Ripple formation on granular beds under the action of a rolling wheel

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Abstract. *Granular surfaces tend to develop lateral ripples under the action of rolling wheels, an effect known as* washboard *or* corrugated *road. We investigated this instability of the flat road surface from the point of view of driven, dissipative granular dynamics. We report the results of both laboratory experiments and soft-particle direct numerical simulations. The experiment consisted of a wheel rolling on sand arranged around the circumference of a rotating table. Above a critical speed, the ripple pattern appears as small patches of travelling waves which eventually spread to the entire circumference. The ripples drift slowly in the driving direction. Interesting secondary dynamics of the saturated ripples were observed, as well as various ripple creation and destruction events. All of these effects are captured qualitatively by 2D soft particle simulations in which a disk rolls over a bed of polydisperse particles in a periodic box. We also discuss a simplified scaling model which gives some insight into the mechanism of the instability.*

Keywords: *washboard road, granular, ripple formation*

1. INTRODUCTION

Ripples which spontaneously appear due to the action of rolling wheels on unpaved roads bedevil transportation worldwide, especially in developing countries [1]. This effect, known as *corrugated* or *washboard* road, can severely limit the usefulness of unsurfaced roads. More generally, the appearance of ripples on a granular surface under tangential stress is reminiscent of other sorts of wind- and water-driven ripples, and of dune formation. This resemblance suggests that this problem, which is well-discussed in the engineering literature [1, 2, 3, 4, 5, 6], might benefit from the simplifications of a physics-oriented approach.

Engineering models of washboard formation range from coupled, damped pendulum models [3, 4] to full continuum simulations of the deformable road surface [6]. Numerous experimental studies have been undertaken, from laboratory scale rigs [2, 3, 4] to full scale road tests [5]. In all cases, however, the engineering goal was to understand all the complexities of the system and to mitigate or eliminate the effect. In contrast, we aim to understand the simplest system that exhibits washboard road and to study it as a nonlinear, pattern forming instability. Only a few theoretical studies of this kind have appeared in the physics literature [8, 7]. In addition to carrying out well instrumented laboratory-scale experiments, we present here the first application of soft-particle Discrete Element Method (DEM) simulations to this problem. We also present a simple theoretical treatment in order to gain some insight into the fundamental mechanism of the instability and to understand the scaling and important non-dimensional groups.

2. DESCRIPTION OF THE EXPERIMENTAL SETUP

The experimental setup is shown in Fig. 1. The road consists of a deep layer of sand arranged on the circumference of a 1 m diameter rotating table. We used natural, rough sand with a grain diameter of 300 ± 100 μm and the bed was typically 50 mm deep. A 100 mm diameter, 20 mm thick hard rubber wheel was attached to a 330 mm long arm in the form of a lever.

The wheel rolled freely on the sand bed as the table rotated at a constant speed. No torque was applied to the wheel other than that produced by its contact with the bed. The table was typically rotated at about 0.6 Hz which corresponds to a horizontal velocity of the wheel $v \approx 2$ m/s.

A potentiometer was attached to the arm to record its angle, and its output was digitized. Simultaneously, a commercial laser displacement profiler[9] was used to record the bed shape. Thus, we could measure the wheel position and ripple shape with a vertical accuracy of 1 mm. Data acquisition was triggered by an optical sensor fixed to the table, and several thousand vertical positions could be measured in one table rotation. During a typical run, the arm inclination remained confined between 70° and 90°, so that the wheel motion was almost vertical. The large angle implies that the natural pendulum frequency of the arm plays no role, and the wheel is merely restored by falling under its own weight. The finite circumference of the table effectively imposes periodic boundary conditions on the ripple pattern, so that its wavelength is quantized since a fully developed washboard pattern contains an integer number of ripples around the table.

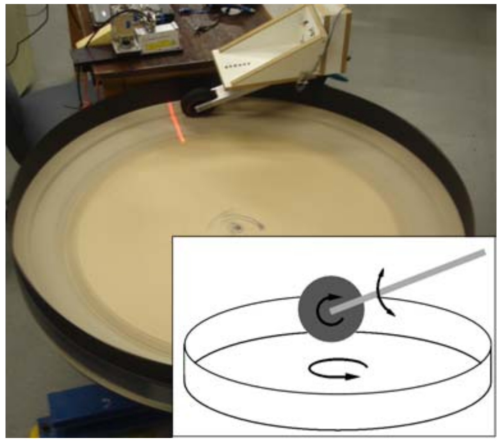

Figure 1. Photo and sketch of the experimental setup. A bed of natural sand is laid on the circumference of a rotating table. A hard rubber wheel attached to an arm is free to bounce and roll on the granular bed. The shape of the sand surface is measured by a laser displacement profiler. The position of the wheel is known from measurements of the arm angle.

