CRITICAL LOADS OF PARTIALLY BURIED COLUMNS UNDER PERIODIC TRANSVERSE EXCITATION

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Abstract. *From the model of a partially buried column proposed in the literature, in which the soil is nonlinear and modeled by Ramberg-Osgood's formulae, the column is excited by both a transverse periodic external force and by an axial constant load. We aim to determine the critical load of such structural element. This load is the maximum force that the column tolerates before any unstabilizing mechanism occurs. Melnikov's method is employed to determine an algebraic expression that relates the column parameters in order to find the values of the critical load. Due to the appearance of a complex integral without an exact analytical solution, Simpson's method is used to solve this integral, and to obtain approximate values of the critical load. Therefore, the values of critical load are compared with the values of the escape load, and then, it is shown that the results obtained from Melnikov's method can be used as a lower bound to predict of the global stability of this structural element. Accordingly, it is of big relevance to practical applications, in the sake of security.*

Keywords: *Partially buried columns, Critical load, Melnikov's method, Escape load, Basin of attraction.*

1. INTRODUCTION

The need of stable and secure structural elements in civil constructions is of great importance. An important structural element that is frequently used is the partially, or totally, buried column. Further, this column can be excited by several different kinds of forces. Analysis of this type of element is made by verifying if it can tolerate the applied loads, considering its geometric parameters, the soil parameters where the element is in, and the loading parameters.

Aiming to analyze the critical load, which is the maximum force, that a partially buried column tolerates, we start from a model developed in Queiroz and Santee (2007) for this structural element. The nonlinear soil is modeled by Ramberg-Osgood's formulae (Greimann *et. al*, 1987) and, then, the necessary criteria to obtain an expression which defines the critical load tolerated by the column, in order to keep this structural element stable, or assure the desired safety, is applied.

This critical load depends on the other parameters of the column, besides the soil where it is in. Additionally it is influenced by periodic external forces, such as: sea waves or the flutter effect of the wind.

Therefore, Melnikov's method is applied to obtain an expression that determines the critical load tolerated by the column. By applying this method, one can obtain the conditions under which the erosion of the basin of attraction starts. So, it furnishes important information about the global stability of such structural element and, then, the maximum load that it can tolerate. So, one can assure that it can tolerate compatible loads with its use and environment.

In order to validate the expression obtained, several tests comparing the critical load with escape load are made. The escape load is calculated by numerical integration algorithms for initial value problems, such Runge-Kutta. We aim to develop a theoretical basis of great relevance to practical applications. So this can ensure a structural project with safety, without being, however, conservative.

2. MATHEMATICAL MODEL FROM THE LITERATURE

The Queiroz and Santee (2007) model considers the column labeled in Fig. (1) and describes the dynamic behavior of this structural element considering its first mode vibration. Then, Hamilton's Principle and variational calculus' tools are employed followed by Euler-Bernoulli's theory that considers the column's height small when compared with its total length L.

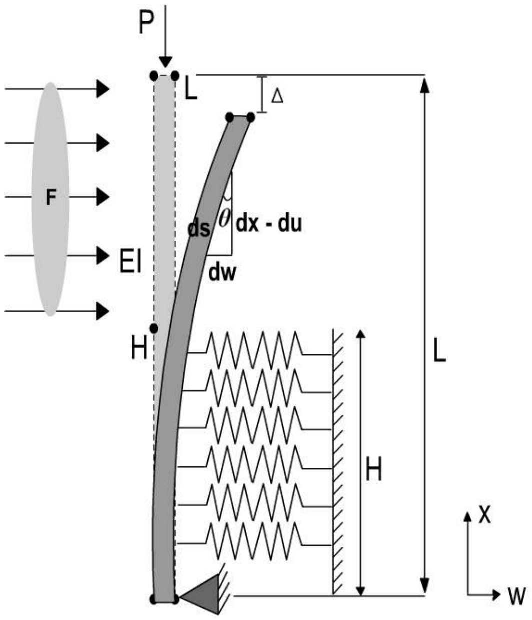

Figure 1. Column's model defined in Queiroz and Santee (2007).

