THE ERROR IN ORIGIN FROM FIRST PRINCIPLES

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Abstract. *In this work we investigate the underlying assumptions involved in the derivation of the parameters presented in the velocity and temperature logarithmic law such as the error in origin directly from the Navier-Stokes equations and the first law of thermodynamics. Our goal is to show that these quantities are unrelated to each other. To do so, we follow and expand theoretical developments in the literature regarding their first principles derivation, and also perform experimental verifications of these quantities for a more complete investigation of the theoretical formulations.*

Keywords: *turbulence, boundary layer, wall function, experimental fluid mechanics.*

1. INTRODUCTION

Turbulent flows over rough surfaces present almost impossible analytical and numerical boundary condition specifications. This means that general methods like dimensional analysis play a crucial role in modelling such flows, since it encapsulates all roughness geometrical (and possibly stochastic) complexity in terms of a few parameters such as roughness length and displacement height. The downside of this procedural simplicity is that, of course, a lot of important and interesting physics are lost due to a lack of first principles formal derivation from the equations of motion. Dimensional analysis – despite its well-known usefulness – takes quantities in just by considering the coherence of their physical dimensions, without any deeper study of the interplay between them. The end result is that these quantities may end up as just curve fitting parameters for experimental data, devoid of a more comprehensive physical interpretation.

In the case of the displacement height, the lack of a formal understanding of its physical interpretation leads to a disagreement about its precise nature, and therefore how to correct determine its value (Perry & Joubert, 1963; Perry et al., 1969; Stearns, 1970; Jackson, 1981; Bottema, 1996; Schaudt, 1998). This problem is also compounded by the fact that the displacement height has two versions: the displacement height for velocity (d) and temperature (d_T) which are frequently used interchangeably in the literature, albeit they stand for different flux densities from the surface roughness (Malhi, 1996).

In this work we investigate whether the displacement height for velocity and temperature – which appear in their respective log-law profiles – can in fact be regarded to be the same quantity. To do so, we detail the underlying assumptions involved in their first principles derivation directly from the Navier-Stokes equation and the first law of thermodynamics (Jackson, 1981; Malhi, 1996). We closely follow the rationale presented in Jackson's derivation of the displacement height, in order to test the hypothesis employed in that paper. We demonstrate that the displacement height can be cast into a new, simpler expression which clearly shows its physical interpretation. Besides, this new expression can be readily compared against experimental and numerical data. The thermal flux extension is then considered by applying the same procedure as in the thermally neutral case. Finally, we use an experimental set-up consisting of hot-wire anemometry in a wind tunnel to further demonstrate – in a controlled environment – how d and d_T are in fact different physical quantities.

2. LOG-LAW OVER ROUGH SURFACES

Despite all its complexity and strong dependency on initial and boundary conditions, all turbulent flows near rough surfaces – except for boundary layers close to separation – share a common logarithmic velocity profile given by (Kundu and Cohen, 2004)

$$\frac{u}{u_\star} = \text{const.}\ \ln\left(\frac{y}{y_0}\right) , \qquad (1)$$

where y_0 is the roughness length, defined as the height at which the velocity profile vanishes, so that the low now depends on the roughness scale and not on the fluid viscosity.

The reason why the displacement height (which is linearly related to the error in origin) enters the log-law above can be illustrated as follows (Sumer, 2005): suppose a rough wall as in Fig. 1 and 2. The vertical coordinate y is not well defined, since it its not clear whether the roughness elements' bottom or top – or some height level between them – should be regarded as the coordinate origin. The displacement height d may simply be defined as an offset – the height from

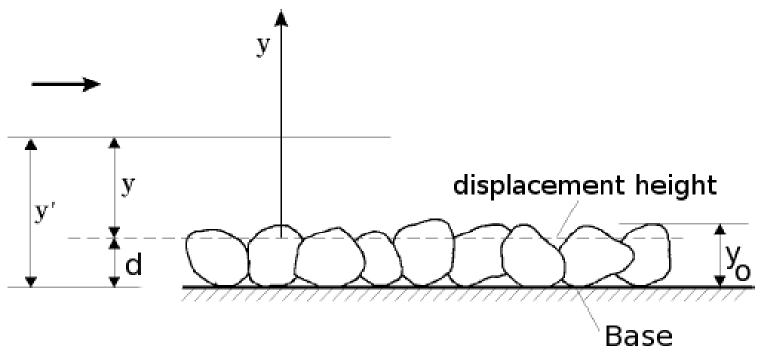


Figure 1. A rough surface and its displacement height. Adapted from Sumer (2005).