3. DESCRIPTION OF THE NUMERICAL SIMULATIONS

We also investigated the washboard formation in 2D DEM simulations. The simulation considers individual deformable disks, rotating and colliding with one another, subject to contact friction and gravity. We used the following physical parameters: particle diameter 8 mm, mass 0.16 g, spring constant $40\mathrm{kNm}^{-1}$, coefficient of restitution is 0.5 and friction 0.3. In the simulation, the wheel was treated like any other disk but its density was 1/5th that of the other disks and its diameter was 12.5 times larger. A constant horizontal velocity v was imposed to the wheel, but it was free to rotate and move vertically. The disks were made slightly polydisperse ($\pm$20% in diameter) in order to avoid crystallisation. In order to mimic the experimental setup, the simulation was made periodic in the horizontal direction. 25 000 small disks were initialized at random positions in the box, and then allowed to fall under gravity to settle into the bed, resulting in a layer 20 diameters thick and 1500 diameters long. The simulations were typically run for 500 passages of the wheel.

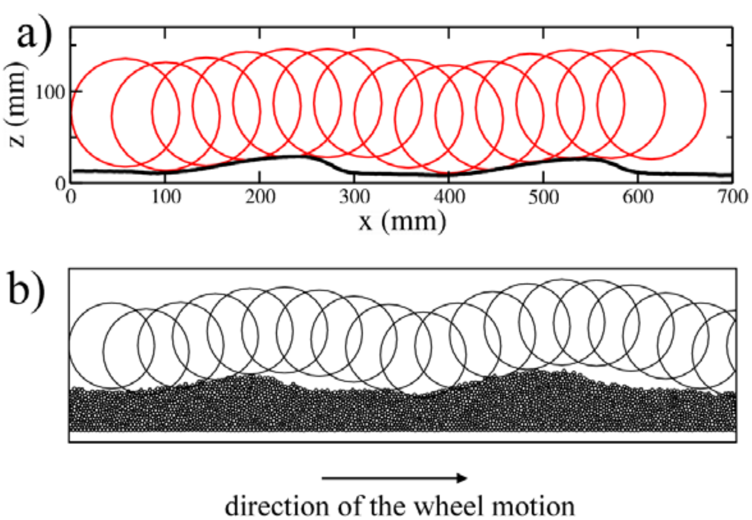

Figure 2. Bed profile and wheel trajectory for typical washboard patterns showing excellent agreement between experiments (a) and DEM simulations (b). Note that the ripples are asymetrical and that the wheel loses contact with the bed at the crest of the ripples.

4. FORMATION OF RIPPLES

The wheel initially rolls smoothly on the flat granular surface. After a few tens of passes, if the velocity is high enough, a small localized ripple starts forming at some position. New ripples rapidly grow from that location, downstream from the initial position. Ripples then grow in height and eventually spread over the whole circumference of the table (in the experiments) or the whole length of the periodic box (in the simulation). A typical pattern is shown in Fig. 2 which displays both experimental and numerical data. There is an excellent qualitative agreement between the experiment and simulation.

The ripples are strongly asymmetrical with the steeper face close to the angle of repose. Individual ripples tend to be separated by flat regions. The height of ripples ranges from a few mm up to 50 mm as the wavelength (or pitch) ranges from 50 mm to 500 mm. Figure 2 also shows the trajectory of the wheel. For large v, the wheel becomes airborne near the crest of each ripple. At lower v, however, a clear washboard pattern can form while the wheel remains constantly in contact with the bed.

As is observed on actual roads [1] we found a critical value of the driving velocity v_c below which the bed remains flat. We find both experimentally and numerically that $v_c \simeq 2.5$ m/s, a value quite similar to that found for real roads. The exact nature of the bifurcation to the rippled state remains unclear, however. The bed becomes extremely sensitive to small, but finite, perturbations for v near v_c, but it is difficult to establish whether v_c represents the onset of a linear instability to infinitesimal perturbations, or whether there is any velocity hysteresis near v_c. Most previous studies [1, 2, 3, 4, 5, 6] have only examined the case of large, artificial initial perturbations and $v \gg v_c$.

