# COMPARISON OF THE KARHUNEN-LOÈVE MODES OBTAINED FROM DISPLACEMENT, VELOCITY AND ACCELERATION RANDOM FIELDS

**Sergio Bellizzi, bellizzi@lma.cnrs-mrs.fr**
Laboratoire de Mécanique et d'Acoustique, CNRS, 31 chemin Joseph Aiguier, 13402 Marseille, France

**Rubens Sampaio, rsampaio@mec.puc-rio.br**
Mechanical Engineering Department – PUC-Rio, Rua Marquês de São Vicente, 225, Gávea, RJ, Brasil - CEP: 22453-900

***Abstract.***
*The Karhunen-Loève (KL) theory establishes that a second-order random field can be expanded as a series involving a sequence of deterministic orthogonal functions with orthogonal random coefficients. The KL theory can be applied to the responses of randomly excited vibrating systems with a view to performing a decomposition in separate variable (time and space) form giving a modal analysis tool. In this paper, an averaging operator involving time and ensemble averages is used to draw up the KL theory. This averaging operator can be applied in stationary cases as well as non-stationary (transient) ones. The KL modes obtained from the displacement field, velocity field, and acceleration field are compared. Stationary as well as transient (non stationary) cases will be considered. The physical interpretation of the KL modes will be also investigated.*

***Keywords:*** *Karhunen-Loève theory, modal analysis, nonlinear random vibrations*

## 1. INTRODUCTION

The Karhunen-Loève (KL) decomposition establishes that a second-order random field can be expanded as a series involving a sequence of deterministic orthogonal functions with orthogonal random coefficients. The deterministic functions, which are also called Karhunen-Loève modes, are the eigenfunction solutions of the Fredholm integral equation, the kernel of which is the autocorrelation (or autocovariance) function of the random field under study. This expansion was developed in the forties by several authors (Loève (1945), Karhunen (1947), and others). It was subsequently investigated and used in many branches of engineering science. Depending on the properties of the random field under study, the use of the expansion, and/or the field of application, this expansion has been given under different names such as Principal Component Analysis (PCA), and Proper Orthogonal Decomposition (POD).

When the term POD is used to denote an expansion, it generally refers to a characterization of the signal based on experimental data. The POD also involves detecting spatially coherent modes in the dynamics of a spatio-temporally varying system by diagonalizing the spatial covariance function of data with respect to an averaging operation (Lumley, 1971) (Sirovich, 1987). In the case of random fields, the averaging operation is taken to be the ensemble average and the POD expansion is called the KL expansion. In the case of spatio-temporal data (not necessarily random ones), the averaging operation is focused typically on the time average. As illustrated in (Graham and Kevrekides, 1996) (Atwell and King, 2001), this is not the only possibility and, when data correspond to a random-response process, the stationarity in time and the ergodicity are required to relate the time average to the ensemble average, or mean operator.

The KL expansion is one of the main tools used to develop the stochastic finite elements method (Ghanem end Spanos, 2003). It is also one of the techniques used to simulate random fields when they are specified by their covariance function and their marginal density probability (Poirion and Soize, 1999) (Ghanem and Spanos, 2003). In vibration analysis the KL modes, or Proper Orthogonal Modes (POMs), advantageously replace the Linear Normal Modes (LNMs) of the underlying linear system (see for example (Steindl et al., 1997) (Ma and Vakakis, 1999) (Trindade et al., 2005)).

The physical interpretation of POMs has also been investigated. These modes have been related to the LNMs of multi-modal free responses of discrete symmetrical systems (Feeny and Kappagantu, 1998), (Kerschen and Golinval, 2002). In these cases, time averaging has been used as the averaging operation in the POD method. Conservative linear systems (discrete and continuous) under random excitation have been studied in (Feeny and Liang, 2003). Linear discrete mechanical systems subjected to Gaussian white-noise excitation have also been addressed in (Kerschen and Golinval, 2002) (Kerschen and Golinval, 2004). In (Bellizzi and Sampaio, 2006), discrete and continuous mechanical systems are studied in the context of stationary as well as transient (non stationary) responses. An averaging operator involving time and ensemble averaging was introduced to obtain the KL expansion in separate variable form from the associated KL expansion.

The purpose of this paper is to compare the KL modes obtained from the displacement field, velocity field, and acceleration field. The displacement, the velocity and the acceleration fields are directly measurable Using the averaging operator involving time and ensemble averaging, stationary as well as transient (non stationary) cases will be considered.

## 2. KARHUNEN-LOÈVE EXPANSION FOR RANDOM VIBRATIONS

In vibration problems, it is often necessary to expand the vibratory field as a series in separate variables (time and space)

$$u(t,\mathbf{x})=\sum_{k=1}^{\infty}a_k(t)\phi_k(\mathbf{x}) \tag{1}$$

where $\phi_k$ are vector functions, and $a_k$ are scalar functions. This expansion, which includes the classical modal expansion, can be used, for example, to analyse the behaviour of the system or to reduce the order of the model of dynamical systems.