which the y coordinate in Eq. (1) is measured. So we can redefine $y \to y - d$. Putting this into Eq. (1) we have

$$\frac{u}{u_\star} = \text{const. } \ln\left(\frac{y-d}{z_0}\right) . \tag{2}$$

As originally formulated by Perry et al. (1969), the error in origin is measured from the top of the roughness elements, as the common practice in aeronautical engineering. On the other hand, the literature in physics and meteorology measures it from the bottom of the roughness elements, hence the name "displacement height".

Both Eqs. (1) and (2) are derived solely through dimensional analysis. In this framework, d is merely an offset to the vertical coordinate, and its value can only be known through fitting to experimental data (Perry & Joubert, 1963; Perry et al., 1969; Schaudt, 1998). However, the displacement height d is the fundamental parameter for flow description over rough surfaces. From this quantity one can measure other parameters such as boundary layer thickness, roughness function and skin-friction velocity (Perry & Joubert, 1963).

Although the displacement height analysis above was centred on the velocity field, the concept can be extended over to the thermal field as well. This enable us to quantify how the heat transfer is affected by the surface roughness and the flow velocity. The temperature profile over the logarithmic region of a turbulent boundary layer is given by (Malhi, 1996)

$$\frac{T_w - T(y)}{t_\star} = \text{const. } \ln\left(\frac{y-d_T}{y_{0T}}\right) , \tag{3}$$

where T_w is the temperature on the surface and $t_\star$ is the skin-friction temperature. As in the velocity case, d_T and y_{0T} are the displacement height and roughness length for temperature.

It is not clear from the framework of dimensional analysis whether d and d_T have anything in common at all. In fact, it is a common practice in the literature to consider both equal to each other. It is thus desirable to know how these quantities depend on the flow parameters and boundary conditions, which will allow for a better estimation of their value.

3. DISPLACEMENT HEIGHT FOR FLOW VELOCITY

In a famous paper, Jackson (1981) provided a new physical interpretation for the displacement height d. Through a global balance of the rate of change of momentum it is argued that d is the level at which the mean drag τ_0 on the surface appears to act, Fig. 2. Following his rationale, we consider a zero pressure-gradient turbulent flow near a idealized rough surface as sketched in Fig. 3. It is assumed that the boundary layer thickness and the horizontal scales are much greater than any dimension of the roughness elements and that the fluid properties are the same at faces AB and CD. Here, δ is a height such that the flow is no longer affected by the roughness elements down below. There is no mean flow crossing the boundary $y = \delta$, and the mean stress acting on this layer must equal τ_0, the average force per unity area acting on the surface.

Starting from the Reynolds-averaged Navier-Stokes equations in two-dimensions

$$\rho\,\partial_x \overline{u}^2 + \rho\,\partial_y\left(\overline{u}\,\overline{v}\right) = -\partial_x \overline{p} + \partial_x T_{11} + \partial_y T_{12} \, , \tag{4}$$

with the total stresses and continuity equations

$$T_{ij} = \mu\left(\partial_j \overline{u}_i - \partial_i \overline{u}_j\right) - \rho\,\overline{u_i^{'}\,u_j^{'}} \, , \tag{5}$$

$$\partial_x \overline{u} + \partial_y \overline{y} = 0 \, , \tag{6}$$

we integrate Eqs. (4) over the control volume in Fig. 3. This is done by first integrating over x then multiplying by y and integrating over y. The final result, as given by Jackson, is

$$M = k\,h\,\tau_0 + \iint_{\text{fluid}} \left[\tau_0 - \left(T_{12} - \rho\,\overline{u}\,\overline{v}\right)\right] \,dx\,dy \, , \tag{7}$$

