BOUNDARY ELEMENT ANALYSIS OF CRACKED THICK PLATES REPAIRED WITH ADHESIVE BONDED PATCHES

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Abstract. *The fracture analysis of cracked thick plates repaired with adhesive isotropic patches using a boundary element formulation is presented. The shear deformable cracked isotropic plate was modelled using the dual reciprocity method (DBEM). To modelling the repair, direct boundary element equations were established using the Reissner plate theory. Coupling action between in-plane and bending effects in the plates is considered. Interaction forces between the cracked plate and the repair were modelled as distributed body forces and treated using two different techniques: the dual reciprocity boundary element method (DRBEM) and the cell method. Coupling equations, based on cinematic compatibility and equilibrium considerations for the adhesive, were established. The crack surface displacements extrapolation technique for stress intensity factors calculation, is presented. Test problems considering circular and rectangular repairs with different plate and repair elastic properties are presented. Good agreement has been achieved when compared with those encountered in the literature.*
Keywords: *Fracture mechanics, cracked plates, dual boundary element method, adhesive patches, dual reciprocity boundary element method.*

1. INTRODUCTION

The Boundary Element Method (BEM) is an atractive numerical alternative to treat fracture problems, mainly to its ability to model continuously high stress gradients without the need of domain discretization. The use of this method in structural analysis has strongly increased since 80s (see Brebbia and Dominguez(1989). Initial works analysing isotropic repair patches in structures using BEM were presented by Young, Cartwritght and Rooke(1988) where cracked plate and the repair using the boundary element method. The shear stresses in the adhesive layer as well as the body forces acting on the plate and on the repair were modeled. Young(1987) modeled the distributed interaction force between the plate and the repair by discretizing the bonded repaired area using internal cells in the BEM formulation. Salgado and Aliabdadi(1998) presents the analysis of metallic thin plates reinforced with bonded isotropic repairs. The reinforced plate was modeled using the dual boundary element method; however the shear stresses in the adhesive layer were modeled as action-reaction body forces interchanged between the plate and the repair. Dirgantara and Aliabadi(2001) presents the dual boundary element method for the analysis of isotropic metallic cracked panles with shear deformation. In this work, boundary hypersingular integral equations and the dual boundary formulation, applying traction integral equation on a face of the crack surface and displacement integral equations in the other face, was used. Later, Wen, Aliabadi and Young(2003) developed a boundary element formulation for the analysis of curved metallic panels with cracks and adhesive isotropic repairs.

In this work, The fracture analysis of cracked thick plates repaired with adhesive isotropic patches using the dual reciprocity method is presented. Coupling equations, based on cinematic compatibility and equilibrium considerations for the adhesive, were established. Test problems considering circular and rectangular repairs with different plate and repair elastic properties are presented.

2. BOUNDARY INTEGRAL FORMULATION FOR PLANE STRESS

The two dimensional boundary integral equation for displacements at the boundary point $\mathbf{x}' \in \Gamma$ that describes membrane effects can be written as (see Brebbia and Dominguez(1989)):

$$c_{ij}^{P}(\mathbf{x}')\, u_\beta(\mathbf{x}') = \int_\Gamma U_{\alpha\beta}^{P}(\mathbf{x}',\mathbf{x})\, t_\beta d\Gamma - \int_\Gamma T_{\alpha\beta}^{P}(\mathbf{x}',\mathbf{x})\, u_\beta d\Gamma + \frac{1}{h_p}\int_A U_{\alpha\beta}^{P}(\mathbf{x}',\mathbf{x})\, f_\beta dA \tag{1}$$

where $\alpha, \beta = 1, 2$ and $c_{ij}^{P}(\mathbf{x}')$ is a function of the geometry at the collocation points that can be determinated by considering rigid body movements. The boundary displacements and tractions for the sheet are denoted by u_α and $t_\alpha (= n_\beta\sigma_{\alpha\beta})$, respectively; displacement and traction fundamental solutions for the plane stress condition are $U_{\alpha\beta}^{P}(\mathbf{x}',\mathbf{x})$ and $T_{\alpha\beta}^{P}(\mathbf{x}',\mathbf{x})$ respectively, $f_\beta(\mathbf{x})$ denote two-dimensional body forces por unit area over a region A of patch and h_p is the thickness of the plate. In this work no others in-plane body forces will be considered.

