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FURTHER DEVELOPMENTS IN UNSTEADY COMPRESSIBLE VORTEX LATTICE METHOD IN TWO DIMENSIONAL MOTION

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Abstract. *Unsteady phenomena like flutter, buffeting, rapid maneuvers in flight and gust entry are usually modeled and studied by a theoretical treatment involving potential flow methods. The resulting equation from this approach is the governing differential equation for general unsteady, non-viscous, potential flow known as equation of wave motion. The disturbance, represented in this equation by the velocity potential, is propagated as wave which spreads at a rate equal to the local speed of sound. Linearization on the basis of small disturbances in a uniform stream of compressible fluid is made upon the equation by the procedure of retaining first order terms. Elementary solutions for this simplified equation recognized as primary extension of the concepts of source, sink, vortex and doublet, used together with boundary conditions associated with the governing equation, enables proper treatment for understanding and tackling unsteady aerodynamic problems. This article presents a procedure for obtaining a numerical solution of aerodynamics coefficients of a thin airfoil in arbitrary motion in a uniform, compressible, subsonic flow field. Distribution of vortex type elementary solutions of the wave equation is used together with a time function that schedules the vortex strength in time to represent in effect the arbitrary vortex moving along a chosen path. A field point is then influenced by the continuous disturbances generated by the vortex with a delay relative to the time of action of the same vortex. A fixed coordinate system in space relative to the body is chosen. The analytical vortex solution is presented, with a comparison of the free vortex type and the bound vortex type, together with the appropriate transformation variables needed to treat the problem.*

Keywords*: unsteady aerodynamics, vortex lattice, compressible flow.*

1. INTRODUCTION

This work is limited to considerations of unsteady potential flows. A thorough analysis, development and classification of the governing differential equations are here given. Theoretical methods for unsteady potential flows in subsonic regime are discussed and a numerical method is indicated to calculate the indicial response of a two dimensional thin profile. Two different methods for analyzing the induced velocity of a point vortex source in unsteady compressible flow are compared and discussed. To the knowledge of the authors, the presentation of the compressible bound vortex singularity, as shown here, has not yet been done. To understand the implications of the addition of time and its effects, a detailed discussion of the boundary conditions, mathematical considerations and transformations, as well as the modeling considerations are necessary. That stated, hereafter, the governing equations are presented with detailed mathematical steps, following overall suggestion by Garrick (1957).

2. BASIC EQUATIONS

2.1. Velocity Potential

Every point of the flow of which the vorticity is zero is said to be in irrotational motion. The condition is, therefore,

$$\nabla \times [q] = 0 \, , \tag{1}$$

being $[q]$ the velocity vector. This relation implies that a scalar velocity potential, ϕ, exists (Milne-Thompson, 1973). Thus when the motion is irrotational the velocity vector is the gradient of a scalar function of position,

$$[q] = \nabla\phi \text{ , and } \nabla \times [q] = \nabla \times (\nabla\phi) = 0 \, . \tag{2}$$

2.2. Continuity Equation

The general equation of continuity can then be stated as, where ρ is the fluid density, and t is time,

$$\frac{\partial\rho}{\partial t} + \nabla(\rho \cdot [q]) = \left(\frac{\partial}{\partial t} + [q]\cdot\nabla\right)\rho + \rho\nabla(\nabla\phi) = \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla^2\phi = 0 \, . \tag{3}$$

2.3. Euler Equation

When Euler's equation of motion is reduced to a single total differential equation and integrated, Lagrange's integral of Euler's equations of motion of an irrotational compressible fluid appears (Lamb, 1945), where p is pressure,

$$\frac{\partial\phi}{\partial t}+\frac{q^2}{2}+\int\frac{dp}{\rho}=0\,, \tag{5}$$

where a function has been included in ϕ, at most varying in time but constant in space. It is assumed that the fluid is barotropic, that is, pressure is a function of density only.

Introducing the acoustic relation, where a is the speed of sound, $a^2 = dp/d\rho$, and taking the gradient in Eq. (5),

$$\nabla\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right)=-\frac{1}{\rho}\cdot\nabla p=-\frac{1}{\rho}\cdot a^2\cdot\nabla\rho\,,\text{ or }-\frac{1}{\rho}\cdot\nabla\rho=\frac{1}{a^2}\cdot\nabla\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right). \tag{6}$$

Differentiating Eq. (5) relative to time,

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial t}=-\frac{a^2}{\rho}\frac{\partial\rho}{\partial t}\,,\text{ or }-\frac{1}{\rho}\frac{\partial\rho}{\partial t}=\frac{1}{a^2}\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial q^2}{2\partial t}\right). \tag{7}$$

Relations in Eq. (6) and Eq.(7) serve to eliminate the density from Eq. (3),

$$\nabla^2\phi=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}+[q]\cdot\nabla\rho\right)=-\frac{1}{\rho}\frac{\partial\rho}{\partial t}-\frac{[q]}{\rho}\nabla\rho\,, \tag{8}$$