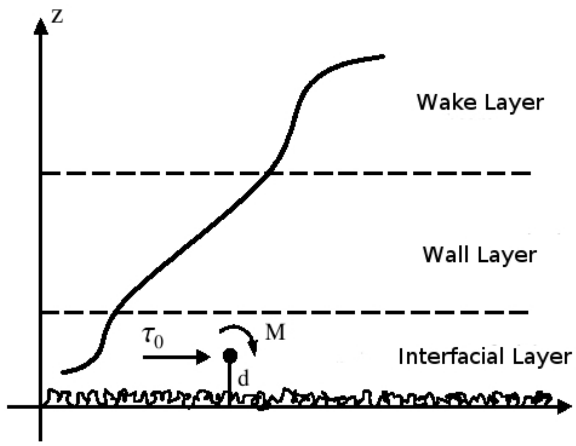


Figure 2. Jackson's flow layers. M is the torque exerted by the flow on the roughness elements.

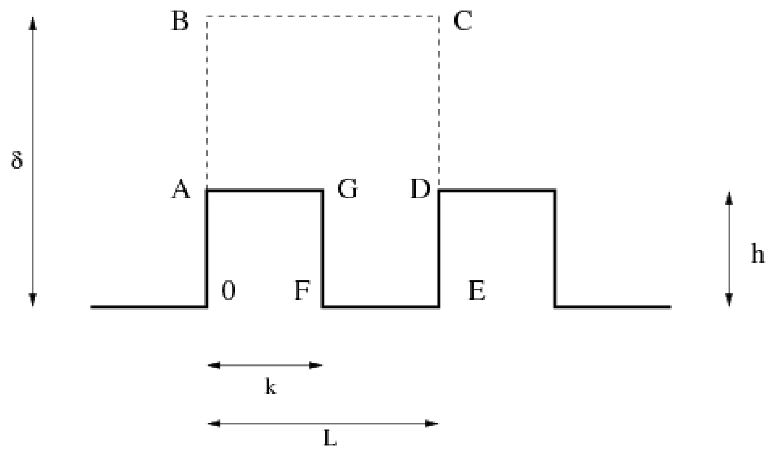


Figure 3. The control volume on an idealized bi-dimensional roughness. The (x, y) coordinate origin is at point 0.

where

$$M \equiv \int_0^h y\,\Delta p\,dy + h\int_0^k T_{12}\,dx\,. \tag{8}$$

Here Δp is the pressure difference between the lateral faces of the roughness elements. The second integral on the above equation represent the torque done by the horizontal stress acting at the top of the elements. Therefore M represents the moment acting on the roughness elements. This must equal the moment acting at the face BC. Dimensionally, we can write this moment as

$$\text{moment}|_{BC} = (L\,\tau_0)\;d\,, \tag{9}$$

where d is a length. Equating Eqs. (7) and (9) we get

$$d - \overline{h} = \frac{1}{L\tau_0}\iint_{\text{fluid}} \left[\tau_0 - (T_{12} - \rho\,\overline{u}\,\overline{v})\right]\;dx\,dy\,, \tag{10}$$

where $\overline{h} \equiv hk/L$ is the average elevation of the surface. d is interpreted by Jackson as the height at which the moment (9) appears to act. It is further interpreted as the displacement height which appears in the log-law Eq. (2).

Eq. (10) provides a full-blown expression for d which should be numerically and experimentally verified. In order to do so, (10) has to be put in a more simplified form. This can indeed be done as follows: the pressure drag between the roughness elements – the first term in the right-hand side of Eq. (8) – is much greater than the surface drag at the top of the elements – the second term in the right-hand side of Eq. (8). Therefore we can approximate M as

$$M \approx \int_0^h y\,\Delta p\,dy\,. \tag{11}$$

This expression has dimension of [FORCE] $\times$ [LENGTH]. Since from Eq. (9) the length scale was identified as the displacement height, we are able to make the following identification

$$\text{[LENGTH]} = d\,, \qquad \text{[FORCE]} = \int_0^h \Delta p\,dy\,, \tag{12}$$

so, from Eq. (11) we can isolate d as

$$d = \frac{\int_0^h y\,\Delta p\,dy}{\int_0^h \Delta p\,dy}\,. \tag{13}$$

In other words, d can be determined by just taking pressure measurements on the roughness elements. The form of the expression above suggests that d may be regarded as the average height in which the drag on the roughness appear to act, or the centroid of the drag on the surface.