Washboard patterns exist over a wide range of parameters. We varied the bed thickness, the grain size and shape, the wheel size, shape and mass. As long as the bed thickness was sufficient to supply enough material for the ripples, in all cases the results were qualitatively identical. All our observations are broadly consistent with previous engineering studies [1, 2, 3, 4, 5, 6], but have the advantage of a simpler suspension system and more modern instrumentation for data acquisition. Perhaps surprisingly, the size and shape of the grains has no effect whatsoever on either the wavelength or the amplitude of the ripples. In experiments, we tried two different natural sands with $d = 300 \pm 100\,\mu\mathrm{m}$ and $3 \pm 0.8\,\mathrm{mm}$, and also replaced the sand with long grain rice. In simulations, we halved the grain size and doubled the bed thickness for the same size wheel. The ripple patterns were identical up to small statistical fluctuations. The mass of the wheel and its suspension strongly affects the pattern. Heavier wheels produce larger amplitude ripples with shorter wavelengths. Wheel diameter, however, seems to be unimportant, and this was tested by also using a non-rotating square *wheel*. Varying the size of the wheel in the simulation, while keeping its mass constant, leaves the pattern unchanged. The insensitivity of the pattern to the wheel and grain sizes raises interesting questions about the scaling of the washboard and are incorporated into the model discussed below.

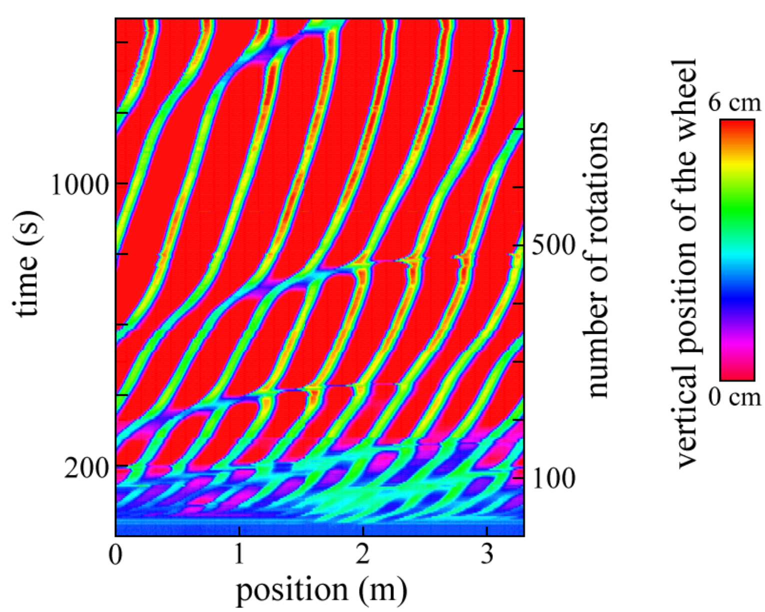

Figure 3. A space-time plot of the vertical position of the wheel for $v = 2$ m s^{-1}. Ripples form at a random position and rapidly spread to the whole circumference of the table. The amplitude of the vertical motion increases as the number of ripples decreases *via* merging events. A drift of the pattern in the driving direction is clearly visible.

5. COARSENING OF THE WASHBOARD PATTERN

We find that both the amplitude and wavelength of the ripples grow initially. They remain roughly proportional to one another as they evolve toward a saturated value which scales with the kinematic lengthscale v^2/g. Thus, as they spread around the table their amplitude and wavelength increases and merging events take place. Figure 3 is a typical space-time plot obtained experimentally with v=2 m/s. Our ripples always travel forwards, whereas ripples on real roads, with driven wheels, have sometimes been observed to travel in both directions [1]. Figure 4(a) shows these data for the run shown in Fig. 3. As the ripple amplitude saturates, the number of ripples drops from 14 to 7. The amplitude increases abruptly each time a ripple disappears. As the velocity is decreased, the ripples can split and the amplitude and wavelength then decreases, but this is not always seen. Figure 4(b) shows that the ripple drift velocity slows significantly as the wavelength increases, as one would expect.

6. INTERNAL STRUCTURE

Using DEM simulations, we can examine aspects of the internal structure of the ripples which are difficult to access experimentally. Fig. 5(a) shows that the slightly polydisperse grains remain well mixed inside the ripple. This demonstrates that size segregation is not crucial to the formation of the ripples, although it is almost certainly present on real roads [1]. Segregation would probably occur in the DEM simulation if the simulations were run for much longer.