Inserting Eq. (6) and (7),

$$\nabla^2\phi=\frac{1}{a^2}\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial q^2}{2\partial t}\right)+\frac{[q]}{a^2}\cdot\nabla\left(\frac{\partial\phi}{\partial t}+\frac{q^2}{2}\right)=\frac{1}{a^2}\left(\frac{\partial^2\phi}{\partial t^2}+[q]\cdot\nabla\frac{q^2}{2}+\frac{\partial q^2}{2\partial t}+\underbrace{[q]\cdot\frac{\partial[q]}{\partial t}}_{\frac{\partial q^2}{2\partial t}}\right), \tag{9}$$

$$\nabla^2\phi=\frac{1}{a^2}\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial q^2}{\partial[q]}\cdot\frac{\partial[q]}{\partial t}+[q]\cdot\nabla\left(\frac{q^2}{2}\right)\right). \tag{10}$$

Where introducing the relations $[q]\nabla(q^2/2)=[q]\cdot[q]\cdot\nabla[q]$ and $\nabla[q]=\nabla^2\phi$, the density is finally eliminated and the governing differential equation for general non-stationary, non-viscous, potential flow is obtained,

$$\nabla^2\phi=\frac{1}{a^2}\left(\frac{\partial^2\phi}{\partial t^2}+2\cdot[q]\cdot\frac{\partial q_c}{\partial t}+q^2\cdot\nabla^2\phi\right),\text{ or in the mnemonic form, }\nabla^2\phi=\frac{1}{a^2}\frac{D_c^2}{Dt^2}(\phi)\,, \tag{11}$$

where the subscript c indicates that q_c is constant in relation to the operations $\partial/\partial t$ and ∇.

The resulting Eq. (11) has taken, thus, similar form of the equation of wave motion, meaning that the disturbance represented by the velocity potential partakes of the local velocity and is propagated as wave which spreads at a rate equal to the local speed of sound.

In a body-fixed Cartesian coordinate system, where $[q]=q(u,v,w)$, the relations in Eq. (11) becomes,

$$\begin{aligned}&\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}+\left(1-\frac{w^2}{a^2}\right)\frac{\partial^2\phi}{\partial z^2}-2\cdot\frac{u\cdot v}{a^2}\frac{\partial^2\phi}{\partial y\partial x}-2\cdot\frac{u\cdot w}{a^2}\frac{\partial^2\phi}{\partial z\partial x}-2\cdot\frac{w\cdot v}{a^2}\frac{\partial^2\phi}{\partial y\partial z}\\&-\frac{2\cdot u}{a^2}\frac{\partial^2\phi}{\partial x\partial t}-\frac{2\cdot v}{a^2}\frac{\partial^2\phi}{\partial y\partial t}-\frac{2\cdot w}{a^2}\frac{\partial^2\phi}{\partial z\partial t}-\frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2}=0\end{aligned}\,. \tag{12}$$

2.4. Perturbation Velocity Potential

The assumption of small perturbation is now assumed to enable linearization of Eq. (11) by retaining first order terms. Linearization of the differential equation and further linearization of boundary conditions of the problem is a classical way of tackling unsteady aerodynamics problems. The simplifying assumptions will not consider the large disturbances occurrences in: including thickness effects, shock flow patterns, near-sonic flow disturbances and stagnation points.

Therefore, linearization on the basis of small disturbances in a uniform stream of a compressible fluid of velocity U consists of considering velocity disturbances small in comparison to U , a , and $U-a$, where $v<<U$, $w<<U$. Also, the pressure differences and density changes must be small in comparison with the main stream pressure and density, that is, $\Delta\rho$ and Δp are small in comparison to ρ_∞ and p_∞, where the infinite subscript refers to free stream conditions.

The value of the velocity is now considered to be a constant, U , in the uniform stream and a disturbance velocity potential, φ , is defined as, $\nabla\varphi=[q]-[U]$, leading to the following linear equation,

$$\frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t}+[U]\cdot\nabla\right)^2\varphi=\nabla^2\varphi\,. \tag{13}$$

When the disturbance potential is considered to correspond to that of a wing immersed in and creating small disturbances to a uniform stream flowing with velocity U in the x direction, in rectangular coordinates, Eq. (12) becomes,

$$\left(1-\frac{U^2}{a_\infty^2}\right)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-2\frac{U}{a_\infty^2}\frac{\partial^2\varphi}{\partial x\partial t}-\frac{1}{a_\infty^2}\frac{\partial^2\varphi}{\partial t^2}=0\,. \tag{14}$$

For $U=0$, the classical wave equation for the propagation of sound is obtained. It can also be transformed to the wave equation by a change of coordinates converting back to a space-fixed system. Using the Galilean transformation,

$$x=\xi+U\cdot t,\quad y=\eta,\quad z=\zeta,\quad t=t. \tag{15}$$

Then, in space fixed coordinates, the disturbance potential created by the wing, moving in the medium otherwise at rest, will take exactly the same form of the wave equation. So making the appropriate transformations,