4. DISPLACEMENT HEIGHT FOR TEMPERATURE

The displacement height for temperature can be seen as the effective height of a heat source inside the roughness sublayer which imparts the heat flux in the logarithmic layer. In order to find a length scale associated with the thermal flux, we are free to repeat the steps used in the velocity case. This new length scale, d_T, will be the analogous of the velocity displacement height d. In the case of heat flux from the surface to the flow, we need to use the heat transfer equation

$$\frac{D}{Dt}\,(\rho\,C_p\,T) = \kappa\,\nabla^2\,T\,, \tag{14}$$

which comes directly from the first law of thermodynamics by using the Boussinesq approximation (Kundu and Cohen, 2004). By using the Reynolds decomposition, this equation can be written as

$$\rho\,C_p\,\left[\overline{u}\,\partial_x\overline{T} + \overline{v}\,\partial_y\overline{T}\right] = \partial_x Q_x + \partial_y Q_y\,, \tag{15}$$

where the heat flux is given by

$$Q_j = \kappa\,\partial_j\overline{T} - \rho\,C_p\,\overline{u_j'\,T'}\,. \tag{16}$$

The integration of Eq. (15) is done in exactly the same fashion as in Eq.(4). Integrating (15) over x and then multiplying by y and integrating over y, we get the following balance equation

$$\int_0^h y\,\left[-\kappa\,\partial_x\overline{T}\right]_k^L\,dy + h\int_0^h \kappa\,\partial_y\overline{T}\,dx + \iint_{\text{fluid}} \left(Q_y - \rho\,C_p\,\overline{v}\,\overline{T}\right)\,dx\,dy = \delta\int_0^L \left[Q_y - \rho\,C_p\,\overline{v}\,\overline{T}\right]^{\delta}\,dx\,. \tag{17}$$

Note that it is a balance equation for the heat flux times a length (*i.e.* a height). This length scale will permit us later on to isolate a height and identify it further as d_T. The first term in the left hand side of the above equation represent the molecular heat flux difference between the roughness elements. The second term in the left hand side is the molecular heat flux coming through the top of the elements. The last term in the left hand side is proportional to the heat flux passing through the fluid inside the control volume. The right hand side represents the heat flux passing through the top face of the control volume. There is no net contribution coming from the lateral faces AB and CD since we supposed that the fluid properties are identical in both faces. Also there is no contribution coming from the bottom surface between the elements since in the balance equation above the heat flux is multiplied by a height, which would vanish at $y = 0$.

Calling the first two terms in the left hand side of the above equation as M_T, we can rewrite (17) as

$$M_T = k\,h\,Q_0 + \iint_{\text{fluid}} \left[Q_0 - \left(Q_y - \rho\, C_p\, \overline{v}\,\overline{T}\right)\right]\, dx\, dy\,, \tag{18}$$

where Q_0 is the heat flux at the surface. Again in analogy to Section 3. we can construct the following quantity at the top surface BC: $(LQ_0)d_T$, where d_T is a length. This is dimensionally reasonable since Eqs. (17) and (18) have dimension of [HEAT FLUX] $\times$ [LENGTH]^2, and the appropriate heat flux quantity at BC is Q_0 and a length is the width L. Just as in Section 3. this quantity at BC must equal M_T, $M_T = (LQ_0)d_T$. Using this in Eq. (18) we have

$$d_T - \overline{h} = \frac{1}{LQ_0} \iint_{\text{fluid}} \left[Q_0 - \left(Q_y - \rho\, C_p\, \overline{v}\,\overline{T}\right)\right]\, dx\, dy\,. \tag{19}$$