The local packing fraction of a pile can be computed from its Voronoï tessellation. This algorithm partitions space into cells corresponding to individual particles. The area of each disk divided by the area of its cell can be interpreted as a local packing fraction. This is shown in Fig. 5(b). The grains located at the edge of the pile (in green) have an infinitely large, open Voronoï cell for which no packing fraction can be defined. The internal grains show no ripple-related structure in their packing, although the overall packing is denser than the initial state of the simulation. Thus, we conclude that varying compaction of the grains is also not essential to the formation of ripples. This contradicts a recent model [8], in

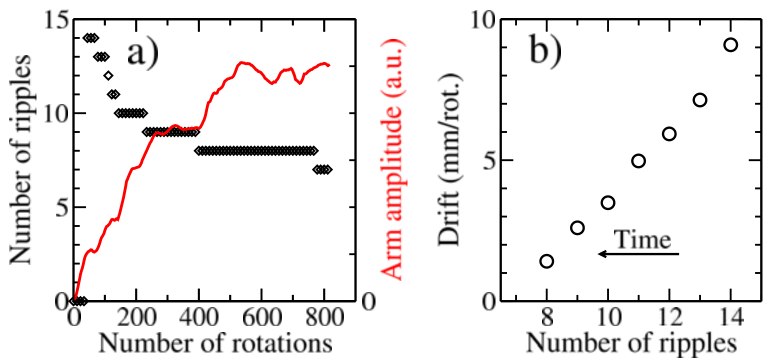

Figure 4. Experimental ripple dynamics from Fig. 3. (a) Time evolution of the number of ripples (diamonds) and of the amplitude of the ripples (solid line). (b) Drift velocity in mm/rotation, as a function of the number of ripples.

which compaction played a central role.

Fig. 5(c) shows the displacement of the grains caused by one pass of the wheel. The configuration is the state before the pass, and the color shows the horizontal distance those grains are about to travel. The displacement is localized to the crest of the ripples, which move on a static bed. Although individual particle displacements can be relatively large (up to 12 diameters), it typically takes 200 to 300 passes of the wheel for a ripple to travel a distance of one wavelength.

7. THEORY

While the dynamics of the wheel and its suspension are simple, the behavior of granular materials is very poorly understood. There is no continuum theory that can be reliably applied to this problem, though phenomenological models exist which can be implemented with large-scale finite-element codes [6]. On the other hand, existing simplified models [7, 8] fail to capture many of the effects we observe. We therefore advance a new model based on surface mass flux similar to some theories of ripples and dunes.

Stresses in a non-cohesive material are related to inertial forces or gravitational forces, so the only dimensional parameters are the 2D bed density ρ (*i.e.* the 3D density multiplied by the width of the wheel) and the acceleration due to gravity g. The experiments and simulations show that the particle size is not important, if it is small enough compared to the wheel and ripples. The primary non-dimensional parameter needed to describe the material is its approximate angle of repose θ_c. If the wheel is supported by buoyancy forces, the penetration of the wheel into the bed is determined by its weight m, which gives rise to a penetration length scale $L_1 = \sqrt{m/\rho}$. There is a secondary dependence on the radius of the wheel R through the ratio R/L_1. The simulations show that this dependence is very weak. The experiments with the non-rotating square wheel, where no such length scale is present, show that this is a secondary effect. The final significant dimensional parameter is v, the horizontal speed of the wheel.

This gives rise to a second penetration length scale $L_2 = mg/\rho v^2$, if the wheel becomes supported by dynamic pressure. The ratio L_1/L_2 behaves like a Froude number $\mathrm{Fr} = (v^2/g)\sqrt{\rho/m}$. This is the only non-dimensional group that can be formed from v, m, g and ρ, thus it should determine the stability of the wheel/bed interaction.