In case of random vibrations, that is when the vibratory field is assumed to be a stochastic field, $\{u(z)\}_{z\in\mathcal{D}}$ in which the domain $\mathcal{D}=\mathcal{D}_t\times\mathcal{D}_{\mathbf{x}}\subset\mathbb{R}\times\mathbb{R}^p$ (with $p=1,2,$ or 3) and $z=(t,x)$, the objective is to expand the vibratory field as a series in separate variables

$$u(t,\mathbf{x})=\sum_{k=1}^{\infty}a_k(t)\phi_k(\mathbf{x}) \tag{2}$$

where $\phi_k$ are deterministic vector functions, and $\{a_k(t)\}_{t\in\mathcal{D}_t}$ are scalar random processes. Usually, $\mathcal{D}_t$ defines the time interval of interest and, without loss of generality, we assume in the sequel that $\mathcal{D}_t=[0,T]$ where $T\in\mathbb{R}^+$.

The KL theory as described in (Bellizzi and Sampaio, 2006) can be used to build series expansions (2) from the correlation function $\mathbf{R}_u(t,t,\mathbf{x},\mathbf{x}')$ of the random field $\{u(t,x)\}_{(t,x)\in\mathcal{D}_t\times\mathcal{D}_x}$. Two cases can be considered depending on the time stationary properties of the random field.

The notations used in this work is the same as in (Bellizzi and Sampaio, 2006). The main points are as follows.

Let $\mathcal{D}$ be a compact subset of $\mathbb{R}^l$ and $\{X(z)\}_{z\in\mathcal{D}}$ a second order stochastic field defined on a probability space $(\Omega,\mathcal{F},P)$ with values in $\mathbb{R}^d$. This random field is a $l$-parameter family on real valued vector, $X(z,\theta)$ for $(z,\theta)\in\mathcal{D}\times\Omega$. Let $L^2(\Omega,\mathbb{R}^d)$ be the Hilbert space of the second-order random vector variables defined on the probability space $(\Omega,\mathcal{F},P)$ with the inner product

$$<\mathbf{Y},\mathbf{Z}>_\Omega=\int_\Omega<\mathbf{Y}(\theta),\mathbf{Z}(\theta)>dP(\theta)=E(<\mathbf{Y},\mathbf{Z}>) \tag{3}$$

where $<.,.>$ denotes the Euclidian inner product in $\mathbb{R}^d$, $dP(\theta)$ is the probability measure, and $E(.)$ denotes the mean, or ensemble average, with respect to the probability measure $P$. The stochastic field can be regarded as a curve in $L^2(\Omega,\mathbb{R}^d)$.

### 2.1 KL expansion based on ensemble averaging

If the covariance function $\mathbf{R}_u(t,t,\mathbf{x},\mathbf{x}')$ does not depend on $t$, the KL expansion given by the classical KL theory takes the form (the equality is achieved in $L^2(\Omega,\mathbb{R}^d)$)

$$u(t,\mathbf{x})=\sum_{k=1}^{\infty}\xi_k(t)\psi_k(t,\mathbf{x})\ \text{ where the functions } \psi_k \text{ solve } \int_{\mathcal{D}_{\mathbf{x}}}\mathbf{R}_u(t,t,\mathbf{x},\mathbf{x}')\psi_k(t,\mathbf{x}')d\mathbf{x}'=\lambda_k\psi_k(t,\mathbf{x}). \tag{4}$$

The functions $\psi_k$ do not depend on the time variable and hence expansion (4) is in the separate variable form and it is optimal to represent the random field $\{u(t,\mathbf{x})\}_{\mathbf{x}\in\mathcal{D}_{\mathbf{x}}}$ for fixed $t\in\mathcal{D}_t$ in the sense that the error term

$$E(\|u(t,.)-\sum_{k=1}^{p}\xi_k(t)\psi_k(.)\|^2_{\ll\gg}) \tag{5}$$

is minimum for each fixed integer $p$. This case corresponds to the well known time stationary case and the function $\psi_k$ are called KL modes..

### 2.2 KL Expansion Based on Time and Ensemble Averaging

If $\{u(z)\}_{z\in\mathcal{D}}$ is not time stationary, a different averaging operator can be used to developp the KL theory.

First, we consider the Hilbert space $\mathcal{L}^2_T(\mathcal{D}_t\times\Omega,\mathbb{R}^d)$ with the inner product given by

$$\prec Y,Z\succ=\mathcal{E}(<Y,Z>)\text{ with }\mathcal{E}(.)=\frac{1}{T}\int_0^T E(.)dt. \tag{6}$$

We can next define the correlation function of the random field $\{u(.,\mathbf{x})\}_{x\in\mathcal{D}_x}$ as

$$\mathcal{R}_u(\mathbf{x},\mathbf{x}')\quad=\quad\mathcal{E}((u(.,\mathbf{x}))(u(.,\mathbf{x}'))^T) \tag{7}$$

which are only spatial variable dependant. Finally, the random field $\{u(.,\mathbf{x})\}_{\mathbf{x}\in\mathcal{D}x}$ can be expanded (using the same arguments as in (Bellizzi and Sampaio, 2006)) as (the equality is achieved in $\mathcal{L}^2_T(\mathcal{D}_t\times\Omega,\mathbb{R}^d)$)