The parameters showed in Fig. (1) are the total length *L,* the height *H*, the axial force *P*, the bending stiffness *EI*, and the transverse force *F*. Then, the motion equation that represents the partially buried column's dynamic behavior, in function of the time *t*, is:

$$\ddot{A} + c\dot{A} + \omega_0^2 A - \beta A\left|A\right| = f\cos\left(\Omega t\right) \tag{1}$$

where the equation's coefficients are:

$$c = \frac{\begin{pmatrix} 5H^9(\beta_e - \beta_d) + 45H^8L(\beta_d - \beta_e) + 180H^7L^2(\beta_e - \beta_d) \\ + 360H^6L^3(\beta_d - \beta_e) + 324H^5L^4(\beta_e - \beta_d) + 104L^9\beta_d \end{pmatrix}}{104mL^9} \tag{2}$$

$$\omega_0^2 = \frac{\begin{pmatrix} 9072EIL^5 + 35H^9E_{ti} - 315H^8LE_{ti} + 1260H^7L^2E_{ti} \\ - 2520H^6L^3E_{ti} + 2268H^5L^4E_{ti} - 3240L^7P \end{pmatrix}}{728mL^9} \tag{3}$$

$$\beta = \frac{3H^7\left(E_{ti} - E_{tf}\right)\begin{pmatrix} 35H^6 - 455H^5L + 2730H^4L^2 \\ - 9464H^3L^3 + 20020H^2L^4 \\ - 24570HL^5 + 14040L^6 \end{pmatrix}}{18928mL^{11}P_u} \tag{4}$$

$$f = \frac{27A_0\left(-H^5 + 5H^4L - 10H^3L^2 + 6L^5\right)}{52mL^7} \tag{5}$$

where: *m* represents the mass per unit length of the column, *E* is the elasticity modulus or Young's modulus, *I* is the moment of inertia of the transversal section, resulting in the bending stiffness *EI*, β_e is the damping parameter for the buried portion, β_d is the damping parameter for non-buried portion, A_0 is amplitude of the harmonic load and **Ω** is the frequency of the external excitation. And, the coefficients of Eq. (1) are named: *c* is the damping factor, ω_0^2 is the stiffness, β is the non-linearity of the system and f is the amplitude of the external excitation force. The soil has E_{ti} as the initial tangent elasticity modulus, E_{tf} as the final tangent elasticity modulus and P_u is the ultimate soil resistance.

The equilibrium points and the basin of attraction of the Eq. (1) are obtained in the next section.

3. COLUMN'S BASIN OF ATTRACTION

Analysis of the critical load, or maximum force that the partially buried column can tolerate, is made by conjecturing that such element is considered perfect, in other words, its geometry does not present imperfections. This way, we are concerned in knowing the basin of attraction of the column represented by Eq. (1).

The basin of attraction is the set of initial condition points that lead to an oscillatory solution around of the stable equilibrium point (Monteiro, 2002). Then, the basin of attraction (grayed area) of Eq. (1) is showed in the following figure.

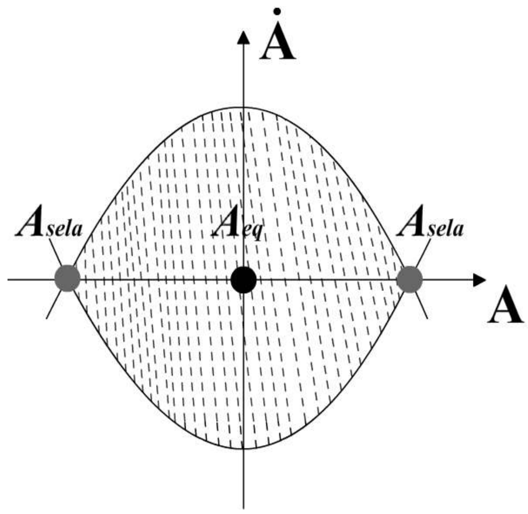

Figure 2. Basin of attraction of the column's model defined by Fig. (1) and Eq. (1).

In agreement with Fig. (2), the central point, A_{eq}, corresponds to the stable equilibrium point and the extreme points, A_{sela}, correspond to unstable equilibrium points, called saddle points. The trajectories that leave one saddle point to the other saddle point, surrounding the basin of attraction, are called heteroclinic orbits.