This expression can be simplified by noting that, as defined above,

$$M_T \equiv \int_0^h y\, \left[-\kappa\, \partial_x \overline{T}\right]_k^L\, dy + h\, \int_0^h \kappa\, \partial_y \overline{T}\, dx\,. \tag{20}$$

so that, following the reasoning in the last section,

$$d_T = \frac{\int_0^h y\, \left[-\kappa\, \partial_x \overline{T}\right]_k^L\, dy + h\, \int_0^h \kappa\, \partial_y \overline{T}\, dx}{\int_0^h \left[-\kappa\, \partial_x \overline{T}\right]_k^L\, dy + \int_0^h \kappa\, \partial_y \overline{T}\, dx}\,. \tag{21}$$

Note that d_T depends only on the molecular heat flux coming from the roughness elements. It is also clear that both d and d_T depend on different physical parameters, and therefore d and d_T are unrelated quantities. This will also be shown in the next section, where they are experimentally verified to have different values for the same flow characteristics.

5. EXPERIMENTAL VERIFICATION

The principal facility used in the experimental simulations was a large wind tunnel situated at the Mechanics of Turbulence laboratory (COPPE/UFRJ – Mechanical Engineering Program). The laboratory was air conditioned with the temperature maintained within an ± 0.5 ^{o}C uncertainty. The basic flow instrumentation consisted of thermo-anemometers and thermocouples.

A general view of the wind tunnel is presented in Figure 4. The test section has an overall length of 10 m, and a cross section area of 0.67 m x 0.67 m. The potential flow velocity can be made to vary from zero to 3.5 m/s with free stream turbulence intensity levels of about 2%. In still air condition, the floor temperature can be raised up to 100 ^{o}C over a 6 m long surface. The heating system is comprised by a series of 6 one-meter independent panels fitted with electrical resistances that furnish a wall temperature controlled variation of 2 ^{o}C. The total heating capacity of each panel is about 0.75 kW/m^2. The whole facility is capable of developing boundary layer gradients of up 60 ^{o}C at uniform mean speeds in the range of 1.5 – 3.5 m/s.

In all experiments, measurements of stream-wise velocity and temperature were made through thermal-anemometers and thermocouples. The velocity measurements were made with DANTEC anemometers of the series 55M. The boundary layer probe was a DANTEC 55P15 model. Reference measurements for the velocity were obtained from a Pitot tube connected to an inclined manometer. The reference mean temperature profiles were obtained from chromel-constantan micro-thermocouples. The probe supports for both the velocity and the temperature probes were mounted on an automatic traverse gear system whose sensitivity is 0.02 mm.

An uncertainty analysis of the data was performed according to the procedure described in Kline (1985). Typically the uncertainty associated with the velocity and the temperature measurements were: U = 0.0391 m/s precision, 0 bias (P = 0.95); T = 0.2 ^{o}C precision, 0 bias (P = 0.99).

The rough surface was constructed with equally spaced transversely rectangular slats made of aluminium. The dimensions of the roughness elements are shown in Figure 5 where K denotes the height, S the length, W the gap and λ the pitch of the protuberances.

To verify the equations deduced in the preceding sections, measurements were taken at one particular station in the wind tunnel, nominally, at 6.500 mm downstream of the settling chamber (see Figure 4).

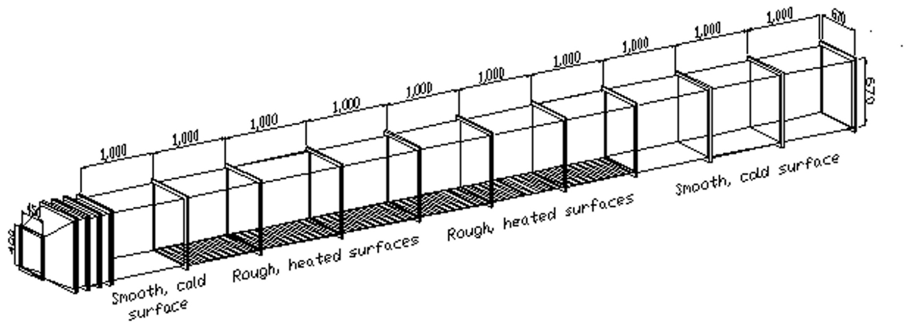


Figure 4. General view of wind tunnel.