$$\forall x\in\mathcal{D}_x,\ u(t,\mathbf{x})=\sum_{k=1}^{\infty}\xi_k(t)\psi_k(\mathbf{x})\text{where the functions }\psi_k\text{ solve }\int_{\mathcal{D}_\mathbf{x}}\mathcal{R}_u(\mathbf{x},\mathbf{x}')\psi_k(\mathbf{x}')d\mathbf{x}'=\lambda_k\psi_k(\mathbf{x}) \tag{8}$$

and $\{\xi_1(t)\}_{t\in\mathcal{D}_t},\{\xi_2\}_{t\in\mathcal{D}_t},\cdots,\{\xi_m\}_{t\in\mathcal{D}_t},\cdots$ are scalar random processes given by $\xi_k(t)=\int_{\mathcal{D}_x}<u(t,\mathbf{x})-m_u(\mathbf{x}),\psi_k(\mathbf{x})>d\mathbf{x}$ with the following orthogonal properties $\mathcal{E}(\xi_{k_1}\xi_{k_2})=0$ if $k_1\neq k_2$ and $\mathcal{E}(\xi_k^2)=\lambda_k$. As in the classical case, the eigenvalues, $\lambda_k$, are related to the mean "energy" of the random field according to the following relation

$$\mathcal{E}(\|u\|^2_{\ll\gg})=\sum_{k=1}^{\infty}\lambda_k. \tag{9}$$

The functions $\psi_k$ will be called $T$-mean KL modes.

Note that if the random field $\{u(t,\mathbf{x})\}_{(t,\mathbf{x})\in\mathcal{D}_t\times\mathcal{D}_x}$ is weakly stationary with respect to the time variable (i.e. if $\mathbf{R}_u(t,t',\mathbf{x},\mathbf{x}')=\mathbf{R}_u(t-t',\mathbf{x},\mathbf{x}')$) then the $T$-mean KL modes does not depend on the parameter $T$ and coincide with the KL modes.

### 2.3 $T$-mean KL modes in practice

The estimation of the $T$-mean KL modes can be obtained from a time and spatial sampling of the random field $\{u(t,\mathbf{x})\}_{(t,,\mathbf{x})\in\mathcal{D}_T\times\mathcal{D}_\mathbf{x}}$. Let consider $x_1,x_2,\cdots,x_N$, $N$ spatial points ($\in\mathcal{D}_\mathbf{x}$) where the random field is sampled in time $M$ times on $[0,T]$ at $t_m=m\Delta t$ with $m=1,\cdots,M$ ($\Delta t$ is the sampling period) and for $R$ independant random events $\theta_r$ for $r=1,\cdots,R$.

Introducing the centered discret field $\mathbf{V}_N(t_m,\theta_r)=\mathbf{U}_N(t_m,\theta_r)-\frac{1}{R}\sum_{s=1}^{R}\mathbf{U}_N(t_m,\theta_s)$, in which $\mathbf{U}_N(t_m,\theta_r)$ is a $dN$-vector line defined by $\mathbf{U}_N(t_m,\theta_r)=\left(u(t_m,\mathbf{x}_1,\theta_r)^T\cdots u(t_m,\mathbf{x}_N,\theta_r)^T\right)$, the spatial covariance matrix having the dimensions $dN\times dN$ can be written as

$$\mathbf{R}=\frac{1}{MR}\mathbf{V}^T\mathbf{V}\text{ where }\mathbf{V}=\begin{bmatrix}\mathbf{V}_{N,M}(\theta_1)\\ \mathbf{V}_{N,M}(\theta_2)\\ \vdots\\ \mathbf{V}_{N,M}(\theta_R)\end{bmatrix}\quad\text{with}\quad\mathbf{V}_{N,M}(\theta_r)=\begin{bmatrix}\mathbf{v}_N(t_1,\theta_r)\\ \mathbf{v}_N(t_2,\theta_r)\\ \vdots\\ \mathbf{v}_N(t_M,\theta_r)\end{bmatrix}. \tag{10}$$

The dimensions of the resulting matrix $\mathbf{R}$ depend neither on the number of realizations $R$ nor on the number of samples $M$.

The $T$-mean KL modes are then approximated at the $N$ spatial points, $\mathbf{x}_n$, by the eigenvectors of $\mathbf{R}$ (which are orthogonal due to the symmetry of $\mathbf{R}$). The quality of the approximation with respect to the continuous formulation (8) depends on the number of spatial points $N$, the sampling period $\Delta t$, and the number of independant events $R$ used. The influence of the parameters will not be discussed here.

From the theoretical point of view, this approach does not require any assumption of stationarity nor ergodicity properties, but in pratice it is not generally possible to measure the vibration for different random events. So without time ergodic assumption, the procedure can only be reasonable implemented from a numerical model and Monte Carlo method.

In case of time ergodic assumption, only a long time trajectory is enough to do the calculations ($R=1$) and the $T$-mean KL modes, which are independent of $T$, can be approximated as the eigenvectors of

$$\mathbf{R}=\frac{1}{M}\mathbf{V}^T\mathbf{V}\text{ in which }\mathbf{V}=\begin{bmatrix}\mathbf{V}_N(t_1)\\ \mathbf{V}_N(t_2)\\ \vdots\\ \mathbf{V}_N(t_M)\end{bmatrix}\text{ and }\mathbf{V}_N(t_m)=\mathbf{U}_N(t_m,\theta_1)-\frac{1}{M}\sum_{k=1}^{M}\mathbf{U}_N(t_k,\theta_1). \tag{11}$$

The expressions (11) is usually used to define the POM when the time averaging is considered as the averaging operator. In this case the KL expansion is optimal only for the data used, whereas in the stochastic time ergodic case the KL expansion is valid for all the events.