In order to modeling cracked plates, the Dual Boundary Element Method (DBEM) will be used. In this method, the displacement integral formulation is written for source points on one crack surface and the traction integral equation on the other surface. Then, using the stress and strain relationships for plane stress, the traction integral equation for two-dimensional problems in a smooth boundary can be derived as (see Dirgantara and Aliabadi(2002):

$$\begin{aligned}\frac{1}{2}t_{\alpha}\left(\mathbf{x}'\right) &= n_{\beta}\left(\mathbf{x}'\right)\int_{\Gamma}U^{P}_{\alpha\beta\gamma}\left(\mathbf{x}',\mathbf{x}\right)t_{\gamma}d\Gamma - n_{\beta}\left(\mathbf{x}'\right)\int_{\Gamma}T^{P}_{\alpha\beta\gamma}\left(\mathbf{x}',\mathbf{x}\right)u_{\gamma}d\Gamma \\ &+ n_{\beta}\left(\mathbf{x}'\right)\frac{1}{h_p}\int_{A}U^{P}_{\alpha\beta\gamma}\left(\mathbf{x}',\mathbf{x}\right)f_{\beta}dA \end{aligned} \tag{2}$$

where $n_\beta(\mathbf{x}')$ is the normal to the boundary evaluated at collocation point. $U^P_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})$ and $T^P_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})$ are the traction fundamental solution for two-dimensional problems.

3. BOUNDARY INTEGRAL FORMULATION FOR PLATE BENDING

If w_α are defined as rotations in the x_α direction, w_3 is the deflection of the plate along x_3, q^P_α and q^P_3 are the distribution of body forces in moment and the out-of-plane body force per unit area, respectively, in the patch area A and p_o is the pressure force applied in the domain of the plate Ω, the boundary integrals for the plate bending problem can be written as:

$$\begin{aligned}c^{P}_{ik}\left(\mathbf{x}'\right)w_{k}\left(\mathbf{x}'\right) &= \int_{\Gamma}W^{P}_{ik}\left(\mathbf{x}',\mathbf{x}\right)p_{k}d\Gamma - \int_{\Gamma}P^{P}_{ik}\left(\mathbf{x}',\mathbf{x}\right)w_{k}d\Gamma + p_{o}\int_{\Omega}W^{P}_{i3}\left(\mathbf{x}',\mathbf{x}\right)d\Omega \\ &+ \int_{A}W^{P}_{ik}\left(\mathbf{x}',\mathbf{x}\right)q^{P}_{k}dA \end{aligned} \tag{3}$$

where $k = 1\ldots3$. $W^P_{\alpha\beta}(\mathbf{x}',\mathbf{x})$ and $P^P_{\alpha\beta}(\mathbf{x}',\mathbf{x})$ are the fundamental solutions for Reissner's plate model (see Van Deer Ween(1982)) and $p_\alpha = M_{\alpha\beta}n_\beta$, $p_3 = Q_\beta n_\beta$. Constant c^P_{ik} has a similar significance with those at in-plane displacement problem.

In a similar way, fracture mechanics problems involving plate bending can be modeled usign DBEM. In this case, the traction equation can be written as:

$$\begin{aligned}\frac{1}{2}p_{i}\left(\mathbf{x}'\right) &= n_{\beta}\left(\mathbf{x}'\right)\int_{\Gamma}W^{P}_{i\beta k}\left(\mathbf{x}',\mathbf{x}\right)p_{k}d\Gamma - n_{\beta}\left(\mathbf{x}'\right)\int_{\Gamma}P^{P}_{i\beta k}\left(\mathbf{x}',\mathbf{x}\right)w_{k}d\Gamma \\ &+ n_{\beta}\left(\mathbf{x}'\right)p_{o}\int_{\Omega}W^{P}_{i\beta 3}\left(\mathbf{x}',\mathbf{x}\right)d\Omega + n_{\beta}\left(\mathbf{x}'\right)\int_{A}W^{P}_{i\beta k}\left(\mathbf{x}',\mathbf{x}\right)q^{P}_{k}dA \end{aligned} \tag{4}$$

where $W^P_{i\beta\gamma}(\mathbf{x}',\mathbf{x})$ and $P^P_{i\beta\gamma}(\mathbf{x}',\mathbf{x})$ are the traction fundamental solution for Reissner's plate as presented in Dirgantara(2002).