$$\begin{cases}\dfrac{\partial^2\varphi}{\partial x^2}=\dfrac{\partial^2\varphi}{\partial\xi^2}\Rightarrow\left(1-\dfrac{U^2}{a_\infty^2}\right)\dfrac{\partial^2\varphi}{\partial x^2}=\dfrac{\partial^2\varphi}{\partial\xi^2}-\dfrac{U^2}{a_\infty^2}\dfrac{\partial^2\varphi}{\partial\xi^2}\\[2ex] \dfrac{\partial^2\varphi}{\partial x\partial t}=\dfrac{\partial^2\varphi}{\partial\xi^2}(1)(-U)+\dfrac{\partial^2\varphi}{\partial\xi\partial t}(1\cdot1)=-U\dfrac{\partial^2\varphi}{\partial\xi^2}+\dfrac{\partial^2\varphi}{\partial\xi\partial t}\Rightarrow\dfrac{-2U}{a_\infty^2}\left(\dfrac{\partial^2\varphi}{\partial x\partial t}\right)=\dfrac{2U^2}{a_\infty^2}\dfrac{\partial^2\varphi}{\partial\xi^2}-\dfrac{2U}{a_\infty^2}\dfrac{\partial^2\varphi}{\partial\xi\partial t}\\[2ex] \dfrac{\partial^2\varphi}{\partial t^2}=\dfrac{\partial^2\varphi}{\partial\xi^2}\cdot U^2+2\dfrac{\partial^2\varphi}{\partial\xi\partial t}(-U)+\dfrac{\partial^2\varphi}{\partial t^2}\cdot1\Rightarrow-\dfrac{1}{a_\infty^2}\left(\dfrac{\partial^2\varphi}{\partial t^2}\right)=-\dfrac{\partial^2\varphi}{\partial\xi^2}\dfrac{U^2}{a_\infty^2}+\dfrac{2U}{a_\infty^2}\dfrac{\partial^2\varphi}{\partial\xi\partial t}-\dfrac{1}{a_\infty^2}\dfrac{\partial^2\varphi}{\partial t^2}\\[2ex] \dfrac{\partial^2\varphi}{\partial y^2}=\dfrac{\partial^2\varphi}{\partial\eta^2}\\[2ex] \dfrac{\partial^2\varphi}{\partial z^2}=\dfrac{\partial^2\varphi}{\partial\zeta^2}\end{cases}\quad, \tag{16}$$

and adding Eq. (16),

Eq. (14) takes the classical wave equation,

$$\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}+\frac{\partial^2\varphi}{\partial\zeta^2}-\frac{1}{a_\infty^2}\frac{\partial^2\varphi}{\partial t^2}=0\,. \tag{17}$$

2.5. Boudary Conditions

The boundary conditions associated with the differential equations are mathematical formulations of several physical statements: surface Boundary Conditions (Total flow tangential to the surface – impenetrable), edge Conditions (Near sharp edges the flow is determined by alteration of the primary non-circulatory pattern. Kutta condition – finiteness of velocity; smooth flow as condition on the rear stagnation points; pressure remains continuous near and vanishes at the edge.), Wake Conditions (Vorticity with the bound circulation vanishes according to Helmholtz-Kelvin theorem. Shed wake remains where it is formed, floats along streamlines. Edge effects and rolling-up of the sheet and general movements at infinity are ignored.), Conditions at infinity (Uniform flow at infinity).

The surface boundary conditions implies that $DF/Dt = 0$, where, $F(x, y, z, t)$ is the moving surface boundary. The wing will be assumed to lie nearly in the x, y plane creating small disturbances to the main stream, that is, small thickness and small camber is considered to obtain the linearized form. So the surface can be expressed as, $F = z - Z_{u,l}(x, y, t) = 0$, where the lower subscripts indicates the upper and lower surfaces. The normal fluid velocity at the upper surface can then be written as,

$$w(x, y, Z_u, t) = \frac{\partial Z_u}{\partial t} + U\frac{\partial Z_u}{\partial x}. \tag{18}$$

By Taylor expansion and neglect of higher order terms the normal velocity will now be $w(x, y, 0, t)$, and the mean camber surface, the camber and the wing thickness are designated by, $z = Z(x, y, t)$, $2Z_c = Z_u + Z_l$, $2Z_t = Z_u - Z_l$, respectively. So, finally, the problems of camber and thickness can be separated out, and the normal velocity expressed for the lower and upper surface results as

$$w(x, y, \pm 0, t) = \frac{\partial Z_c}{\partial t} + U\frac{\partial Z_c}{\partial x} \pm \left(\frac{\partial Z_t}{\partial t} + U\frac{\partial Z_t}{\partial x}\right) = w_c \pm w_t . \tag{19}$$

Considering that the pulsating wing and the steady and unsteady flow associated with thickness have been separated out, it finally results that the profile in unsteady flow is represented by its mean camber surface, and the surface boundary condition is finally resumed as,

$$w_c(x, y, 0, t) = \frac{\partial Z_c}{\partial t} + U\frac{\partial Z_c}{\partial x}, \tag{20}$$

which is the normal velocity induced by the movement of the mean camber of the body. Since the normal velocity is symmetrical in relation to the mean camber, its value at $z = 0$ is $w = \left(\frac{\partial \varphi}{\partial Z}\right)_{Z=0}$.