## 3. $T$-MEAN KL MODES OF VIBRATORY RANDOM FIELDS

Vibration analysis using $T$-mean KL theory can be independently developed from the displacement field, the velocity field, the acceleration field, and also from displacement-velocity field, as done in (Bellizzi and Sampaio 2007). It is essential then to understand the relationship among the $T$-mean KL modes obtained using displacement, velocity, or acceleration fields, as well as the relation between the $T$-mean KL modes and the Linear Normal Modes (LNM).

### 3.1 Linear case

The linear case has been considered from the point of viewed of displacement fields in many papers in particular for the KL modes (stationary case). Considering the $T$-mean modes, the main result, see (Bellizzi and Sampaio, 2006), is that for a discret damped mechanical system under white noise excitation, the $T$-mean KL modes obtained from the random displacement field coincide with the linear normal modes if the modal matrix diagonalizes the damping matrix and the covariance matrix of the excitation. In (Bellizzi and Sampaio, 2007), this result is extended to $T$-mean KL modes obtained from the random displacement-velocity field.

### 3.2 Nonlinear case

One rather interesting known result is the relation between KL modes defined from the response of the nonlinear system and tke KL modes defined from the response of the equivalent linear system obtained using the method of statistical linearization as described in (Roberts and Spanos, 1990).

Let us consider the nonlinear system

$$\dot{\mathbf{Z}}(t) = \mathbf{G}(\mathbf{Z}(t)) + \mathbf{F}(t) \tag{12}$$

with external random excitation. A suitable equivalent linear system relationship between $\mathbf{Z}(t)$ and $\mathbf{F}(t)$ can be written as follow

$$\dot{\mathbf{Z}}(t) = \mathbf{L}_{eq}\mathbf{Z}(t) + \mathbf{F}(t) \tag{13}$$

where the matrix constant $\mathbf{L}_{eq}$ is determined by

$$\min_{\mathbf{L}} E(\parallel \mathbf{G}(\mathbf{Z}(t)) - \mathbf{L}\mathbf{Z}(t) \parallel^2). \tag{14}$$

For the non-linear system (12), when there exists a stationary, ergodic probability measure, it can be shown (Kozin, 1987) that the stationary covariance matrix of the nonlinear response (12) is identical to the stationary covariance matrix of the equivalent linear response (13).

Assuming the existence of stationary response to (12), the KL modes obtained from the stationary response of the non-linear system agree with the KL modes obtained from the stationary response of the equivalent linear system.

We will now consider the case of transient (or non-stationary) responses. Let consider the non-linear system

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{G}(\mathbf{U}(t)) = \mathbf{F}(t),\ t \in [0, T] \tag{15}$$

$$\mathbf{U}(0) = \mathbf{U}_0,\ \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0. \tag{16}$$

A suitable equivalent linear system relationship between $\mathbf{U}(t)$ and $\mathbf{F}(t)$ can be written as follows:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}_{eq}\mathbf{U}(t) = \mathbf{F}(t),\ t \in [0, T] \tag{17}$$

where the constant matrix $\mathbf{K}_{eq}$ is determined by

$$\min_{\mathbf{K}} \mathbb{E}(\parallel \mathbf{G}(\mathbf{U}(.)) - \mathbf{K}\mathbf{U}(.) \parallel^2) \tag{18}$$

with $\mathbb{E}(.) = \frac{1}{T}\int_0^T E(.)dt$. This criterion differs from the stationary one given by (14). It can be used to obtain an equivalent linear system with a constant matrix. This linearization method differs from that described in [16] in the case of non-stationary responses where the equivalent linear system is a time-varying linear system.

As in (Roberts and Spanos, 1990), the condition required to obtain optimum can be written as follows

$$\mathbb{E}(\mathbf{U}(.)\mathbf{U}^T(.))\mathbf{K}_{eq}^T = \left[\mathbb{E}(G_1(\mathbf{U}(.))^T\mathbf{U}(.)) \cdots \mathbb{E}(G_d(\mathbf{U}(.))^T\mathbf{U}(.))\right] \tag{19}$$

where $\mathbf{G}(\mathbf{U}) = (G_1(\mathbf{U})G_2(\mathbf{U}) \cdots G_d(\mathbf{U}))^T$.

It is then interesting to ask when the $T$-KL modes obtained from the non-linear transient response (15) agree with the $T$-KL modes obtained from the transient response of the equivalent linear system (17).