In order to obtain the values of these points and the exact expressions of the heteroclinic orbits, one can consider the model free of the non-conservatives forces, that is, the damping factor and the external excitation force must be null. And, this is an assumption that Melnikov's method requires. So, the Eq. (1) becomes:

$$\ddot{A} + \omega_0^2 A - \beta A|A| = 0 \tag{6}$$

As in the equilibrium points, including the saddle points, the velocity is null, the acceleration is null, too. Then, it has:

$$\begin{aligned} &\omega_0^2 A - \beta A|A| = 0 \Rightarrow \\ &A_{eq} = 0 \\ &A_{sela} = \pm \frac{\omega_0^2}{\beta} \end{aligned} \tag{7}$$

The value of the saddle point will be used by Melnikov's Method as an initial condition to obtain the Melnikov function that defines the critical load tolerated by the column, because these points lay in the solutions which are the heteroclinic orbits.

The heteroclinic orbits form the boundary between the initial conditions that lead the solution to be stable and unstable (*escape* of the solution, Santee and Gonçalves, 2006). Further, it has a practical application of the Melnikov's method on the studied problem.

4. MELNIKOV'S METHOD

Melnikov's method is an important tool for verifying the structure's global stability (Moon, 1992; Santee, 1999). This method foresees the conditions of the first transverse crossing between the stable and unstable manifolds, and this is the point that the basin of attraction starts to be eroded and, consequently, the column starts to lose stability. The method starts from the Melnikov function that measures the distance between these manifolds when this distance is small (Guckenheimer and Holmes, 1983; Moon, 1992).

A manifold is defined as a solution that leaves or arrives to saddle points. It is stable when, at a given the initial condition (points of the basin of attraction), it approaches the saddle point as $t\to\infty$, and it is unstable when it approaches the saddle point as $t\to-\infty$. Melnikov's method is discussed as follows.

4.1. Applying Melnikov's method

Start from the column's motion equation, Eq. (1), in which the non-conservative forces are all removed (Moon, 1992). These forces are removed to obtain the exact expression of the solutions that become the heteroclinic orbits. Following, Eq. (6) is integrated in the displacement, A, to obtain the column's energy expression, that is,

$$\int\left(\dot{A}\frac{d\dot{A}}{dA}+\omega_0^2 A-\beta A\left|A\right|\right)dA=\int 0\,dA \tag{8}$$

where it is, by the chain rule:

$$\ddot{A}=\dot{A}\frac{d\dot{A}}{dA} \tag{9}$$

The result of the above integral, Eq. (8), is defined as:

$$\frac{1}{2}\dot{A}^2+\frac{1}{2}\omega_0^2 A^2-\frac{1}{3}\beta A^2\left|A\right|=c \tag{10}$$

where c is a new constant of integration.

We are interested in the initial condition defined by the saddle point to obtain the desired solution. Such solutions surrounds the basin of attraction, in other words, they are the heteroclinic orbits searched. Since the velocity is null in the saddle point, the respective saddle point determined by Eq. (7) is replaced in energy equation, Eq. (10), obtaining the value of the constant c, because this is the value of the energy on the saddle point. Thus,

$$\begin{gathered}\frac{1}{2}\omega_0^2 A_{sela}^2-\frac{1}{3}\beta A_{sela}^2\left|A_{sela}\right|=c=\frac{E_{sela}}{m}\Rightarrow\\ \frac{E_{sela}}{m}=\frac{\omega_0^6}{6\beta^2}\end{gathered} \tag{11}$$

We want here to determine the solution of the Eq. (10) in order to obtain the heteroclinic orbits. And for the solution of the, well known, first order ordinary differential equation the variables separation technique is applied to solve it (Boyce and Diprima, 2002; Zill, 2003). Therefore, the following expression is reached:

$$\frac{dA}{\sqrt{2}\sqrt{\frac{E_{sela}}{m}-\frac{1}{2}\omega_0^2 A^2+\frac{1}{3}\beta A^2\left|A\right|}}=\pm dt \tag{12}$$

And, both sides of equation above, Eq. (12), are integrated considering Eq. (11). So, one obtains:

$$\frac{-2LN\left[\dfrac{\sqrt{\left(2\beta A+\omega_0^2\right)}-\sqrt{3}\omega_0}{\sqrt{\beta A-\omega_0^2}}\right]}{\omega_0}=\pm t+k \tag{13}$$

where LN is the neperian logarithm, and k is a new integration constant which is set to make $t=0$, when A is zero, $A=0$, and the velocity is maximal, that is, $\dot{A}=\dot{A}_{\max}$. Therefore, we obtain the solution $A(t)$ for the assumption that the structure is perfect:

$$A(t)=\frac{\omega_0^2\left(e^{2\omega_0 t}(4\sqrt{3}-7)+4e^{\omega_0 t}(2-\sqrt{3})-1\right)}{\beta\left(e^{2\omega_0 t}(4\sqrt{3}-7)-2e^{\omega_0 t}(2-\sqrt{3})-1\right)} \tag{14}$$