The disturbance pressure field given by

$$-\frac{p}{\rho_\infty} = \frac{\partial \varphi}{\partial t} + U\frac{\partial \varphi}{\partial x} \tag{21}$$

is zero everywhere except at the lifting surface, where the pressure difference between lower and upper surface, $P = p_u - p_l$, positive downwards, is given by, since $p_u = -p_l$,

$$P(x, y, t) = p(x, y, +0, t) - p(x, y, -0, t) = -2\rho_\infty\left(\frac{\partial \varphi}{\partial t} + U\frac{\partial \varphi}{\partial x}\right). \tag{22}$$

So the mathematical Problem consist of determining the disturbance potential, φ, from the differential equations and subsequently the velocity, $[q]$, and pressure, $[p]$, fields. Boundary conditions must be specified at the surface, $Z = 0$, at infinity and at the trailing edge. On the plane $z = 0$, ahead of the wing,. $\varphi = 0$, $\partial\varphi/\partial t = 0$ and $\partial\varphi/\partial x = 0$. At the wing, the induced velocity is given by Eq. (20) and its value at $Z = 0$. After the wing the pressure, given in Eq. (21), is zero. So with the known normal velocity distribution, w, over the wing surface the unknown surface distribution of potential or pressure should be determined.

3. ELEMENTARY SOLUTIONS

Solutions appearing in relation to Green's Theorem and Kirchhoff's principle that satisfies Laplace's and the wave equations are of special significance and are elementary solutions recognized as extensions of the concepts of sources and doublets in hydrodynamics. And when velocity potential is involved it may be considered as flow singularities. So a discussion of Green's theorem, a form of divergence theorem, is demanding for the theory of the potential for boundary value problems in unsteady aerodynamics. Green's Theorem states that for any two functions φ and ψ which with their first derivatives are finite, single-valued, and continuous, the following relation is valid,

$$\int_V \left(\psi \cdot \nabla^2 \varphi - \varphi \cdot \nabla^2 \psi\right) dV = \int_S \left(\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n}\right) dS \, . \tag{23}$$

This expression reduces a volume integral into a surface integral where $\partial/\partial n$ denotes the outward normal to the surface S. Applying this theorem to the case of a potential, φ_p, at a field point P expressed in terms of the boundary surface values of φ (Laplace's solution in a volume V) and $\partial\varphi/\partial n$, and ψ the specially chosen function $1/r$, gives, as in Morino (1974) for both steady and unsteady cases,

$$\varphi_p = \frac{1}{4\pi} \int_S \left(\frac{\partial \varphi}{\partial n} \frac{1}{r} - \varphi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right) dS \, . \tag{24}$$

Solutions are now build up by integration of surface distributions of elementary unit source solutions and of the doublet solution with axis in the direction n, respectively,

$$\varphi_1 = -\frac{1}{4\pi \cdot r} \text{ and } \varphi_2 = -\frac{1}{4\pi} \frac{\partial}{\partial n}\left(\frac{1}{r}\right), \tag{25}$$

where r is the geometric distance from the field point (x, y, z) to the source disturbance (ξ, η, ζ), $r = \sqrt{(x-\xi)^2 + (y-\zeta)^2 + (z-\zeta)^2}$. The strength of the source is determined from the total efflux of fluid of unit density through a small sphere about the source.

For the wave equation the argument $t-(r/a_\infty)$ corresponding to the time of action of the disturbances whose influence reach the field point at time t, retarding the designation of the potential, is introduced,

$$\varphi_p = \frac{1}{4\pi} \int_S \left(\frac{1}{r} \left(\frac{\partial \varphi}{\partial n}\right)_{t-(r/a_\infty)} - \frac{\partial}{\partial n} \frac{\varphi\left[t-(r/a_\infty)\right]}{r} \right) dS \, , \tag{26}$$

where the fundamental solutions (Morino, 1974) corresponding to the source and doublet are , respectively

$$\varphi_1 = -\frac{1}{4\pi \cdot r} f\left(t - \frac{r}{a_\infty}\right) \text{ and } \varphi_2 = -\frac{1}{4\pi} \frac{\partial}{\partial n}\left(\frac{1}{r} f\left(t - \frac{r}{a_\infty}\right)\right). \tag{27}$$

The source solution represents spherical waves issuing from a disturbance in a medium at rest. The strength of the source and doublet are given by the arbitrary functions $f(t)$.

3.1. Uniformly Moving Sources

The method for resolving the fundamental differential equation with the corresponding fundamental solutions for the case of moving source strength consists of the scheduling of the source strength in Eq. (27) in time and position to represent the arbitrary source moving along a chosen path. Let the arbitrary function $f(t)$ in Eq. (27) be defined as an impulse occurring at time $t = \tau$, that is, $f(\tau)\delta(t-\tau)$, where $\delta(\tau) = 0$ for $\tau \neq 0$, and unit area for $\tau = 0$, called the Dirac's function.