## 4. APPLICATION TO CONTINUOUS SYSTEMS

As an example we will discuss a continuous beam, with two types of boundary conditions, with a pure nonlinear restoring force, $F(t, z) = Du(t, z_f)^3\delta(z - z_f)$, with D a material constant, $u(t, z_f)$the displacement at $z = z_f$, and a localized excitation force $\delta(z - z_e)f(t)$. The beam model is is reduced to a truncated series

$$u(t, z) = \sum_{i=1}^{d} \varphi_i(z)x_i(t) \tag{20}$$

where $\varphi_i(z)$ denote the modal functions (with $\int_0^L \varphi_i(z)\varphi_j(z)dz = \delta_{ij}$ with $L$ denotes the length of the beam) and $x_i(t)$ denote the modal components. We assume that the modal components solve the following second order differential equations

$$\ddot{\mathbf{X}}(t) + \boldsymbol{\Theta}\dot{\mathbf{X}}(t) + \boldsymbol{\Omega}^2\mathbf{X}(t) + \lambda \left( \sum_{i=1}^{d} \varphi_i(z_f)x_i(t) \right)^2 \boldsymbol{H}\mathbf{X}(t)) \quad = \quad \mathbf{G}(t),\ t \in [0,T] \tag{21}$$

where $\boldsymbol{\Omega}^2 = \text{diag}(\omega_i^2)$, $\boldsymbol{\Theta} = \text{diag}(2\tau_i\omega_i)$ ($\omega_i$ denote the modal frequencies and $\tau_i$ the associated modal damping ratios), $\boldsymbol{H}$ is a $d \times d$ matrix with general term $\varphi_i(z_f)\varphi_j(z_f)$ and the component of the modal excitation vector $\mathbf{G}(t)$ are related to the physical excitation by $g_i(t) = \varphi_i(z_e)f(t)$. We will assume that, $\{f(t)\}_{\mathcal{D}}$ is a white-noise random process. Note that the covariance matrix of the modal exciation being not diagonal, even in the linear case, the KL expansion could differ from the modal expansion.

The displacement, velocity and acceleration fields were obtained from (20) solving, from given excitation histories $f(t)$, equation (21) numerically (using the Newmark method). The excitation histories were simulated using the procedure described in (Poirion and Soize, 1988).

Given a spatial discretisation $z_k = k\Delta z$ for $k = 1, \cdots, N$ with $\Delta z = L/N$, the objective is to compare the

- the $T$-mean KL modes obtained from the transient displacement field over $[0,T]$;
- the $T$-mean KL modes obtained from the transient velocity field over $[0,T]$;
- the $T$-mean KL modes obtained from the transient acceleration field over $[0,T]$;
- the $T$-mean KL mode obtained from the equivalent linear system;
- the modes obtained from POM analysis of some displacement trajectories

The $T$-mean KL modes of the transient non-linear response were computed using the method described section 2.3 The simulated data were also used to estimate $\mathbf{K}_{eq}$ solving Eq. (19) and $T$-mean KL modes of the transient response of the equivalent linear system (17) were computed from the covariance matrix function obtained solving the associated differential equations (Lyapounov equations) of (17).

The parameters values were: $L = 0.6$, $EI = 1.4$, $\rho S = 0.1620$, $d = 12$, $\tau_i = \tau = 0.01$ and $d = 12$ with

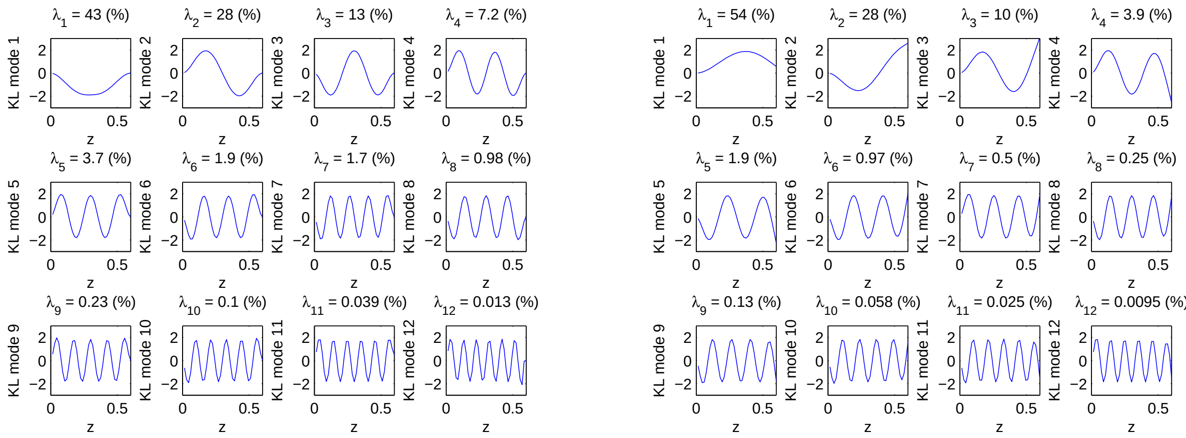
- in the clamped-clamped beam case, $z_f = 0.6$, $D = 10^7$, $z_e = 0.05$ (all modes were excited and the correlation coefficient between pairs of modal components were equal to 1),
- in the clamped-free beam case, $z_f = 0.3$, $D = 10^8$, $z_e = 0.05$ (all modes were excited and the correlation coefficient between pairs of modal components were equal to 1).

In both cases, the equation (21) was solved using the frequency sample $f_e = 8000$Hz, zero initial values and with $S_f = 1$, $T = 1$ (which correspond to approximately four fundamental periods of the smaller resonance frequency), $N = 40$, $M = 1000$ (number of independent events or trajectories).

