustrate the application of the correspondence principle to the problem of a viscoelastic cantilever beam with constant load in its end.

The use of the correspondence principle on the generalized Hooke's law will be defined as numerical correspondence principle, NCP. In this method the stress and/or strain will be obtained through a numeric method (i.e. the finite element method), considering the material with purely elastic behavior. The main advantages of numeric correspondence principle is the possibility of solution problems by the CP where analytic elastic solution is not available.

3. PVC VISCOELASTIC CHARACTERIZATION

In this work the tensile creep test was used to characterize the behavior of polyvinyl chloride (PVC), being the objective of characterization test to supply data to determine the material parameters.

3.1. The creep test

The creep test consists of measuring the time dependent strain resulting from the application of a steady uniaxial stress. The norm ASTM D 2990-01 (ASTM, 2001) establishes the requirements and the necessary procedure for the accomplishment of the creep test in plastics.

In this work, the experimental apparatus used for accomplishment of the creep test is presented in the figure 2 . The experimental apparatus consists basically of a system to apply the load to the test body, a stove for temperature control and a measurement system for accompaniment of test body deformation. The test body deformation is measured through the use of EXCEL strain gage and HBM measuring amplifier system.

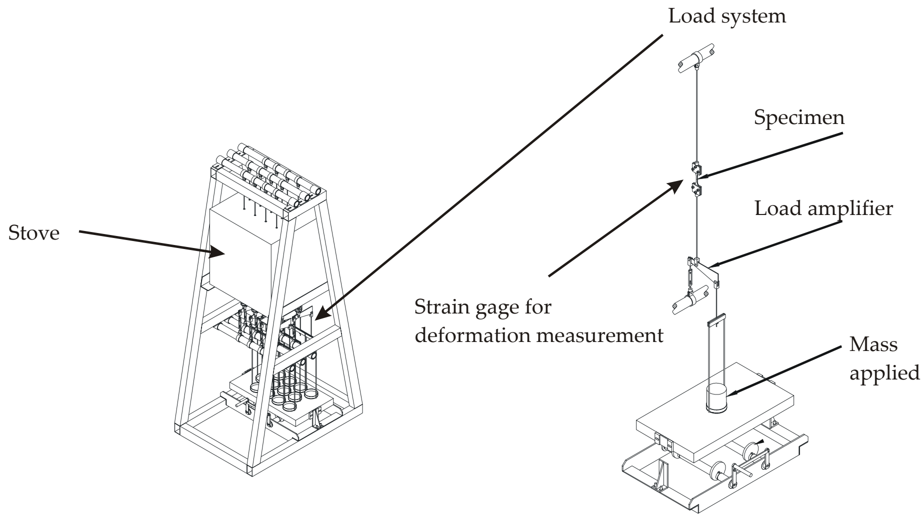


Figure 2. Experimental apparatus used in the creep test.

3.2. The adjust of material parameters

With base in the deformation of specimen, measured through the creep test, determination of the creep compliance parameters is obtained through an adjustment of experimental points to the mathematical model of material behavior. A process of nonlinear optimization is used. The optimization process consists in the minimization of least-square function

$$F(x,t) = \min_x \frac{1}{2} \sum_{i=1}^{n} \left(J(x,t_i) - J_{m_i} \right)^2 \tag{5}$$

where $J(t)$ given values by mathematical model of creep compliance and $J_m(t)$ the value of creep compliance obtained experimentally, i.e., $J_m(t)=\varepsilon(t)/\sigma_0$, where $\varepsilon(t)$ is the measured deformation in the specimen and σ_0 is the magnitude tension applied in the specimen. The illustration (3) presents the results of creep test bodies and fittings to the classic and fractional rheological models given by the equations (2) and (3).

4. UNCERTAINTY ANALYSIS

In the structural analysis, it becomes necessary to determine the related parameters with respect to the geometry component, the applied load, the material properties, and the contour conditions of the problem through of measurement processes. The main uncertainty sources associated with the measurement processes are: instrument resolution; inherited uncertainty; environmental factors like variations and thermal gradients; vibrations; influence operator and measurement procedure, etc..