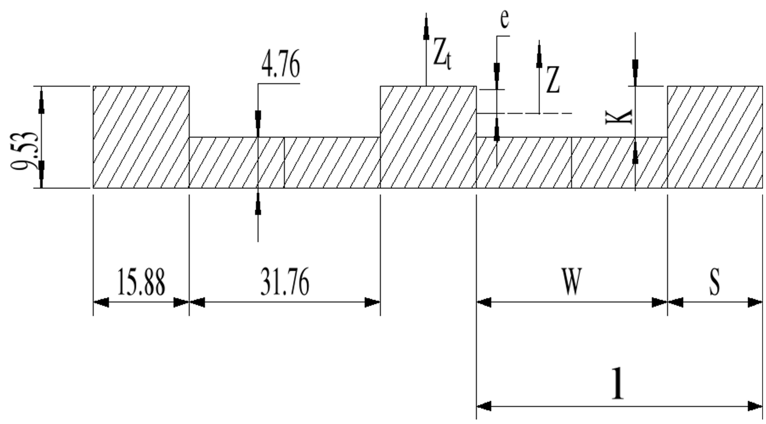


Figure 5. Geometry of roughness protuberances. Dimensions are in millimeters.

At the testing station, the experimental flow conditions were as shown in Table 5.where δ^* denotes the boundary layer displacement thickness, θ the boundary layer momentum thickness and G the Clauser parameter. This table incorporates the friction-velocity, a parameter whose determination will be explained next.

Surface	U_∞ [m/s]	$u_\star$ [m/s]	δ^* [mm]	θ[mm]	G	q_w [kW/m^2]
Rough	3.0	0.161	28.79	19.15	8.53	0.75

Table 1. Experimental conditions.

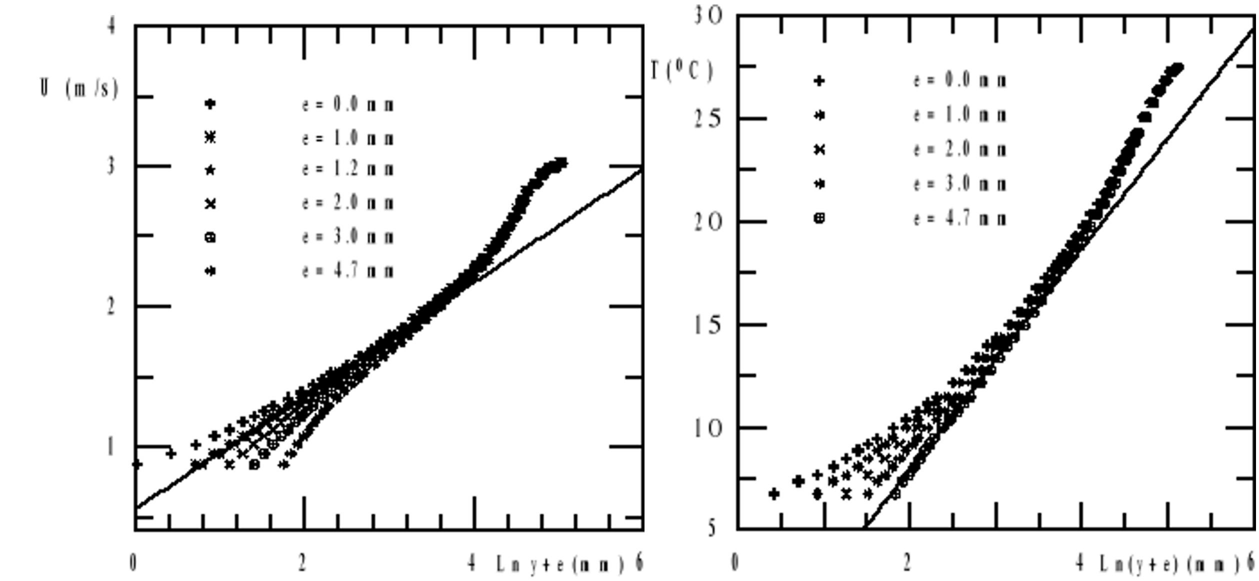
To determine the velocity error in origin, ε, the method of Perry and Joubert (1963) was used. Thus, velocity profiles were plotted in log-linear graphs in dimensional coordinates. Next, the normal distance from the top of the roughness elements was incremented by 0.1 mm and a straight line fit was applied to the resulting points. The best fit was chosen by searching for the maximum coefficient of determination, R-squared. Others statistical parameters were also observed, the residual sum of squares and the residual mean square. Normally, a coefficient of determination superior to 0.99 was obtained.