Let a succession of impulses one following the other in a path act at points, given in space fixed coordinates by $\xi(\tau), \eta(\tau), \zeta(\tau)$. The effect in time t of all impulses that acts before time t is,

$$\varphi = -\frac{1}{4\pi} \int_{-\infty}^{t-r/a_\infty} \frac{f(\tau)}{r} \delta\left(t - \tau - \frac{r}{a_\infty}\right) d\tau \, . \tag{28}$$

The integral in Eq. (28) will have nonzero values only for values of τ that satisfies

$$t-\tau=\frac{r}{a_\infty}=\frac{1}{a_\infty}\sqrt{\left(x-\xi(\tau)\right)^2+\left(y-\eta(\tau)\right)^2+\left(z-\zeta(\tau)\right)^2}\ . \tag{29}$$

$t-\tau$ expresses the distance between the source point and the field point in terms of the time travel of the outgoing waves. When considering the case of uniform motion with velocity U in the negative x direction, let sources be located on the ξ axis and flow consecutively one after the other at positions, $\xi=-U\tau$, $\eta=0$, $\zeta=0$.

So the distance between source point and field point is

$$r=\sqrt{\left(x+U\tau\right)^2+\left(y\right)^2+\left(z\right)^2}\ . \tag{30}$$

Defining the variable θ as,

$$-\theta=t-\tau-\frac{r}{a_\infty}, \tag{31}$$

by the selective property of the pulse function Eq. (28) will have a value only for $\theta=0$, given by

$$\varphi=-\frac{1}{4\pi}\left(\frac{f(\tau)}{r}\frac{d\tau}{d\theta}\right)_{\theta=0}. \tag{32}$$

Equation (30) and Eq. (31), putting $\theta=0$, results in a quadratic equation for τ. Choosing the solution that leads to $\tau<t$, gives

$$\tau=\frac{1}{\beta^2}\left(t+\frac{Ux}{a_\infty^2}-\frac{R}{a_\infty}\right), \tag{33}$$

where

$$R=\sqrt{\left(x+Ut\right)^2+\beta^2\left(y^2+z^2\right)},\qquad \beta^2=1-M_\infty^2,\quad\text{and}\qquad M_\infty=\frac{U}{a_\infty}. \tag{34}$$

And also, taking the derivative in Eq. (31),

$$\frac{1}{r}\frac{d\tau}{d\theta}=\frac{a_\infty}{a_\infty\cdot r+U\left(x+U\tau\right)}. \tag{35}$$

Replacing r by its characteristic value, $r=a_\infty\left(t-r\right)$, and replacing τ by Eq. (33), the following relation is obtained,

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{1}{R}. \tag{36}$$

Hence, from Eq. (32),

$$\varphi=-\frac{1}{4\pi}\frac{1}{R}f\left(\left(t+\frac{Ux}{a_\infty^2}-\frac{R}{a_\infty}\right)\frac{1}{\beta^2}\right) \tag{37}$$

Expressing the results in terms of a field point x_0, y_0, z_0 of a coordinate system moving uniformly along with the source located at ξ_0, η_0, ζ_0, the following transformation is made,

$$x+Ut=x_0-\xi_0,\ y=y_0-\eta_0\ \text{and}\ z=z_0-\zeta_0. \tag{38}$$

After performing this substitution, the moving source solution is obtained, equivalent to the one obtained by Condon (1958), where R and D are called amplitude radius and phase radius respectively,

$$\varphi_1 = -\frac{1}{4\pi \cdot R} f\left(t - \frac{D}{a_\infty}\right) = -\frac{1}{4\pi \cdot R} f\left(t - \tau_1\right),$$

$$R = \sqrt{\left(x-\xi\right)^2 + \beta^2\left(y^2+z^2\right)} \quad \tau_1 = \frac{D}{a_\infty} = \frac{-M_\infty\left(x-\xi\right)+R}{a_\infty \cdot \beta^2} \quad \beta^2 = 1 - M_\infty^2 . \tag{39}$$

4. FREE VORTEX IN A COMPRESSIBLE TWO-DIMENSIONAL FLOW

In an incompressible flow the perturbation generated by a vortex filament is sensed immediately by the whole field. In a compressible flow the perturbation is sensed by a field point with a delay. The propagation occurs in all directions and travels with the same speed as the sound velocity of the flow, a_∞. Since the vortex is free, superimposed to this movement is the velocity of the flow, U, and the vortex moves in the direction of the flow velocity. Based on the equivalency of the three dimensional vortex filament with the punctual two dimensional vortex (Miranda, 2006) it is possible to define the time function of the delay necessary to account the perturbations of the infinite vortex filament. Figure 1 shows the instant that a perturbation generated at point A reaches the field point P.

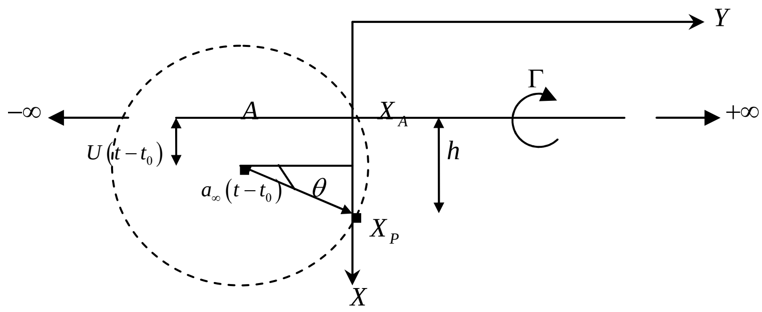

Figure 1. Diagram of the free vortex

From Fig. 1:

$$\cos\theta = \frac{\sqrt{\left[a_\infty\left(t-t_0\right)\right]^2 - \left[h - U\left(t-t_0\right)\right]^2}}{a_\infty\left(t-t_0\right)} . \tag{40}$$