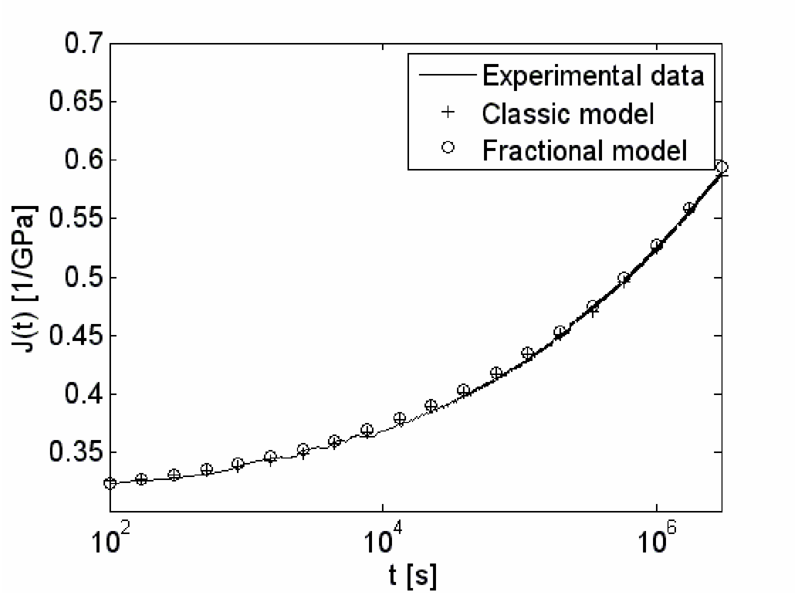


Figure 3 Results of curve-fitting of experimental data to classic and fractionary material models.

4.1. The method of Monte Carlo simulation

A measurement/simulation model [17] is expressed by a functional relationship *f*:

$$y = f(x_1, x_2, ..., x_n), \tag{6}$$

where y is a single (scalar) output quantity and **x** represents the N input quantities ($x_1, x_2, ..., x_n$). The method of uncertainty propagation used in this work the is the method of Monte Carlo simulation (MCS). The Monte Carlo simulation is a methodology which allows to make use of deterministic analysis in context with stochastic analysis. If all input model parameters ($x_1, x_2, ..., x_n$) are described by a probability density functions (PDF), an algorithm can be used to generate a input vector $x_j=[x_1, x_2, ..., x_n]^T$. Each element x_i of this vector should be generated in agreement with its PDFs. Applying the generated vector x_j to the equation (6), the corresponding output y_j is obtained. If this simulation process is repeated M times ($M >> 1$), the output is a vector $[y_1, y_2, ..., y_M]^T$ that can be considered as a sample answer population. Through this sample answer population system, a PDF can be determined and describes the behavior of the answer system (ISO, 2004).

4.2. Uncertainty analysis in the material characterization process

The main uncertainty sources of creep test are related with the determination of specimen geometry, applied load, control of the environmental conditions (temperature, humidity and vibrations), material characteristics (material cracks, orthotropic material characteristic, aging effects, residual stress, etc.), test procedure, test apparatus limitations and curve-fitting process (figure 6).

4.3. Uncertainty associated with fitted parameters

The material parameters are obtained from a nonlinear curve-fitting process. The influence of measurement uncertainty deformation (using strain gages**) is incorporated into the material parameters uncertainty**.

In agreement with references (Baker and Cox, 2004, Bevington, 1969), the uncertainty (covariance) matrix associated with fitted parameters is approximated by

$$C = \sigma^2 \left(J^T J\right)^{-1} \tag{7}$$

where σ^2 the measured points variance and J Jacobian matrix evaluated in the solution x^*. The standard uncertainties associated with fitted parameters $u(a_j)=(C(j,j))^{1/2}$, i.e., the square roots of the diagonal elements of the uncertainty matrix C. If an estimate a priori of σ is not available, then the variance can be esteemed for

$$\sigma = \frac{\|f\|}{\sqrt{n-m}} \tag{8}$$

where m is the number of adjusted parameters, $\|f\|$ represents the norm of the residue and n is the number of measured points.