The determination of ε (= 1.2 mm) is illustrated in Figure 6(a).

Having found ε, we can then use the gradient of the log-law to determine $u_\star$. Another method to obtain $u_\star$ is the momentum integral equation. This latter method, however, is very sensitive to any three-dimensionality of the flow and the determination of the derivatives of the various mean flow parameters is a highly inaccurate process.

Let us now turn to the temperature data. In the present experiment, the total heat flux provided to the roughness elements was 0.75 kW/m^2. Temperature profiles were measured at the same station of the velocity profiles. The error in origin for the temperature profile, ε_T, was also determined through the method of Perry and Joubert (1963). Then, the gradient of the temperature log-law was used to find the friction temperature, and, consequently, the local wall heat flux.

The temperature error in origin was found to be ε_T = 4.7 mm, and its calculation is illustrated in Figure 6(b). The experimental results are summarized below, again showing the difference between the values of d and d_T for the same



(a) (b)

Figure 6. Determination of ε and ε_T according to the method of Perry and Joubert (1963, 1969). (a) curves were drawn for values of ε = 0, 1, 1.2, 2, 3 and 4.7 mm. Resulting $u_\star$ = 0.1612 m/s, d = 3.57mm, y_0 = 0.252mm; (b) Curves were drawn for values of ε_T = 0, 1, 2, 3 and 4.7 mm. Resulting $t_\star$ = 2.36^oC.

experimental conditions.

• **velocity profile**

$$\frac{u}{u_\star} = \frac{1}{\kappa} \ln\left(\frac{y-d}{y_0}\right) , \tag{22}$$

where κ is the von Karmann constant and $u_\star = 0.161$ m/s.

y_0 [mm]	ε [mm]	d [mm]
0.252	1.2	3.57

Table 2. Experimental conditions: velocity profile.

• **temperature profile**

$$\frac{T_w - T}{t_\star} = \frac{1}{\kappa_T} \ln\left(\frac{y-d_T}{y_{0T}}\right) , \tag{23}$$

where $\kappa_T = 0.44$ and $t_\star = 2.362$ ^{o}C.

y_{0T} [mm]	ε_T [mm]	d_T [mm]
1.693	4.7	0.07

Table 3. Experimental conditions: temperature profile.

6. REFERENCES

Bottema, M., 1996, “Roughness parameters over regular rough surfaces: experimental requirements and model validation”, Journal of Wind Engineering & Industrial Aerodynamics, Vol. 64, 249–265.

Jackson, P. S., 1981, “On the displacement height for the logarithmic velocity profile”, J. Fluid Mech., Vol. 11, 15–25.

Kline, S. J., 1985, “The purpose of uncertainty analysis”, J. Fluids Eng., Vol.107, 153–160.

Kundu, P.K., Cohen, I.M., 2004, “Fluid mechanics”, third edition, Elsevier Academic Press, 759 p.

Malhi, Y., 1996, “The behaviour of the roughness length for temperature over heterogeneous surfaces”, Q. J. R. Meteorol. Soc., Vol. 122, 1095–1125.

Sumer, B. M., 2005, “Lecture notes on turbulence”, Technical University of Denmark, 153 p.

Perry, A. E. and Joubert, P. N., 1963, "Rough-wall boundary layers in adverse pressure gradients", J. Fluid Mechanics, Vol. 17, 193–211.

Perry, A. E., Schofield, W. H. and Joubert, P. N., 1969, "Rough-wall turbulent boundary layers", J. Fluid Mechanics, Vol. 37, 383–413.

Schaudt, K. J., 1998, "A new method for estimating roughness parameters and evaluating the quality of observations", Journal of Applied Meteorology, Vol. 37, 470–476.

Stearns, C. R., 1970, "Determining surface roughness and displacement height", Boundary-Layer Meteorol., Vol. 1, 102–111.