The velocity induced in a given point by a filament vortex is given by the integration of the Biot-Savart law along the whole filament:

$$\overrightarrow{dW}_{P\overline{AB}} = -\frac{\Gamma}{4\pi} \cdot \frac{\overrightarrow{dl} \times \vec{r}}{\left|r\right|^3} = -\frac{\Gamma}{4\pi} \cdot \frac{r \cdot \sin\theta}{\left|r\right|^3} dl \Rightarrow W_{P\overline{AB}} = -\frac{\Gamma}{4\pi \cdot h} \cdot \left(\cos\theta_1 + \cos\theta_2\right), \tag{41}$$

Substituting Eq. (40) in Eq. (41),

$$W_{P\overline{AB}} = -\frac{\Gamma}{2\pi \cdot h} \cdot \frac{\sqrt{\left[a_\infty\left(t-t_0\right)\right]^2 - \left[h - U\left(t-t_0\right)\right]^2}}{a_\infty\left(t-t_0\right)} . \tag{42}$$

If $\left(t-t_0\right) = h/\left(U + a_\infty\right)$, $f\left(t\right) = 0$ .And if $\left(t-t_0\right) < h/\left(U + a_\infty\right)$, $f\left(t\right)$ is a complex number. This gives the influence of all perturbations that have reached the field point in a given instant. Figure 2 shows the values, for different Mach numbers, of the induced velocity $W_{P\overline{AB}}$ as a function of time, given $t_0 = 0; h = 10; \Gamma = 1; U = 1$, where these dimensional values can be used according to any appropriate dimension, for the results are dimensionless.

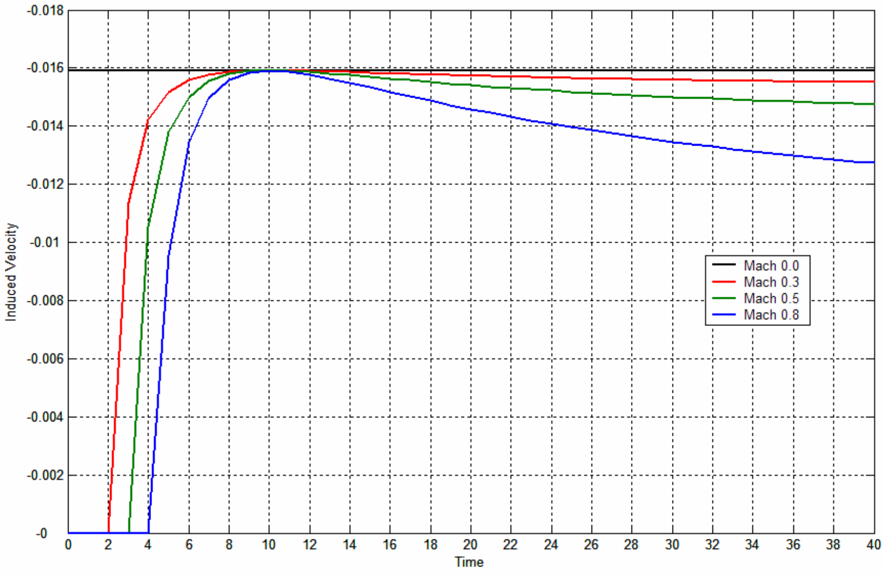

Figure 2. Time dependence of the Induced Velocity, in free vortex approach.

5. BOUNDED VORTEX IN A COMPRESSIBLE TWO-DIMENSIONAL FLOW

When the Galilean transformation is applied (Eq. 15) the vortex is now fixed to the new coordinate system and it is not free to move in the direction of the free stream velocity. The center of the vortex remains bounded as the perturbation propagates through the field. As time follows, perturbations generated in different positions of the new coordinate axis reaches a determined fixed point in the field. When a perturbation reaches this fixed point, the vortex that generated the perturbation starts contributing to the total induced velocity at that point. From Fig. 3 and Galilean transformation (Eq. 15),

$$\cos\theta = \frac{(y-y_a)}{\sqrt{\left[x-U(t-t_0)\right]^2+(y-y_a)^2}}, \tag{43}$$

and by Eq. (41),

$$W_t = -\frac{\Gamma}{4\pi}\frac{1}{x}\left[\frac{(y-y_a)}{\sqrt{\left[x-U(t-t_0)\right]^2+(y-y_a)^2}} - \frac{(y-y_b)}{\sqrt{\left[x-U(t-t_0)\right]^2+(y-y_b)^2}}\right]. \tag{44}$$