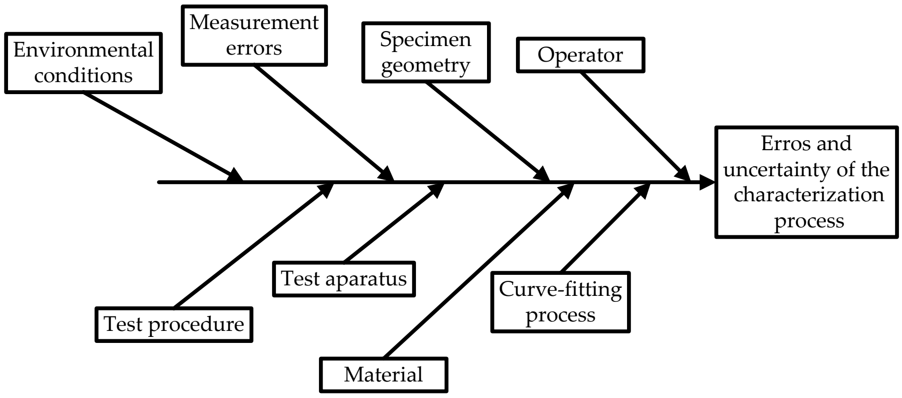


Figure 4. The main uncertainty sources of characterization process.

The tables 1 and 2 present the values and uncertainty levels of adjusted parameters for Maxwell model with 11 parameters and fractional Zener model, respectively.

Table 1. Classic material parameters obtained by nonlinear curve-fitting.

Parameters	Mean	Standard deviation ($U_{68\%}$)	$U_{68\%}$/mean (%)	Units
J_0	3,23E-10	3,61E-12	1,12E+00	*1/GPa*
J_1	1,09E-11	2,10E-13	1,93E+00	*1/GPa*
J_2	3,13E-11	4,83E-13	1,54E+00	*1/GPa*
J_3	5,53E-11	5,83E-13	1,06E+00	*1/GPa*
J_4	5,67E-11	4,37E-13	7,72E-01	*1/GPa*
J_5	1,41E-10	4,83E-13	3,43E-01	*1/GPa*
τ_1	4,27E+02	1,29E+01	3,02E+00	*1/s*
τ_2	4,96E+03	1,52E+02	3,07E+00	*1/s*
τ_3	5,04E+04	8,43E+02	1,67E+00	*1/s*
τ_4	5,00E+05	3,80E+03	7,61E-01	*1/s*
τ_5	2,00E+06	4,95E+03	2,48E-01	*1/s*

Table 2. Fractionary material parameters obtained by nonlinear curve-fitting.

Parameters	Mean	Standard deviation ($U_{68\%}$)	$U_{68\%}$/mean (%)	Units
E_0	3,33E+09	5,26E+06	1,58E-01	*Pa*
E_1	2,73E+08	2,07E+06	7,58E-01	*Pa*
α	2,53E-01	9,45E-04	3,73E-01	$Pa.s^{\alpha}$
τ	6,99E+10	1,14E+07	1,63E-02	

4.4. Uncertainty analysis in the interconversion process

The software ANSYS becomes necessary the interconversion of creep compliance and Poisson's ratio (both obtained experimentally by creep test), in Bulk modulus

$$K(t) = K_{\text{inf}} + \sum_{i=1}^{n} K_i e^{-t/\tau_i} \tag{9}$$

and shear modulus

$$G(t) = G_{\text{inf}} + \sum_{i=1}^{n} G_i e^{-t/\tau_i} \tag{10}$$

being G_{inf}, K_{inf}, K_i, G_i, and τ_i material parameters.

The parameters uncertainties of creep compliance and Poisson's ratio, will be propagated through interconversion process to the parameters of bulk and shear modulus.

The uncertainty propagation through the interconversion process is accomplished using the method of Monte Carlo simulation. The interconversion process is repeated many times in order to compose the uncertainty of bulk and shear modulus. The table 3 show the values of bulk and shear modulus of PVC to 30 ºC obtained by interconvertion process.

Table 3. Bulk and Shear parameter's model obtained by interconversion process.