The velocity is obtained of the Eq. (14) which is differentiated in t. Then, this gives:

$$v(t)=\dot{A}(t)=-\frac{6\omega_0^3 e^{\omega_0 t}\left(e^{2\omega_0 t}(15\sqrt{3}-26)-\sqrt{3}+2\right)}{\beta\left(e^{2\omega_0 t}(4\sqrt{3}-7)-2e^{\omega_0 t}(2-\sqrt{3})-1\right)^2} \tag{15}$$

It is necessary to consider the equation of the model to be rewritten as a system of first order equations for the application to the Melnikov functions, like this:

$$\begin{aligned}\dot{A}&=v\\ \dot{v}&=-cv-\omega_0^2 A+\beta A\left|A\right|+f\cos\left(\Omega t\right)\end{aligned} \tag{16}$$

And, the non-conservative forces, that is, the damping and the external excitation force, are considered so small that the system of first order equations is expressed as:

$$\begin{aligned}\dot{A}&=\frac{\partial E}{\partial v}+\zeta g_1\\ \dot{v}&=\frac{\partial E}{\partial A}+\zeta g_2\end{aligned} \tag{17}$$

where $g=g(A,\ v,\ t)=(g_1,\ g_2)$ is a periodic vector, ζ is a small perturbation parameter and, $E(A,\ v)$ is a total energy's system free of the non-conservatives forces.

Thus, the Melnikov function, that gives a measure of separation between the stable manifold and the unstable manifold, is given by the following equation:

$$M\left(t_0\right)=2\int_0^{\infty} g^*.\nabla E\left(A^*, v^*\right)dt \tag{18}$$

where $g^*=g(A^*,\ v^*,\ t+t_0)$, $A^*(t)$ and $v^*(t)$ are the solution of the heteroclinic orbits considering the saddle point of the system. Therefore, we obtain from the above equations:

$$g^*=\begin{Bmatrix} g_1\\ g_2\end{Bmatrix}=\begin{Bmatrix} 0\\ f\cos(\Omega t)-cv\end{Bmatrix} \tag{19}$$

$$E\left(A^*, v^*\right)=\begin{Bmatrix}\dfrac{\partial E}{\partial A}\\ \dfrac{\partial E}{\partial v}\end{Bmatrix}=\begin{Bmatrix}\beta A\left|A\right|-\omega_0^2 A\\ v\end{Bmatrix} \tag{20}$$

Consequently, it results in the following integral which solved return the Melnikov function, that is,

$$M(t_0) = 2\int_0^\infty v(t+t_0)\big(f\cos(\Omega(t+t_0)) - cv(t+t_0)\big)dt \tag{21}$$

where *v(t)* is the velocity obtained of the Eq. (15). And, in agreement with Melnikov's method, the crossing between the manifolds occur when the Melnikov function is zero, that is, $M(t_0)=0$ for one or more values of t_0, because it means that the distance between them is null.

The Melnikov function, Eq. (21), determines the expression for the critical load or maximum force that the structural element can tolerate according Melnikov's Method. Since at the moment of the closing of this article we did not obtain the exact solution of the integral given by Eq. (21), we used a numerical method to solve the equation, specifically Simpson's compound rule (Burden and Faires, 2003).

4.2. Comparing the critical load with escape load

In order to verify if the expression obtained is reliable to determine the global stability of the structure, we compare the values of the escape load with the critical load, maximum force, considering the moment that the crossing between the stable and unstable manifolds occur.

Thus, considering a practical example of a partially buried column, and to show the differences between the forces we consider the following configurations (Santee and Gonçalves, 2006; Queiroz and Santee, 2007): A circular cylindrical steel column with length $L=35$ m, mass density $m=15$ kg/m, bending stiffness $EI= 5672067$ Nm², external diameter $D=10$ cm, thickness $t=1$ cm, buckling load $P=7500$ N, damping $c \cong 0.05\sqrt{4\omega_0^2}$. And, considering the soil of soft clay, it has: $Eti=4037$ Pa, $Etf=0$ Pa, $n=1$, $Pu=50$ N/m. The initial value of f is $f=0.01$ N.