In Figs. 1-3, the twelve $T$-mean KL modes obtained from the transient displacement, velocity and acceleration field, respectively, are plotted for the two boundary conditions. On the left is the clamped-clamped, and on the right the clamped-free case. For each $T$-KL mode, the repartition of the modal energy is also given. In both cases, the shapes of the first $T$-mean modes obtained from the displacement field look like to the LNMs of the underlying linear system. On the contrary, the last $T$-mean modes obtained from the acceleration field look like to the LNMs of the underlying linear system. In this case, the contribution of these modes to the total energy is very small. This behaviour is related to the frequency repartiton energy wich differ from the displacement field to the acceleration field. Hence the modal analysis based on KL theory applied to the velocity field or to the acceleration field has to be manipulated carefully.

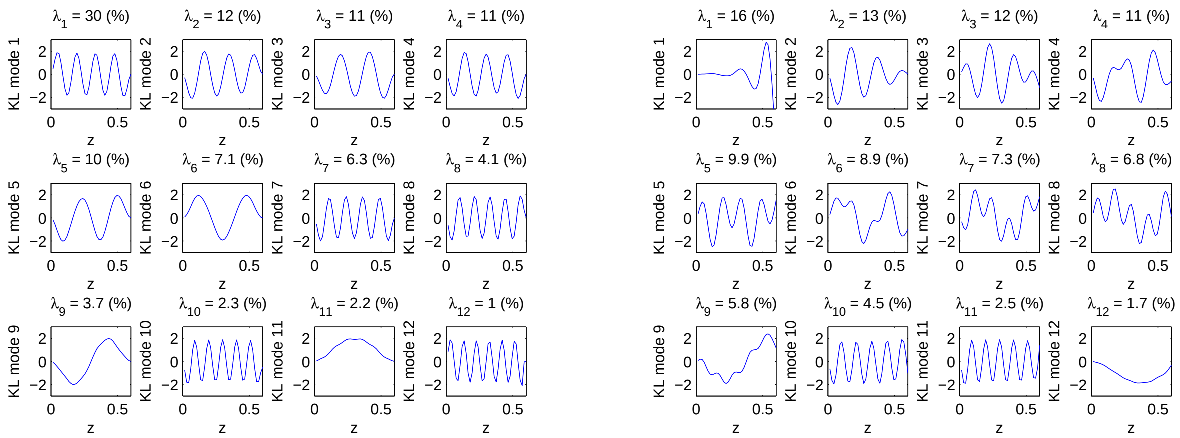
In Figs. 4, the four first modes, respectively, of the underlying linear clamped-clamped beam, the corresponding modes obtained using the $T$-mean KL expansions of the transient diplacement responses of the non-linear system and those of the equivalent linear system are compared. First of all, we can observe that the POMs obtained with the two systems (the non-linear and the equivalent linear system) are very similar. The result which holds true when we are looking for the stationary responses using the averaging operation ($E(.)$) seems to be reasonably true in the case of transient response using the averaging operation ($\mathbb{E}(.)$). Of course, the proof of this concordance still needs to be established theoretically. As mentioned above, the non-linear effect appears to be more significant in the first two modes, and to be less pronounced in the higher order modes. We have also plotted, in these figures, several eigenvectors obtained from single realizations of displacement history. These modes were computed using the direct method described in section 2.3 with the parameter value $R = 1$. The eigenvectors obviously differ from the POMs as well as from the LNMs. Depending on the realization, the difference with respect to the POMs can be significant (see Fig. 4).

The same comments can be made for the clamped-free case (see Fig. 5).



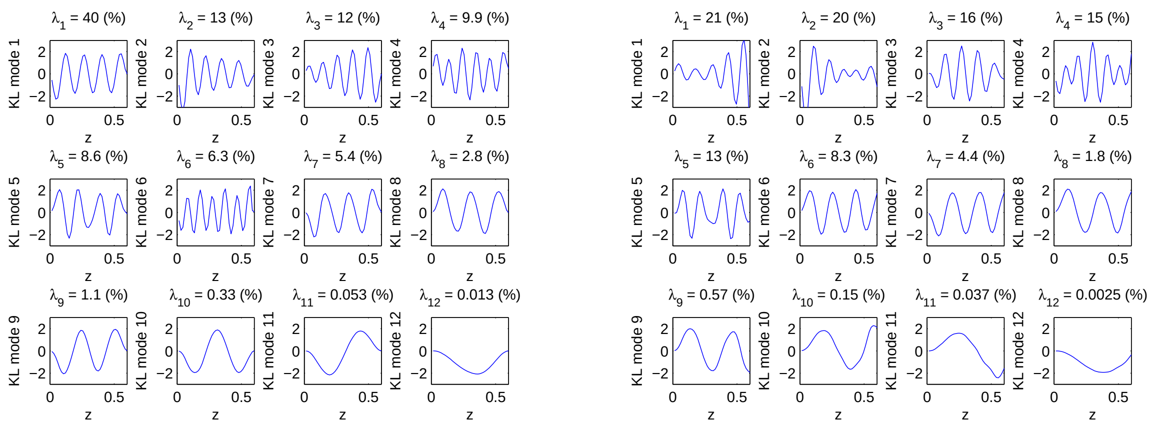

(a) (b)