We hypothesize that the instability results from a switch between highly dissipative displacement supporting forces and weakly dissipative dynamic supporting forces. In the absence of any special spring/dashpot suspension, the saturated ripples involve ballistic trajectories, so the natural length scale for the ripple wavelength λ is the kinematic lengthscale $L = v^2/g$. Then $\lambda/L = f_\lambda(\mathrm{Fr})$, where f_λ is a relatively weak function of Fr. We expect the saturated amplitude of the

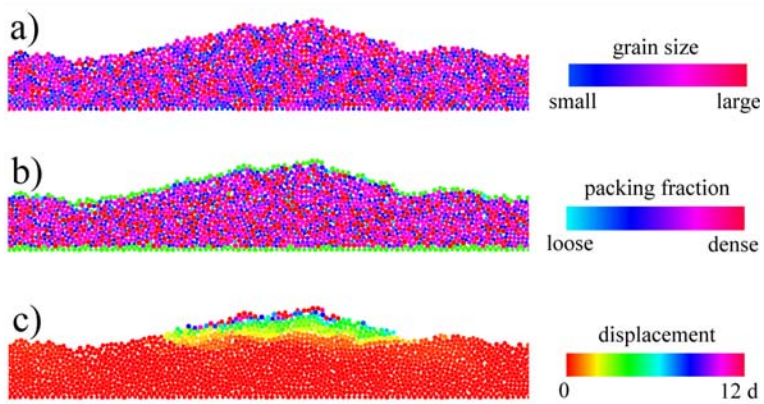

Figure 5. The internal structure of the ripples, using DEM simulation. (a) grain size, (b) local packing fraction, (c) displacement between two consecutive passes of the wheel. The ripples show no segregation and although the bed is on average more compact than initially, no domains of high packing fraction are found. The particle displacement is localized at the crests of the ripples.

ripples A to be determined by λ and the angle of repose θ_c, so that $A/L = f_A(\mathrm{Fr})$, with $f_A(\mathrm{Fr}) \propto \tan\theta_c$.

A continuum theory can be constructed by assuming that the increment to the deformation h of the bed is small with each pass of the wheel. If z is the height of the wheel relative to the bed and x is the horizontal coordinate, the acceleration of the wheel, when it is in contact with the surface, is $v^2(z+h)_{xx}$, and thus

$$mv^2(z+h)_{xx} = N - mg\cos\phi, \tag{1}$$

where ϕ is the mean angle of the suspension arm from the horizontal, N is the normal force. The bed height h obeys a mass conservation equation $h_t + q_x = 0$, where q, the volume flux, is given by

$$q = zv\ f_q\left(h_x; \left[\frac{zg}{v^2}\right], \left[\frac{Ng}{\rho v^4}\right]\right), \tag{2}$$

where f_q is a non-dimensional flux function that should diverge as $h_z \to \pm\tan\theta_c$, and which depends only on the local non-dimensional groups zg/v^2 and $Ng/\rho v^4$. The key to the model is the normal force

$$N = \begin{cases} 0 & z > 0 \\ -z\delta\rho v^2 + f_N(z_x)z^2\rho g & z < 0 \end{cases} \tag{3}$$

δ is a geometric factor giving the dynamic lift that may depend on z/R. $f_N(s)$ is a switch function which is zero for $s > 0$ and rapidly reaches a maximum value for $s < 0$, representing the indentation resistance of the bed.

This system can be non-dimensionalised using length scale v^2/g, timescale v/g and force scale $\rho v^4/g$ to give

$$z_{xx} + h_{xx} = N\mathrm{Fr}^2 - \cos\phi, \tag{4}$$

$$h_t + \left[z f_q(h_x; z, N)\right]_x = 0, \tag{5}$$

$$N = -z\delta + f_N(z_x)z^2 \quad z < 0. \tag{6}$$

This non-dimensionalisation does not depend on any wheel properties, which only enter through the Froude number. The crucial effect driving the instability lies in the normal force. For a roughly flat bed, $N \approx \cos\phi/\mathrm{Fr}^2$ so that $z \propto 1/\mathrm{Fr}^2$. Thus, as Fr increases, the $-z\delta$ term dominates over $f_N(z_x)z^2$. This is the key to the instability since $-z\delta$ is a conservative force so that the wheel can bounce back from the bed. It is this bounce that produces the phase lag between q and h which is necessary for instability.

8. CONCLUSION

In conclusion, in all our experiments and simulations (with various wheel and grain sizes, materials) ripples formed when the rotation rate was above a critical value, showing that the instability is very robust. We studied the coarsening process and measured the time evolution of the wavelength and the amplitude. DEM simulations have shown that neither compaction nor segregation processes are responsible for the instability, thus invalidating pre-existing theories. We suggested that the onset of this instability is controlled by the non-dimensional parameter $v^2/g\sqrt{\rho/m}$ although further experiments are needed to validate this model. Washboard road will no doubt continue to annoy drivers for as long as there are unpaved roads and wheels to roll over them. We hope to obtain a more general understanding of it as an interesting example of nonlinear pattern formation in granular media.

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