FRACTURE ANALYSIS OF REISSNER PLATES USING THE DUAL BOUNDARY ELEMENT METHOD

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Abstract. *The application of the dual boundary element method (DBEM) for the analysis of cracks in shear deformable plates, is presented. Equations for the DBEM, including the direct boundary and the traction boundary equation for Reissner plates, were established. Taylor expansions (combined with Telles transformation and element sub-division) were used to deal with hypersingular kernels arising in the traction equation. Continuity conditions for the existence of Haddamard and Hölder principal-value integrals are discussed. The stress intensity factors (SIF) were calculed using the crack surface displacements extrapolation. A general crack modelling strategy is presented and implemented in MATLAB™. Test problems, including comparisons to reported solutions, are presented. Results demonstrate that DBEM is a reliable method for fracture mechanics analysis in plate bending problems.*

Keywords: *Fracture mechanics, cracked plates, dual boundary element method, Reissner plates, shear deformable plates*

1. INTRODUCTION

The Finite Element Method (FEM) has been extensively used for the analysis of fracture problems, mainly due to its generality. In the case of laminar structures FEM analysis is carried out developing plate elements based on the Kirchhoff or Reissner-Mindlin plate theories. However, this generality could introduce expensive computational costs, mainly in problems that involve stress field singularities (like those found near to the crack tip) that requires very fine discrtization around of the singular point. The Boundary Element Method (BEM) is an atractive numerical alternative to treat fracture problems, mainly to its ability to model continuously high stress gradients without the need of domain discretization. Additionaly crack propagation problems can be analised with simple techniques that only require aditional elements at crack front and incremental solvers. The use of this method in structural analysis has strongly increased since 80s (Brebbia and Dominguez(1989)).

At present, the BEM analysis of Reissner's plates is based on fundamental solutions developed using the H$\ddot{o}$mander method as presented on the work of Vander Wee$\ddot{e}$n(1982). Rashed, Aliabadi and Brebbia(1999) presents the hypersingular boundary element formulation for plate bending analysis based on the Reissner's theory. In this work, first order Taylor expansion was used to deal with the strong singularity, whereas rigid body considerations, together with the Taylor expansion, were used to compute the hypersingular kernels. Wen, Aliabadi and Young(2003) presents the analysis of stiffened cracked plates using the dual boundary element method (DBEM). In this work rectangular stiffened plate containing a single crack and double cracks were analized using this method. In the work of Dirgantara and Aliabadi(2001) the dual boundary element formulation for fracture mechanics analysis of shear deformable shells, was presented. Shell formulation was formed by coupling boundary element formulation for shear deformable plate and two-dimensional plane stress elasticity. Recently, Guimaraẽs and Telles(2007), presents a numerical Green's function technique for crack analysis in Reissner's plates. The technique produces a plate bending fundamental Green's function that automatically includes embedded cracks to be used in the classical boundary element method.

This work presents the Dual Boundary Element Method applied to plate fracture analysis considering bending moments and shear forces. The hypersingular equations for Reissner's plate bending are developed obtaining the traction equations. Different types of singularities that appears in these equations and their treatment using the Taylor series expansion methodology are identify. The dual boundary element method is presented for the treatment of fracture mechanics problems and a general methodology is expossed. Finally, the stress intensity factors for bending problems are defined. Numerical examples are presented and preliminary conclusions are stablished.

2. REISSNER'S PLATE EQUATIONS

Consider an arbitrary plate of thickness, h with a domain Ω and boundary Γ in the x_i space. The $x_1 - x_2$ plane is assumed to be located at the middle surface $x_3 = 0$. The generalized displacements are denoted as w_i, where w_α denotes rotations (ϕ_{x1} and ϕ_{x2}) and w_3 denotes the transverse deflection w (see Rashed(1999)).

The equilibrium equations can be formed by considering the equilibrium of a typical differental plate element under uniform load q (per unit area), as regarded positive when applied in the x_3 direction. The equilibrium of moments about the x_1 and x_2 axis and the equilibrium of forces in the x_3 direction can be written as follows:

$$M_{\alpha\beta,\alpha} - Q_\alpha = 0 \tag{1}$$
$$Q_{\alpha,\alpha} + q = 0 \tag{2}$$

The stress resultant-strain relationships are derived using the basic minimun principle for the stresses as presented in Reissner(1947). The resultant tensor moment $M_{\alpha\beta}$ and the normal shear vector Q_α is given by:

$$M_{\alpha\beta} = D\frac{1-v}{2}\left(2\chi_{\alpha\beta} + \frac{2v}{1-v}\chi_{\gamma\gamma}\delta_{\alpha\beta}\right) \tag{3}$$

$$Q_\alpha = D\frac{1-v}{2}\lambda^2\psi_\alpha \tag{4}$$

where:

$$\begin{aligned} 2\chi_{\alpha\beta} &= w_{\alpha,\beta} + w_{\beta,\alpha} \\ \psi_\alpha &= w_\alpha + w_{3,\alpha} \end{aligned} \tag{5}$$

D represents the flexural rigity, v is th poisson's ratio, $\chi_{\alpha\beta}$ is the curvature tensor and ψ_α are the transversal shear strains. Equation (4) represents the generalised Hooke's law for Reissner's plates.