In this example, the column is buried halfway, *50%*, of its total length, then, the motion equation's column is given by:

$$\ddot{A} + 0.40\dot{A} + 16.17A - 46.82A|A| = f\cos(\Omega t) \tag{22}$$

Based on Eq. (22), one gets the following figure which shows the values obtained by these forces, critical and escape loads, considering different values of frequency.

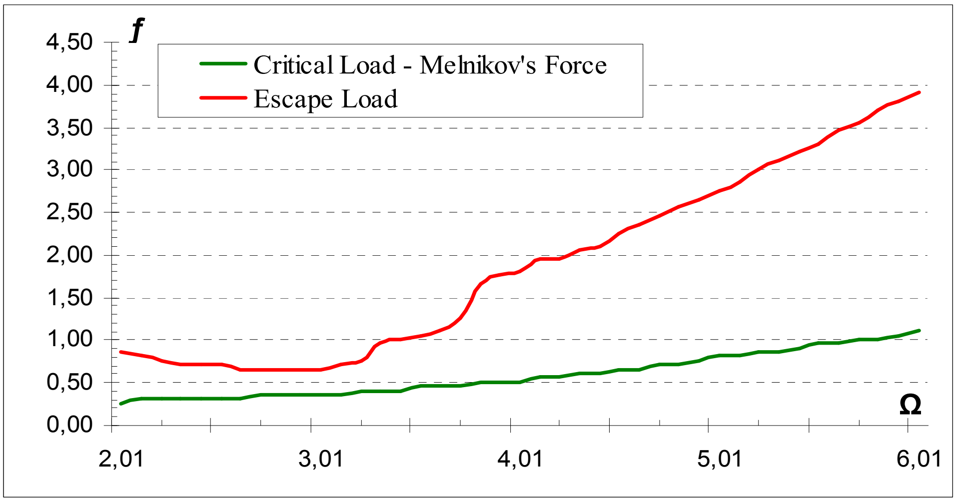

Figure 3. Comparison between critical load, Melnikov's force, and the escape load for $0.5\,\omega_0 < \Omega$ *(rad/s)* $< 1.5\,\omega_0$ considering the Eq. (22).

The critical load, Melnikov's force, is always below of the escape load in agreement with Fig. (3). Therefore, the value of Melnikov's force can be used as a lower bound estimate for the global stability's structure, because after this point, it has that the basin of attraction starts to be eroded, and this erosion can up until the total loss of stability of this element.

In this another example, the column is buried 70% on the soil. Then, it has:

$$\ddot{A} + 0.81\dot{A} + 60.81A - 343.21A|A| = f\cos(\Omega t) \tag{23}$$

The coefficients on the Eq. (23) have their values sufficiently increased in relationship to Eq. (22). It is one of the factors that shown the great sensibility that the column has in relationship to the parameters of modeling, column/soil/applied load.

The following figure keeps the same standard in relationship to previous figure where the critical load and escape load are compared considering different values of frequency of the external excitation (Ω).