Parameters	mean	Standard deviation ($U_{68\%}$)	$U_{68\%}$/mean (%)	Parameters	mean	Standar deviation ($U_{68\%}$)	$U_{68\%}$/mean (%)
G_0 (GPa)	$1{,}16.10^9$	$1{,}34.10^7$	1,16	K_3(GPa)	$3{,}82.10^8$	$1{,}16.10^7$	3,03
G_1 (GPa)	$3{,}85.10^7$	$1{,}15.10^6$	2,98	K_4 (GPa)	$3{,}35.10^8$	$9{,}63.10^6$	2,88
G_2 (GPa)	$9{,}85.10^7$	$2{,}53.10^6$	2,57	K_5 (GPa)	$4{,}41.10^8$	$1{,}13.10^7$	2,55
G_3 (GPa)	$1{,}37.10^8$	$2{,}90.10^6$	2,12	τ_1 (1/s)	$4{,}12.10^2$	$1{,}24.10^1$	3,00
G_4 (GPa)	$1{,}20.10^8$	$2{,}28.10^6$	1,90	τ_2 (1/s)	$4{,}54.10^3$	$1{,}40.10^2$	3,08
G_5 (GPa)	$1{,}58.10^8$	$2{,}13.10^6$	1,35	τ_3 (1/s)	$4{,}37.10^4$	$7{,}28.10^2$	1,66
K_0 (GPa)	$3{,}23.10^9$	$7{,}91.10^6$	2,45	τ_4 (1/s)	$4{,}36.10^5$	$3{,}30.10^3$	7,57
K_1 (GPa)	$1{,}08.10^8$	$3{,}96.10^6$	3,69	τ_5 (1/s)	$1{,}56.10^6$	$4{,}67.10^3$	2,98
K_2 (GPa)	$2{,}75.10^8$	$9{,}17.10^6$	3,33				

4.5. Uncertainty propagation through simulation process

The method of Monte Carlo simulation was used to propagate the input parameters uncertainties through the simulation model. The finite element commercial software ANSYS possesses the "Probabilistic Design System" tool, which uses the MCS to propagate the input parameters uncertainty through the finite element model of component. There is a possibility of creating a metamodel (i.e., a "model of the model") to the problem by the response surface method (RSM), which allows an analysis by Monte Carlo simulation computationally more efficient.

The simulation through the finite element method is a nonlinear problem. Therefore, the analysis with high number of degree-of-freedom through MCS requests a high computational time because MCS needs to repeat the problem simulation several times. One of the contributions in this work is the alternatives development to make possible the uncertainty propagation of viscoelastic structural problems through MCS. **The main proposed solutions for the problem are the use the response surface method (RSM), the analysis using the models supplied for the classic, and the numeric correspondence principle associated to MCS.** The case study presented will demonstrate the proposal methodology.

5. CASE STUDY: STRAIN ANALYSIS OF A VISCOELASTIC CANTILEVER BEAM

This case study is about of uncertainty analysis involved in the strain simulation for a certain point in the surface of viscoelastic cantilever beam subjects a constant load in its end (figure 5). The three-dimensional finite element model, used in the solution by PDS tool, is presented in the figure 6.

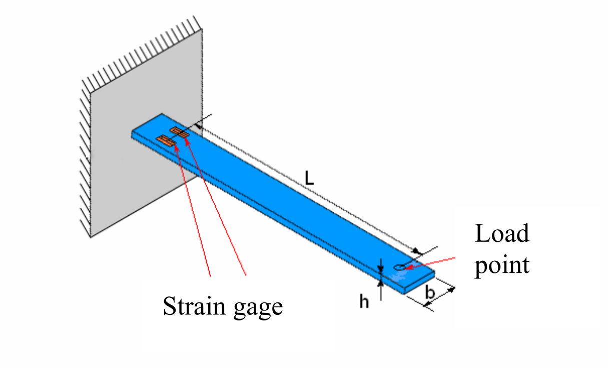


Figure 5. Viscoelastic cantilever beam.