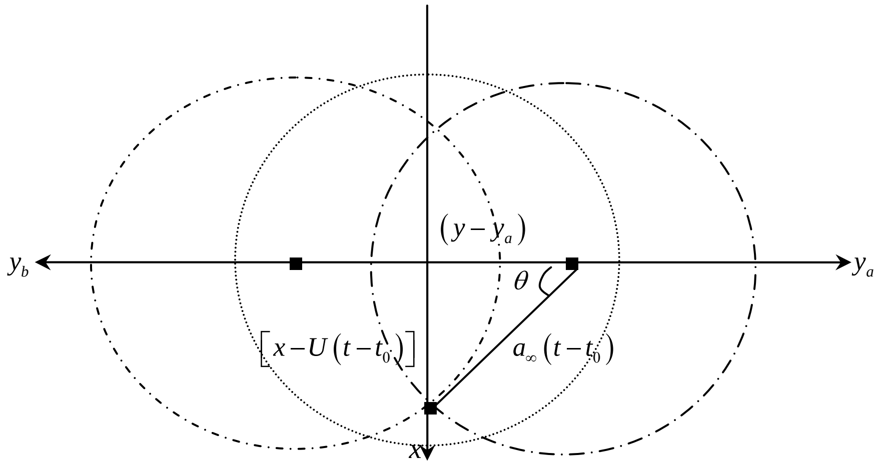

Figure 3. Diagram of the bound vortex

Also from Fig. 3,

$$(y-y_a)=\sqrt{a_\infty^2(t-t_0)^2-[x-U(t-t_0)]^2}, \text{ and } (y-y_b)=\sqrt{a_\infty^2(t-t_0)^2-[x-U(t-t_0)]^2}. \quad (45)$$

So,

$$W_t=-\frac{\Gamma}{4\pi}\frac{\beta}{x}\left[\frac{\beta(y-y_a)}{\sqrt{[x^2+\beta(y-y_a)]^2}}-\frac{\beta(y-y_b)}{\sqrt{[x^2+\beta(y-y_b)]^2}}\right]=-\frac{\Gamma}{2\pi}\frac{\beta}{x}\left[\frac{\beta(y_a)}{\sqrt{[x^2+\beta(y_a)]^2}}\right]=-\frac{\Gamma}{2\pi}\frac{\beta}{x}\frac{\beta y_a}{\sqrt{x^2+\beta^2y_a^2}}, \quad (46)$$

For the bounded vortex, all coordinates transformations applied in section 2 of this article are valid and the time necessary for the perturbation to reach a certain field point may be expressed by Eq. (39), reminding that applying for the case where a determined field point lies in the x axis, that is, $z-\zeta=0$, $y-\eta=\eta=y_a$ and $x-\xi=x$,

$$\tau_1=\tau=\frac{-M_\infty(x-\xi)+R}{a_\infty\cdot\beta^2}=\frac{-M\cdot x+\sqrt{x^2+\beta^2y_a^2}}{a\beta^2}, \quad (47)$$
$$(a\beta^2\tau+M\cdot x)^2-x^2=\beta^2y_a^2.$$

So,

$$W_t=-\frac{\Gamma}{2\pi}\frac{\beta}{x}\frac{\beta y_a}{\sqrt{x^2+\beta^2y_a^2}}=-\frac{\Gamma}{2\pi}\frac{\beta}{x}\frac{\sqrt{(a\beta^2\tau+M\cdot x)^2-x^2}}{a\beta^2\tau+M\cdot x}. \quad (48)$$

Figure 4 shows the values, for different Mach numbers, of the induced velocity W_t as a function of time, given $t_0=0; x=10; \Gamma=1; U=1$, valid for any appropriate dimension, since the results are dimensionless.

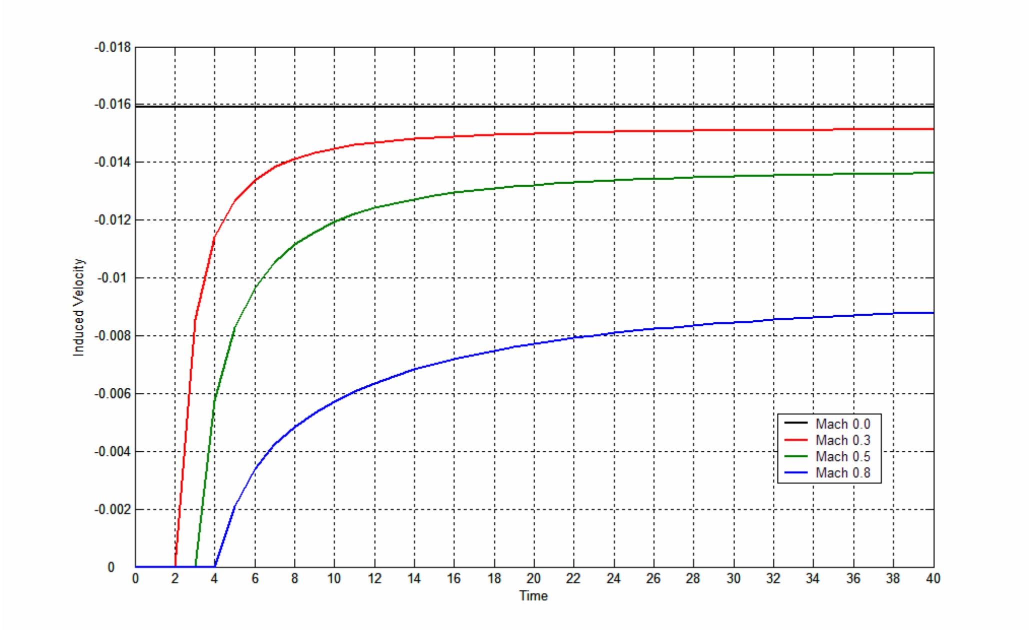

Figure 4. Time dependence of the Induced Velocity, in bound vortex approach.