Figure 1. T-mean KL modes of the displacement field obtained from the transient response over $[0, T]$, (a) clamped-clamped, (b)clamped-free



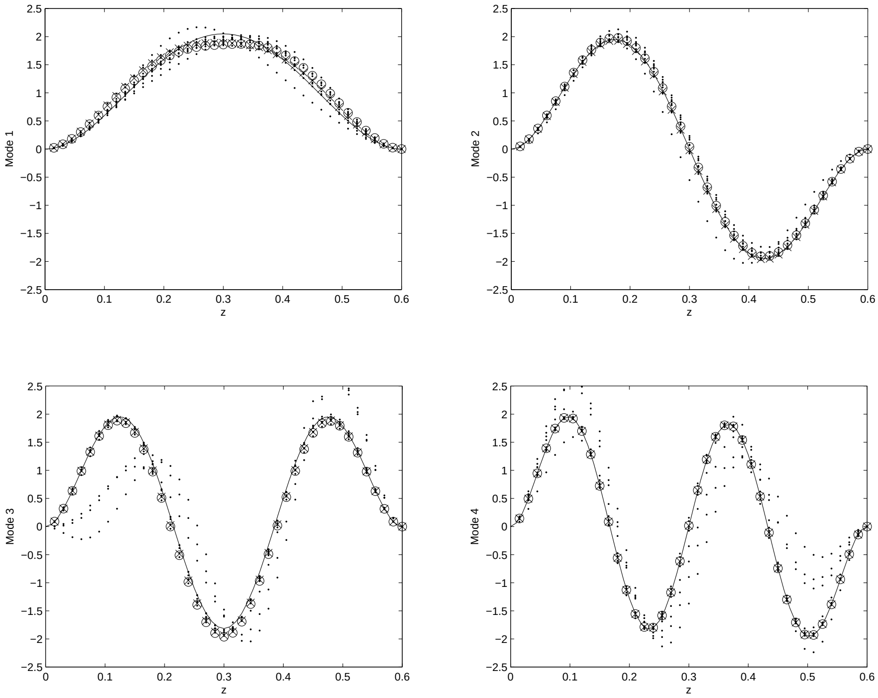

(a) (b)

Figure 2. T-mean KL modes of the velocity field obtained from the transient response over $[0, T]$, (a) clamped-clamped, (b)clamped-free



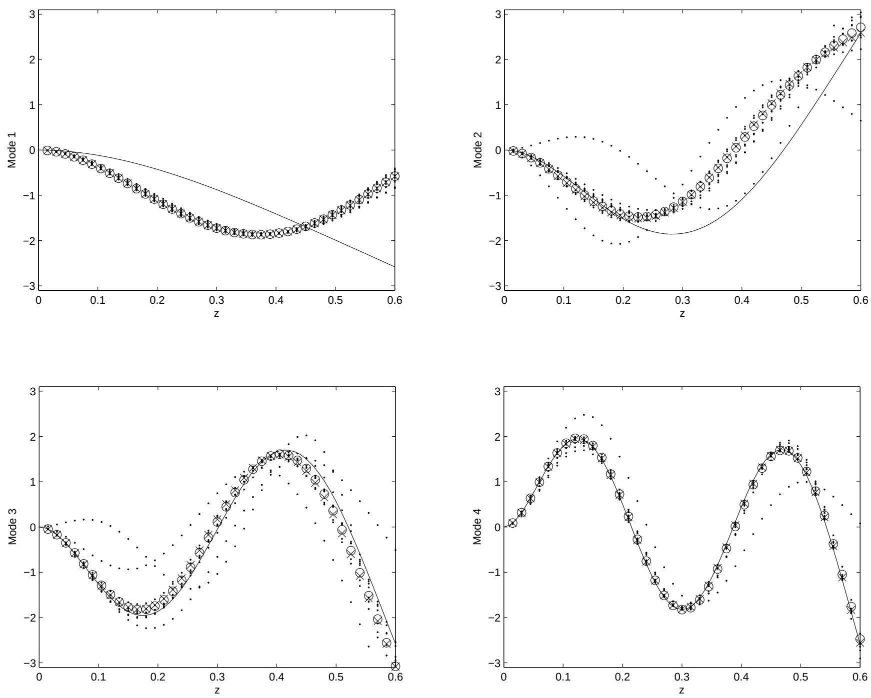

(a) (b)

Figure 3. T-mean KL modes of the acceleration field obtained from the transient response over $[0, T]$, (a) clamped-clamped, (b)clamped-free




(a) (b) (c) (d)

Figure 4. The four first modal functions (solid line) of linear clamped-clamped beam, the corresponding $T$-mean KL modes obtained from the transient displacement field over $[0, T]$ ($\circ$), the corresponding $T$-mean KL modes obtained from the equivalent linear system ($\times$) and the corresponding modes obtained from POM analysis of some displacement trajectories (dotted lines).




(a) (b) (c) (d)

Figure 5. The four first modal functions (solid line) of linear clamped-free beam, the corresponding $T$-mean KL modes obtained from the transient displacement field over $[0, T]$ ($\circ$), the corresponding $T$-mean KL modes obtained from the equivalent linear system ($\times$) and the corresponding modes obtained from POM analysis of some displacement trajectories (dotted lines).