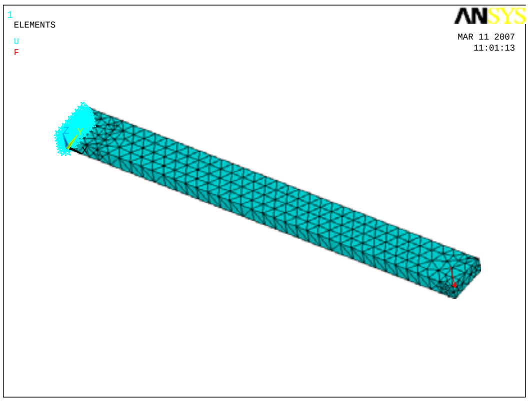


Figure 6. Finite element model of the cantilever beam.

An experimental test was realized to comparing the experimental results with the value simulated using the correspondence principle and finite element method. The beam is subject to a constant load due to the mass applied in its end. Due the viscoelastic material behavior, the beam will be deformed continually. Four strain gages were used in the beam (two strain gauges in the tensile surface and two strain gauges in the compression surface). The figure 7 shows the experimental apparatus. The figure 8 presents the experimental and simulated results.

Figure 7. Apparatus used in the experimental test.

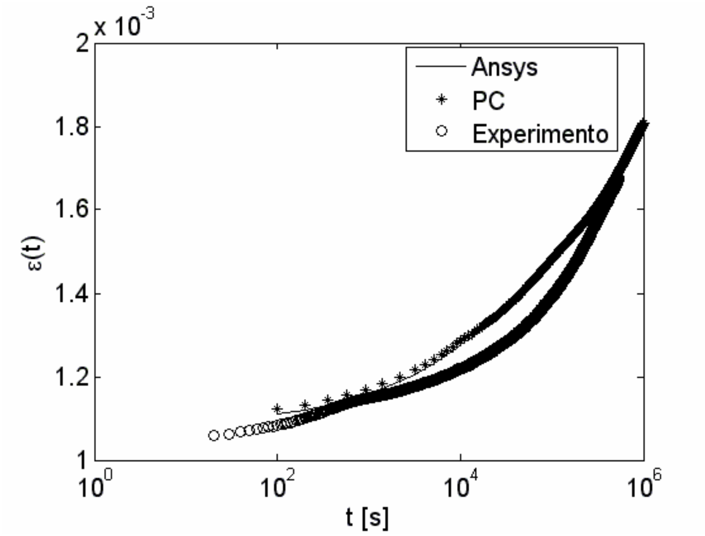


Figure 8. Deformation in the viscoelastic cantilever beam obtained experimentally and numerically via correspondence principle and finite element method.

5.1. Modeling aspects

The Monte Carlo simulation method, associated with the model supplied by the classic correspondence principle, requests the determination of the elastic analytic solution. The strain in a cantilever beam subjects the a load at the end is given by the equation

$$\varepsilon(x) = \frac{6}{bh^2 E}(L - x) \tag{11}$$

where L length of beam, x point of measured the deformation, E elastic modulus, b and h width and thickness of beam, respectively. For a step function load history, the solution of a viscoelastic problem by the classic correspondence principle is given for

$$\varepsilon(x,t) = \frac{6}{bh^2}(L - x)J(t) \tag{12}$$

5.2. Input parameters characterization

5.2.1. Uncertainty associated with a determination of geometrical parameters

This study needs to determine the length, width and thickness of the traverse section beam. The main uncertainty sources acting in the determination of the geometry of beam are:

- Resolution of measurement systems (measurement scale and micrometer);
- Inherited uncertainty of calibrated measurement systems;
- Repeatability of measurement results;
- Uncertainty in determination of the PVC thermal coefficient, used for the uncertainty quantification due the material thermal dilation effects;
- The own beam geometry;

The combined uncertainty of measured geometry parameters on test body is computed as:

$$u_c = \sqrt{u_R^2 + u_{\mathrm{Re}}^2 + u_{cal}^2 + u_{\Delta T}^2} \tag{13}$$

where u_c combined standard uncertainty, u_R uncertainty due the instrument resolution, u_{Re} uncertainty due the repeatability of measurement results, u_{cal} inherited uncertainty of instrument calibration and $u_{\Delta T}$ the uncertainty due the temperature variation of stove.