When the bounded vortex approach is used, the convergence for the value for the induced velocity is faster than the free vortex approach, as seen in Fig. 5. Both methods converges to the same value of induced velocity and are valid for applications. Comprehension of both elucidates the mechanism of propagation of vortices at compressible subsonic regime.

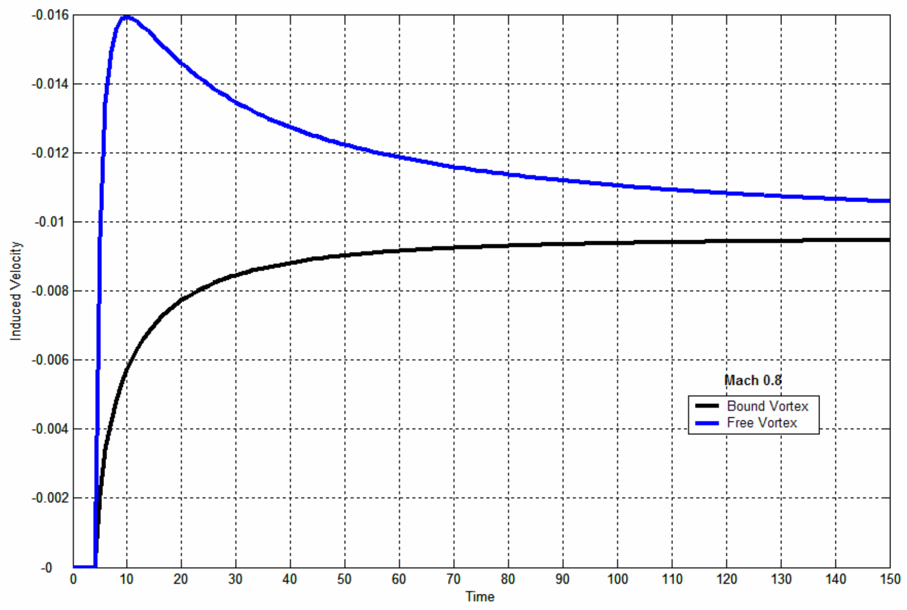

Figure 5. Comparison between bounded and free vortex approach.

A numerical solution for the aerodynamics coefficients of a thin airfoil in arbitrary motion in a uniform, compressible, subsonic flow field can now be obtained. The change in circulation that accompanies all changes in the profile's motion can be modeled by shedding vortices from the trailing edge of the profile. The values for vortices, used in the numerical calculation, should be determined considering the results for the induced velocity explained in this article. Detailed numerical steps for free vortex approach are shown in Lavagna (1992) and Hernandes (2003). A future work by the authors of this article will develop the numerical steps for the bounded vortex.

5. CONCLUDING REMARKS

Results and considerations of the bound vortex approach here presented inserts in the context of the work on unsteady aerodynamics and the elementary solutions used in the compressible vortex lattice method. The results shown in this article are a further step to understanding potential flow methods in unsteady aerodynamics. Comparisons made between different approaches regarding unsteady two-dimensional vortex formation are given here to enlighten the difference between free and bounded vortex formation. The results represent the theoretical values for unsteady flows and do not require further validation. Detailed numerical steps and overall comparison of unsteady aerodynamics coefficients are left to presentations in future articles.

6. REFERENCES

Condon, E. U., Odishaw, H., 1958, "Handbook of Physics". McGraw-Hill Book Company, inc., NY.pp. 3-117.

Garrick, E., 1957 "Aerodynamic Components of Aircraft at High Speeds", section F "Nonsteady Wing Characteristic", Prinston University press, New Jersey, pp. 658-680.

Hernandes, F. and Soviero, P.A.O., 2003, "Modelo Numérico para Perfis Finos em Escoamento Compressível Não Permanente", Master Thesis, Instituto Tecnológico de Aeronáutica, São José dos Campos, SP, Brazil.

Lamb, H.,1932 "Hydrodynamics", Ed. Dover, NY, pp 2-8.

Lavagna, L.G.M. and Soviero, P.A.O., 1992, "Aerodinâmoca da Placa Plana em Movimento Não-Permanente", Master Thesis, Instituto Tecnológico de Aeronáutica, São José dos Campos, SP, Brazil.pp. 332-340.

Miranda, F. I. and Soviero, P.A.O., 2006, "Resposta Indicial de Asas Finitas em Escoamento Compressível Subsônico", Master Thesis, Instituto Tecnológico de Aeronáutica, São José dos Campos, SP, Brazil. pp. 49-55.

Morino, L., 1974, "A general Theory of Unsteady Compressible Potential Aerodynamics", NASA CR-2464, National Aeronautics and Space Administration, Washington, D.C., December 1974. pp. 8-15.

Milne-Thompsom, L.M.,1973, "Theoretical Aerodynamics", Ed. Dover, NY,pp. 52.