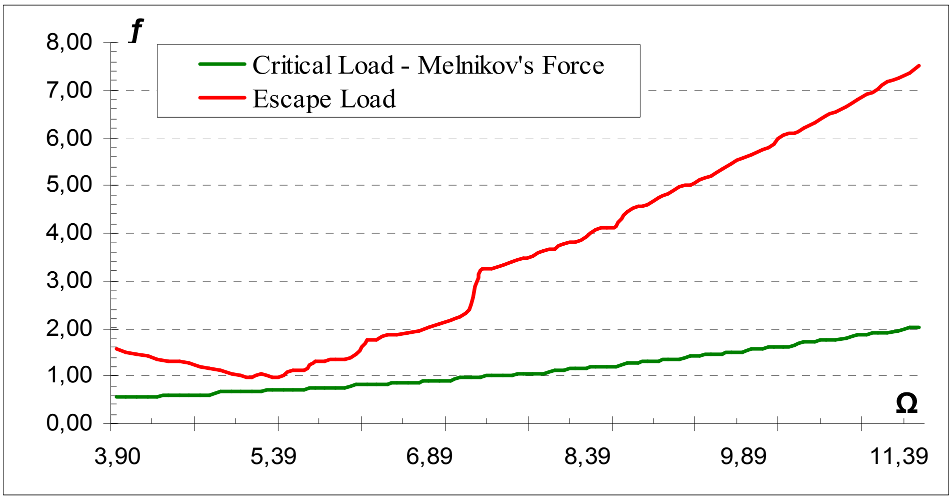

Figure 4. Comparison between the loads for $0.5\,\omega_0 < \Omega$ *(rad/s)* $< 1.5\,\omega_0$ considering the Eq. (23).

In agreement with Fig. (4), the result obtained is similar to obtained for Eq. (22), however the values of the amplitude of the external excitation (*f*) are bigger than the previous figure (Fig. (3)). The reason is that the coefficients on Eq. (23) are bigger than the on Eq. (22) and, then, the soil act powerfully on column such it need of great vibrations for occur the *escape*.

In this final example, 90% of the column is buried in the soil. Thus,

$$\ddot{A} + 1.34\dot{A} + 179.01A - 1392.72A|A| = f\cos(\Omega t) \tag{24}$$

The Fig. (5) shows the result that is similar with the previous. It observes, again, the sensibility that the motion equation's column has with regard to parameters of modeling used, considerably, the depth of the column on soil.

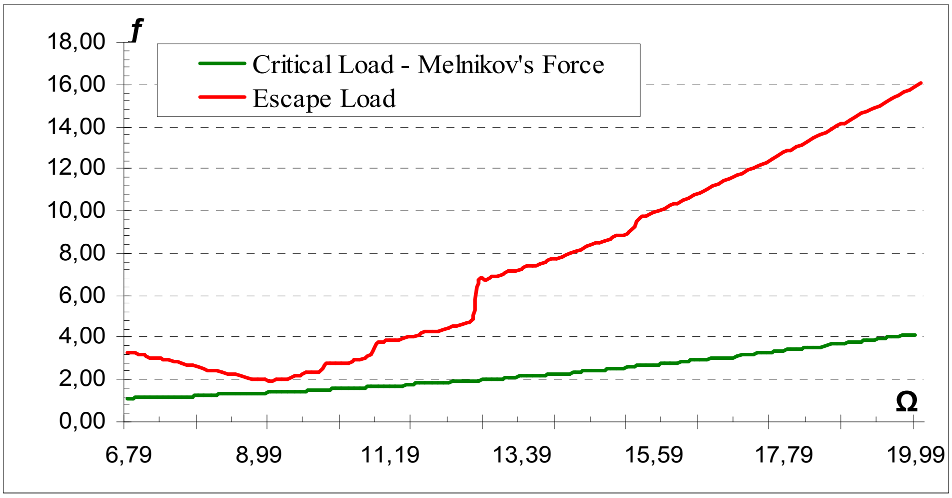

Figure 5. Comparison between the loads for $0.5\,\omega_0 < \Omega$ *(rad/s)* $< 1.5\,\omega_0$ considering the Eq. (24).

Several tests have been made for different values of H, and the results obtained keep the same pattern showed in this paper. Then, the results are validated and shown that the criterion used has its theoretical/practice importance.

5. CONCLUSIONS

From a mathematical model described in the literature for a partially buried column, this paper developed and tested a criterion to safely predict the column's load capacity as a function of the various parameters of the column-load-soil system. When applying Melnikov's method we aimed in writing the transverse load parameter as a function of the other system parameters.

However, due to the complexity involved in one of the integrations it was necessary to rely on a numerical solution, namely Simpson's rule. From the results obtained, we have shown that Melnikov's method gives us a lower bound estimate to the load capacity of the column, above which the structure starts to lose stability.

Further studies are being elaborated to integrate Eq. (21). The method that seems most promising is the use of the residual theorem from the theory of complex variables. This will give us an algebraic expression for the critical load. The algebraic expression obtained by this method can be further used to develop a reliable design criterion for this type of structural element.

Moreover, another works have been done comparing the results obtained by Melnikov's criterion with Bifurcation Theory's criterion. Also, we have searched a new methodology/criterion capable of define with great verity the critical load that the element tolerate compatible with the environment inserted.

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