The tables 4 and 5 present the main uncertainty sources and uncertainty budget for width and thickness obtained through the use of a DIGIMESS micrometer with resolution of 10 µm. It's presented in the table 6 the uncertainty budget of measurement beam length through the use of a measuring scale with resolution of 10 µm.

Table 4. Uncertainty budget associated with the determination of the width.

Symbol	Uncertainty source	Distribution	Value	Divider	$u_{68\%}$	v_{eff}[1]
R	Resolution (m)	Rectangular	1,00E-05	1/(2√3)	2,89E-06	∞
Re	Repeatibility (m)	Normal	1,32E-04	1/√n	4,65E-05	16
Cal	Calibration erros (m)	Rectangular	2,00E-06	1/√3	1,15E-06	∞
dT	Thermal dilatation (m)	Rectangular	3,38E-05	1/√3	1,95E-05	∞
u_c	Combined uncertainty (m)	Normal			5,05E-05	134

Table 5. Uncertainty budget associated with the determination of the thickness.

Symbol	Uncertainty source	Distribution	Value	Divider	$u_{68\%}$	v_{eff}
R	Resolution (m)	Rectangular	1,00E-05	1/(2√3)	2,89E-06	∞
Re	Repeatibility (m)	Normal	2,39E-05	1/√n	5,80E-06	16
Cal	Calibration erros (m)	Rectangular	2,00E-06	1/√3	1,15E-06	∞
dT	Thermal dilatation (m)	Rectangular	1,27E-05	1/√3	7,34E-06	∞
u_c	Combined uncertainty (m)	Normal			9,86E-06	134

Table 6. Uncertainty budget associated with the determination of the length.

Symbol	Uncertainty source	Distribution	Value	Divider	$u_{68\%}$	v_{eff}
R	Resolution (m)	Rectangular	1,00E-05	1/(2√3)	2,89E-06	∞
Re	Repeatibility (m)	Normal	1,91E-04	1/√n	6,03E-05	9
Cal	Calibration erros (m)	Rectangular	2,00E-06	1/√3	1,15E-06	∞
dT	Thermal dilatation (m)	Rectangular	3,56E-04	1/√3	2,05E-04	∞
u_c	Combined uncertainty (m)	Normal			2,14E-04	1430

5.2.2. Uncertainty associated with the applied load

The load is applied through a deadweight. The main uncertainty sources in the applied load quantification are:

- The misalignment in the load application;
- Uncertainty associated with gravitational acceleration;
- Resolution balance;
- Repeatability of measurement results of applied mass ;
- Inherited uncertainty of calibrated balance.

The table (7) presents the uncertainty budget of the applied mass determination. The table (8) shows the input parameters uncertainty of the beam simulation model.

Table 7. Uncertainty budget associed with the determination of the mass aplied the beam.

Symbol	Uncertainty source	Distribution	Value	Divider	$u_{68\%}$	v_{eff}
R	Resolution (m)	Rectangular	1,00E-03	1/(2√3)	2,89E-04	∞
Re	Repeatability (m)	Normal	6,74E-04	1/√n	4,94E-05	10
Cal	Calibration errors (m)	Rectangular	8,75E-04	1/√3	5,05E-04	∞
u_c	Combined uncertainty (m)	Normal			6,16E-04	844

[1] v_{eff} is the effective degrees of freedom [17]

Table 8. Random input parameters for simulation model.

Symbol	Uncertainty source	Distribution	Mean	$U_{68\%}$	$U_{68\%}$/mean (%)
m	Mass applied (kg)	Normal	8,75E-01	6,16E-04	7,04E-02
L	Length (m)	Normal	3,39E-01	2,11E-04	6,24E-02
b	Width (m)	Normal	3,21E-02	2,94E-05	9,18E-02
h	Tickness (m)	Normal	1,20E-02	8,23E-06	6,86E-02
θ	Strain gauge misalignment (º)	Rectangular	0,00E+00	5,77E-01	Inf
ν	Poisson’s ratio	Rectangular	3,80E-01	1,10E-02	2,89E+00
g	Gravitational acceleration (m/s^2)	Rectangular	9,81E+00	5,00E-06	5,10E-05

5.3. Uncertainty propagation through simulation model

The table 9 shows the results by method of Monte Carlo simulation using tree simulation model: (a) model supplied by the classic correspondence principle (PC+SMC); (b) model supplied by numeric correspondence principle (SMC+PCN); (c) model supplied by the Response Surface Method (RSM). The table (10) shows the results through the method of simulation of Monte Carlo by the fractionary material model using the model supplied by PC and PCN.