4. BOUNDARY ELEMENT EQUATIONS FOR THE PATCH

Similary, the displacements of a point **x**' for the patch are given by:

$$c^{R}_{ij}\left(\mathbf{x}'\right)u_{\beta}\left(\mathbf{x}'\right) + \int_{\Gamma}T^{R}_{\alpha\beta}\left(\mathbf{x}',\mathbf{x}\right)u_{\beta}d\Gamma = \frac{1}{h_R}\int_{A}U^{R}_{\alpha\beta}\left(\mathbf{x}',\mathbf{x}\right)f_{\beta}dA \tag{5}$$

for two-dimensional problems, and:

$$c^{R}_{ik}\left(\mathbf{x}'\right)w_{k}\left(\mathbf{x}'\right) + \int_{\Gamma}P^{R}_{ik}\left(\mathbf{x}',\mathbf{x}\right)w_{k}d\Gamma = \int_{A}W^{R}_{ik}\left(\mathbf{x}',\mathbf{x}\right)q^{R}_{k}dA \tag{6}$$

for plate bending problems. In this equation, $\alpha = 1, 2$. ${q_\alpha}^R$ and q^R_3 are distributed body moments and out-of-plane body force by unit area, respectively, generated by interaction with the adhesive layer (superindex R refers to repair).

5. COUPLING EQUATIONS

Isotropic plate equations has fifteen unknowns variables: five displacements (or tractions) at any boundary points and five unknowns displacements and five interaction body forces at any point in the repair region. In addition, ten unknows appears at repair: five displacements (at boundary and domain) and five interacion body forces (at domain). In this way we have twenty five unknows in the problem. Equations (1) throught Eq. (9) represents only fifteen equations. Ten aditional equations must be provided. Aditional equations can be written if cinematic compatibility between plate's and repair and the equilibrium conditions at adhesive layer, are considered. In this way a total of twenty five equations could be written.

The equilibrium of forces acting in the adhesive layer can be written as:

$$f_\alpha^P + f_\alpha^R = 0$$

$$q_3^P + q_3^R = 0$$

$$q_\alpha^P + q_\alpha^R + f_\alpha^R \left(h_A + \frac{h_P + h_R}{2} \right) = 0 \tag{7}$$

Where h_A represents the thickness of the adhesive.

The shear force $\tau_{3\alpha}^A$, acting at interior of adhesive layer can be written as:

$$\tau_{3\alpha}^A = f_\alpha^R = \frac{\mu_A}{h_A} \left(u_\alpha^R - \frac{h_R}{2} w_\alpha^R \right) - \left(u_\alpha^P + \frac{h_P}{2} w_\alpha^P \right) \tag{8}$$

where μ_A is the shear modulus of the adhesive and h_A its thickness. Finally, we can consider that deflexion and rotation angles at coincident points at plate and repair can be related as:

$$\begin{aligned} w_3^P &= w_3^R \\ q_\alpha^P &= C \left(w_\alpha^R + w_\alpha^P \right) \end{aligned} \tag{9}$$

where, $C = D(1-\nu)\lambda^2/2$. In this way, Eq.(7) through Eq.(9) represents ten aditional equations obtained by considering eqauilibrium and cinematic compatibility conditions in the adhesive layer.

6. DOMAIN INTEGRAL TECHNIQUES

6.1 DRBEM integration technique

Equations Eq.(3) to Eq.(6) contains domain integrals. In this work, the Dual Reciprocity Method was used to the treat these integrals where the body forces are approximated as:

$$f_\beta = \sum_{n=1}^{N} f_n(r)\, \alpha_\beta^n \tag{10}$$

$$q_k = \sum_{n=1}^{N} f_n(r)\, \alpha_k^n \tag{11}$$

where α^n is a set of unknown coefficients, $f_n(r)$ denotes a known approximating functions and N is the total number of collocation points. In this work $f(r) = 1 + r$ for the two-dimensional case and $f(r) = r - \lambda^2 r^3/3^2$, where r is the distance from the collocation point to the DRM point.