## 5. CONCLUSION

In this present study, the randomly excited vibrating system responses have been analyzed using the KL theory based on an averaging operator involving time and ensemble averages. This approach permits the analysis of stationary responses as well as non-stationary responses. In the stationary case, this approach coincides with the classical KL theory.

The vibration analysis using $T$-mean KL theory has been independently developed from the displacement field, the velocity field and the acceleration field. The $T$-mean KL modes obtained using displacement, velocity, or acceleration fields have been compared, and the relations between the $T$-mean KL modes and the Linear Normal Modes (LNM) have been analyzed. Moreover, in the non-linear case, the $T$-mean KL modes have been compared to the $T$-mean KL modes of the equivalent linear system obtained using the statistical linearization method.

## 6. ACKNOWLEDGEMENTS

This work was supported by the Brazilian Agency Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), by the International Cooperation Project Capes-Cofecub, $n^o 476/04$, and Faperj.

## 7. REFERENCES

Arnold, L., 1973, "Stochastic differential equations: theory and applications", J. Wiley & Sons.

Atwell, J.A. and King, B.B., 2001, "Proper orthogonal decomposition for reduced basis feedback controllers for parabolic equations", Mathematical and Computer Modelling, Vol. 33, pp. 1-19.

Bellizzi, S., and Sampaio, R., 2006, "POMs analysis of randomly vibrating systems obtained from Karhunen-Loève expansion", Journal of Sound and Vibration, Vol. 297, pp. 774-793.

Bellizzi, S., and Sampaio, R., 2007, "Analysis of randomly vibrating systems using Karhunen-Loève expansion", Proceedings of the ASME International Design Engineering Technical Conferences & Computers and Information in Engineering Conference, Las Vegas, Nevada, USA.

Feeny, B.F. and Liang, Y., 2003, "Interpreting proper orthogonal modes of randomly excited vibration systems", Journal of Sound and Vibration, Vol. 265, pp. 953-966.

Feeny, B.F. and Kappagantu, R., 1998, "On the physical interpretation of proper orthogonal modes in vibrations", Journal of Sound and Vibration, Vol. 211(4), pp. 607-616.

Ghanen, R. and Spanos, P.D., 2003, "Stochastic Finite Elements: a Spectral Approach", Dover Publications, (Rev. edition).

Graham, M.D. and Kevrekidis, I.G., 1996, "Alternative approachs to the Karhunen-Loève decomposition for model reduction and data analysis", Computers Chem. Engng, Vol. 20(5), pp. 495-506.

Kerschen, G. and Golinval, J.C., 2002, "Physical interpretation of the proper orthogonal modes using singular value decomposition", Journal of Sound and Vibration, Vol. 249(5), pp. 849-865.

Kerschen, G. and Golinval, J.C., 2004, "Comments on Interpreting proper orthogonal modes of randomly excited linear vibration systems", Journal of Sound and Vibration, Vol. 274, pp. 1091-1092.

Kozin, F., 1988, "The method of statistical linearization for non-linear stochastic vibration", In: Ziegler F. and Schueller editors, Proceedings of Nonlinear Stochastic Dynamic Engineering Systems (IUTAM Symposium Innsbruck/Austria 1987),Berlin: Springer-Verlag.

Lumley, J.L., 1971, "Stochastic tools in turbulence", Academic Press.

Ma, X. and Vakakis, A.F., 1999, "Karhunen-Loève decomposition of the transient dynamics of a multibay truss", AIAA Journal, Vol. 37(8, pp. 939-946.

Poirion, F. and Soize, C., 1988, "Simulation numérique des champs stochastiques gaussiens homogènes et non homogènes", La Recherche Aérospatiale, Vol. 1, pp. 41-61.

Poirion, F. and Soize,C., 1999, "Monte Carlo construction of the Karhunen-Loève expansion for non-Gaussian random fields", In: N. Jones, R. Ghanen editors, Proceedings of Thirteenth ASCE Engineering Mechanics Conference, CDrom.

Roberts, J.B. and Spanos, P.D., 1990, "Random Vibration and Statistical Linearization", Dover Publications.

Sakamoto, S. and Ghanem, R., 2002, "Simulation of multi-dimensional non-gaussian non-stationary random fields", Probabilistic Engineering Mechanics, Vol. 17, pp. 167-176.

Sirovich, L., 1987, "Turbulence and the dynamics of coherent structures part I: coherent structures", Quarterly of Applied Mathematics, Vol. 45(3), pp. 561-571.

Steindl, A., Troger, H. and Zemann, J.V., 1997, "Nonlinear Galerkin methods applied in the dimension reduction of vibrating fluid conveying tubes", In: M.P. Paidoussis, A.K. Bajaj, T.C. Corke, T.M. Farabee, and F. Williams, D.R. Hara editors, Proceedings of ASME Symposium on Fluid-Structure Interaction, Aeroelasticity, Flow-Induced Vibration and Noise, (Dallas, EUA), AD53-1, pp. 273-280.

Trindade, M.A., Wolter, C. and Sampaio, R., 2005, "Karhunen-Loève decomposition of coupled axial/bending vibrations of beams subject to impacts", Journal of Sound and Vibration, Vol. 279, pp. 1015-1036.

## 8. Responsibility notice

The author(s) is (are) the only responsible for the printed material included in this paper