Table 9. Obtained random output parameter for a classic material model.

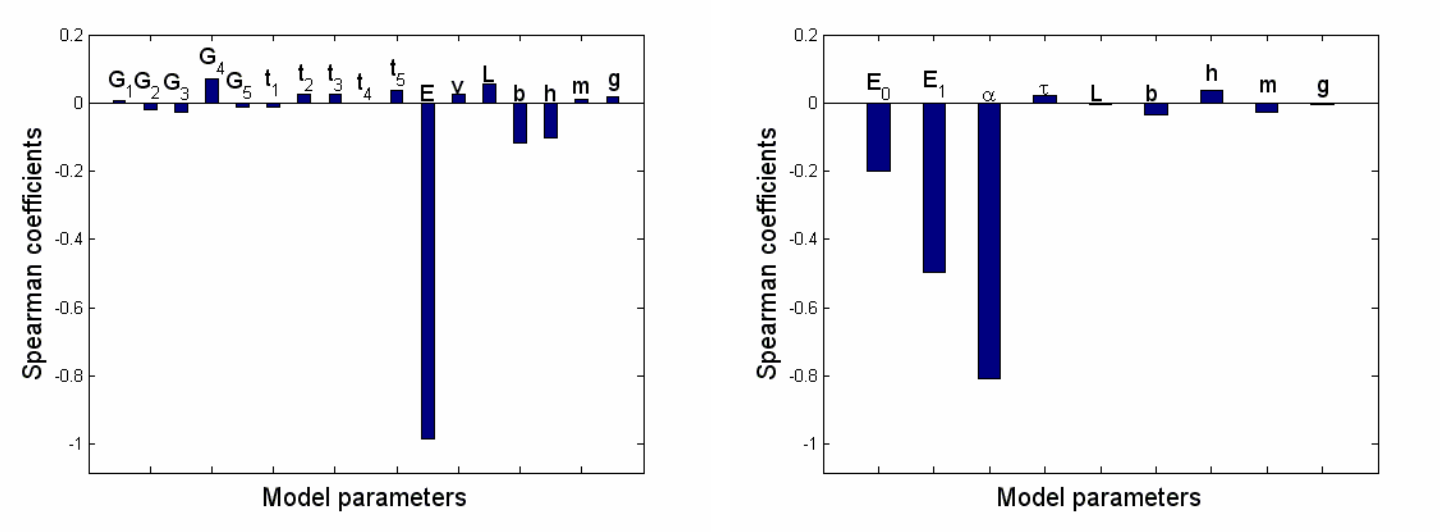
Method	Mean (m)	$U_{68\%}$ (m)
PC+SMC	1,17E-3	±1,28E-5
PCN+SMC	1,16E-3	±1,27E-5
RSM+SMC	1,16E-3	±1,30E-5

Table 10. Obtained random output parameter for a fractionary material model.

Method	Mean (m)	$U_{68\%}$ (m)
PC+SMC	1,179E-3	±3.07E-6
PCN+SMC	1,169E-3	±3.76E-6

5.4. Sensibility analysis

The sensibility analysis was accomplished using the Spearman rank order correlation coefficients [11]. It is presented in the figure (11) the result for a classic and fractional material model using the classic correspondence principle associated to the MCS. The objective of the sensibility analysis is present the parameters that have a larger contribution in the uncertainty of output model.



(a) (b)

Figure 11. Spearman rank order correlation coefficients by solution obtained MCS associated the classic correspondence principle: (a) classic material model; (b) fractionary material model.

6. CONCLUSIONS

The analysis and propagation of uncertainties are essential for an evaluation of the reliability output of simulation model. The results of uncertainty propagation by the method of Monte Carlo simulation associated at classic and numeric correspondence principle have demonstrated a great agreement among the results, making possible the use of MCS in the viscoelastic materials analysis.

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