Applying the DRM technique, the domain integrals for two-dimensional problems can be written as (see Dirgantara(2002)):

$$\frac{1}{h_P} \int_A U_{\alpha\beta}^P(\mathbf{x}',\mathbf{x})\, f_\beta(\mathbf{x})\, dA = \sum_{n=1}^{N} \alpha^n \left[c_{\alpha\beta}^P(\mathbf{x}')\, \hat{u}_\beta^n(\mathbf{x}') - \int_\Gamma U_{\alpha\beta}^P(\mathbf{x}',\mathbf{x})\, \hat{t}_\beta^n(\mathbf{x})\, d\Gamma + \int_\Gamma T_{\alpha\beta}^P(\mathbf{x}',\mathbf{x})\, \hat{u}_\beta^n(\mathbf{x})\, d\Gamma \right] \tag{12}$$

for the displacement equation, and

$$\frac{n_\beta(\mathbf{x}')}{h_P} \int_A U_{\alpha\beta\gamma}^P(\mathbf{x}',\mathbf{x})\, f_\gamma(\mathbf{x})\, dA = \sum_{n=1}^{N} \alpha^n \left[\frac{1}{2}\hat{t}_\alpha^n(\mathbf{x}') - n_\beta(\mathbf{x}') \int_\Gamma U_{\alpha\beta\gamma}^P(\mathbf{x}',\mathbf{x})\, \hat{t}_\gamma^n(\mathbf{x})\, d\Gamma + n_\beta(\mathbf{x}') \int_\Gamma T_{\alpha\beta\gamma}^P(\mathbf{x}',\mathbf{x})\, \hat{u}_\gamma^n(\mathbf{x})\, d\Gamma \right] \tag{13}$$

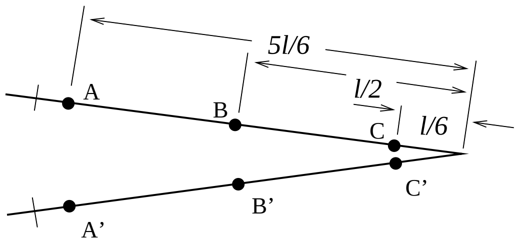

Figure 1. Crack tip element

for the traction equation. For bending plate equations, domain integrals can be written as:

$$\int_A U^P_{ki}(\mathbf{x}',\mathbf{x})\, q_k(\mathbf{x})\, dA = \sum_{n=1}^{N} \alpha^n_k \left[c^P_{ik}(\mathbf{x}')\, \hat{w}^n_{k\beta}(\mathbf{x}') - \int_\Gamma U^P_{\alpha\beta}(\mathbf{x}',\mathbf{x})\, \hat{p}^n_{k\beta}(\mathbf{x})\, d\Gamma + \int_\Gamma T^P_{\alpha\beta}(\mathbf{x}',\mathbf{x})\, \hat{w}^n_{k\beta}(\mathbf{x})\, d\Gamma \right] \tag{14}$$

for displacement equation, and

$$n_\beta(\mathbf{x}') \int_A U^P_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})\, q_k(\mathbf{x})\, dA = \sum_{n=1}^{N} \alpha^n_k \left[\frac{1}{2}\hat{p}^n_{k\alpha}(\mathbf{x}') - n_\beta(\mathbf{x}') \int_\Gamma U^P_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})\, \hat{p}^n_{k\gamma}(\mathbf{x})\, d\Gamma + n_\beta(\mathbf{x}') \int_\Gamma T^P_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})\, \hat{w}^n_{k\beta}(\mathbf{x})\, d\Gamma \right] \tag{15}$$

for the traction equation.

6.2 Cell domain integration

In the cell integration method the domain integral can be expressed as:

$$\frac{1}{h_S} \int_{\Omega_R} U^{*S}_{ij}(\mathbf{x}',\mathbf{x})\, f_j(\mathbf{x})\, d\Omega_R \cong \frac{1}{h_S} \sum_{k=1}^{ncells} \int_{\Omega_k} U^{*S}_{ij}(\mathbf{x}',\mathbf{x})\, f_j(\mathbf{x})\, d\Omega_k \tag{16}$$

and the integration is carried out on each cell. Using bi-quadratic isoparametric approximation proposed in this work we can write:

$$\frac{1}{h_S} \sum_{k=1}^{ncells} \int_{\Omega_k} U^{*S}_{ij} f_j(\mathbf{x})\, d\Omega_k \cong \frac{1}{h_S} \sum_{k=1}^{ncells} \left[\int_{\Omega_k} \underline{\mathbf{U}}^* \underline{\mathbf{N}} d\Omega_k \right] \mathbf{a}_k \tag{17}$$

where, $\underline{\mathbf{N}}$ is the matrix of bi-quadratic Lagrange shape functions and $\mathbf{a}_k = \left\{ \mathbf{u}^S_d, \mathbf{u}^R \right\}^T$ is the vector of nodal displacements at cell k. In this vector, $\mathbf{u}^S_d$ refers to sheet displacement at Ω_R and $\mathbf{u}^R$ refers to repair displacements. Similar expressions can be stablish in the case of plate bending equations.

7. STRESS INTENSITY FACTOR CALCULATIONS

For plate problems, considering bending and plane tension, the stress intensity factors can be represented by superposition of five stress intensity factors (SIF's), two due to membrane loads and three due to bending and shear loads. In terms of displacements on the crack surfaces they can be written as:

$$\{K\} = \frac{1}{\sqrt{r}} \mathbf{C} \{\Delta w\} \tag{18}$$

where K is a vector containing the five stress intensity factors. Using the extrapolation technique and discontinuos quadratic boundary elements for modeling crack surfaces, SIF can be calculated as:

$$\{K\}^{tip} = \frac{r_{AA'}}{r_{AA'} - r_{BB'}} \left(\{K\}^{BB} - \frac{r_{BB'}}{r_{AA'}} \{K\}^{AA'} \right) \tag{19}$$

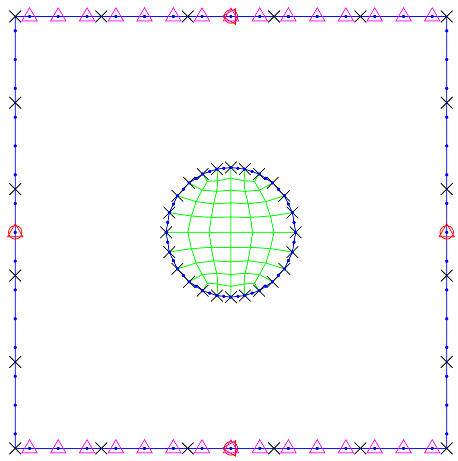

Figure 2. BEM for plate with adhesive isotropic patch

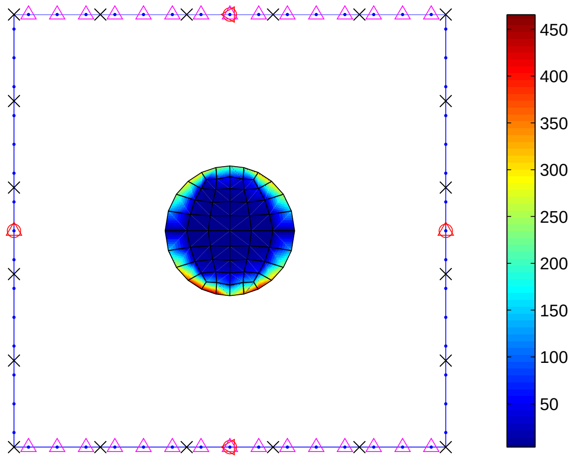

Figure 3. Shear stress distribution in the adhesive layer for isotropic patch

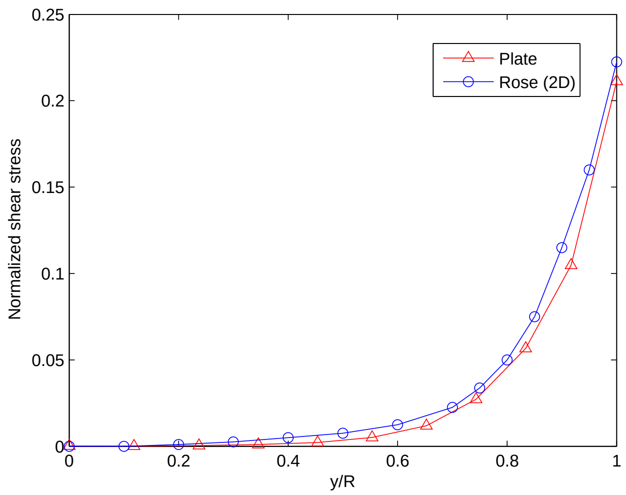

Figure 4. Shear stress distribution in the adhesive layer along y-axis for isotropic patch

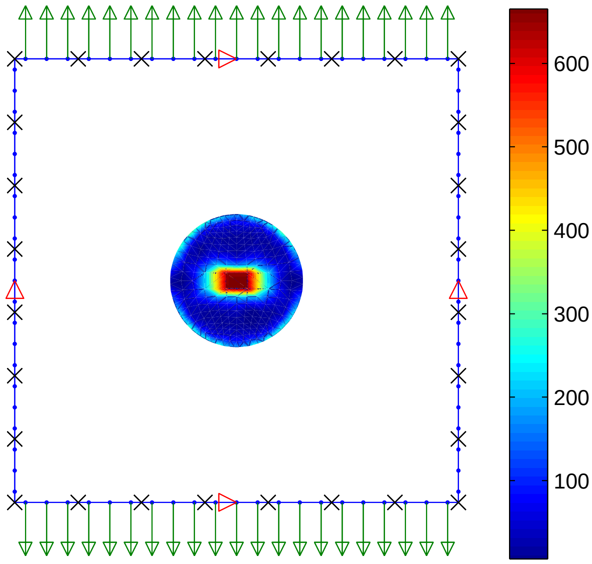

Figure 5. Shear stress distribution in the adhesive layer for a cracked plate

8. NUMERICAL EXAMPLES

8.1 Plate with adhesively bonded circular patch

A square isotropic plate with adhesively bonded isotropic circular patch will be analyzed and shear stress distribution in the repair zone and will be compared with the theoretical solution given by Rose(2002) for isotropic sheet repaired with isotropic repair. The wide of the plate is $200mm$, thickness $1.5mm$ and it is subject to in-plane load σ_0. The material constants are chosen as $E = 70GPa$, $\nu = 0.3$. A circular isotropic patch of radius $R = 30mm$ is bonded to the sheet over the region $Ax_1^2 + x_2^2$. The patch has the same material as the plate with thickness $h_R = 1.5mm$. The adhesive layer has thickness $h_a = 0.15mm$ and shear modulus $G_a = 0.6GPa$. The boundary of the plate is subdivided into 28 quadratic discontinuos elements and 24 elements at boundary patch (see Fig. 2). 56 continuos and constant bi-quadratic cells has been used. Simply supported conditions are applied to the plate (see Fig. 2). Figure 3 shows the shear stress distribution in the adhesive layer and figure 4 presents the normalized shear stress in the adhesive along y-axis obtained compared with analytic solution given by Rose(2002). As expected, maximum shear stress is presented at patch border in direction of y-axis. This result agree with those obtained by the analytical model proposed by Rose(2002), where the adhesive shear stress decays exponentially from ends of the patch, i.e., the load transfer effectively occurs over a strech length at the ends of patch for the 2D case. The results presented in figure 4 shows that bending response of plate and repair has a little influence in the adhesive shear stress magnitued, as the shear stress curve obtained is similar to the 2D-case.

8.2 Cracked plate with adhesively bonded circular patch

Figure 5 presents shear stress distribution in the adhesive layer for the same problem but considering a cracked plate. Length of this crack is considered as $2a = 30mm$. Mechanical properties and dimensions for plate, repair and adhesive are the same. Shear stress gradients appear near crack's border where the difference between plate and repair displacements is higher, as expected. Due to out-of-plane bending induced by load eccentricity, the shift of the neutral plane, repairs experiencie a bending moment contributing to a considerable increase in stress-intensity factor, because the oppening effect presented in this case. For this case the stress intensity factor obtained was 1685.9 $MPa.m^{1/2}$, compared with 1462 $MPa.m^{1/2}$ obtained using a 2D-model. This difference can be atributed to bending effect generated by the eccentric load.

9. CONCLUSIONS

A boundary element formulation for modelling cracked plates repaired with isotropic patches was presented. The cracked plate was modelled with the DBEM and the patch was modelled with the BEM. The interaction between the plate and the patch was modeled considering shear body forces uniformly distributed on the interaction zone using a linear elastic relationship. The cell domain integration and the dual reciprocity have been used to treat the domain integrals that arise in the formulation due to shear interaction forces. Was found that the number of boundary element used to discretized plate boundary and the repair boundary was little effect in the results obtained. Results show good agreement when compared with those obtained in the literature. It can be concluded that the new formulation can be used with reasonable accuracy to study the mechanical behaviour of adhesively bonded repairs. Boundary element formulation presented was implemented into computational program developed in MATLAB(R) under Windows XP. This program permits to analize isotropic sheets and plates with or without cracks, repaired with isotropic bonded adhesive repairs. Although examples are not presented here, the strategy used for the definition of the patch region allows for problems with partial debonding to be modelled in a straightforward manner.

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11. Responsibility notice