3. BOUNDARY INTEGRAL FORMULATION FOR REISSNER PLATES

The integral equation can be derived by considering the integral representation of equations (1) and (2) via the following integral identity:

$$\int_\Omega \left[(M_{\alpha\beta,\beta} - Q_\alpha) W_\alpha^* + (Q_{\alpha,\alpha} + q) W_3^*\right] d\Omega = 0 \tag{6}$$

where W_i^* $(i = \alpha, 3)$ are weighting functions. Integrating by parts and making use of the relationships in Eq.(4) and applying the Green's second identity for the $M_{\alpha\beta}$ integral gives (see Dirgantara and Aliabadi(2001)):

$$\begin{aligned} \int_\Gamma \left(W_j^* p_j - P_j^* w_j\right) d\Gamma &+ \int_\Omega \left[W_3^* + \frac{v}{(1-v)\lambda^2} W_{\theta,\theta}^*\right] q d\Omega \\ &+ \int_\Omega \left[\left(M_{\alpha\beta,\beta}^* - Q_\alpha^*\right) w_\alpha + Q_{\alpha,\alpha}^* w_3\right] d\Omega = 0 \end{aligned} \tag{7}$$

This equation represents the generalized Betti's reciprocal theorem for Reissner plates; It has to be noted that the weighting functions can be chosen to represents arbitrarily state. This state is defined for concentrated generalised loads: two bending moments ($i = \alpha = 1, 2$) and one concentrated shear force ($i = 3$) at an arbitrary point $\mathbf{x}' \in \Omega$. By chosing weighting function as:

$$\begin{aligned} M_{i\alpha\beta,\beta}^*(\mathbf{x}', \mathbf{x}) - Q_{i\alpha}^*(\mathbf{x}', \mathbf{x}) &= -\delta(\mathbf{x}', \mathbf{x})\delta_{i\alpha} \\ Q_{i\alpha,\alpha}^*(\mathbf{x}', \mathbf{x}) &= -\delta(\mathbf{x}', \mathbf{x})\delta_{i3} \end{aligned} \tag{8}$$

and making use of the Dirac delta property: $\int_\Omega \delta(\mathbf{x}', \mathbf{x}) w_i(\mathbf{x}) d\Omega = w_i(\mathbf{x}')$, equation (7) can be written as:

$$\begin{aligned} c_{ij}(\mathbf{x}') w_j(\mathbf{x}') + \int_\Gamma P_{ij}^*(\mathbf{x}', \mathbf{x}) w_j(\mathbf{x}') d\Gamma &= \int_\Gamma W_{ij}^*(\mathbf{x}', \mathbf{x}) p_j(\mathbf{x}') d\Gamma \\ &+ \int_\Omega \left(W_{i3}^*(\mathbf{x}', \mathbf{x}) - \frac{v}{(1-v)\lambda^2} W_{i\alpha,\alpha}^*(\mathbf{x}', \mathbf{x})\right) q d\Omega \end{aligned} \tag{9}$$

where $c_{ij}(\mathbf{x}')$ are the jump terms arising from the terms of $O(1/r)$ in the kernel P_{ij}^*, $W_{ij}^*(\mathbf{x}', \mathbf{x})$ and $P_{ij}^*(\mathbf{x}', \mathbf{x})$ are the two-point fundamental solution kernels for the displacements and the tractions respectively. It represents the displacement or the tractions at the point $\mathbf{x}$ in the direction j due to unit load applied at collocation point $\mathbf{x}'$ at the direction i. The expressions for these kernels are given by Vander Weeën(1982). W_{ij}^* is a weakly singular and P_{ij}^* has a strong (Cauchy

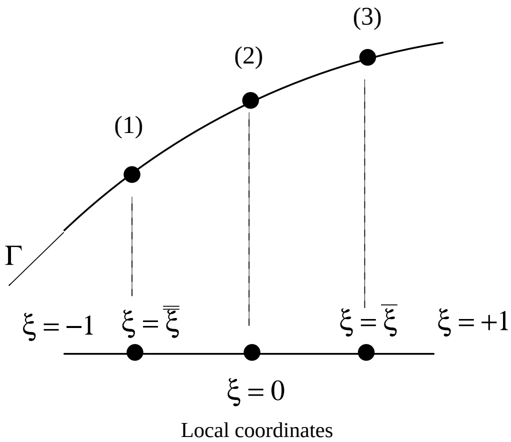

Figure 1. General quadratic element

principal value) singularity $O(1/r)$. Equation (9) represent three integral equations (two (i=1, 2) for rotations and one for deflection).

The domain integral in Eq.(9) can be transferred to the boundary (by applying the divergence theorem), in the case of a uniform load (q = constant) to give:

$$\int_{\Omega} \left(W_{i3}^{*}\left(\mathbf{x}',\mathbf{x}\right) - \frac{v}{(1-v)\lambda^{2}} W_{i\alpha,\alpha}^{*}\left(\mathbf{x}',\mathbf{x}\right) \right) q d\Omega = q\int_{\Gamma} \left(V_{i,\alpha}\left(\mathbf{x}',\mathbf{x}\right) - \frac{v}{(1-v)\lambda^{2}} W_{i\alpha}^{*}\left(\mathbf{x}',\mathbf{x}\right) \right) n_{\alpha} d\Gamma \tag{10}$$

where V_i^* are the particular solutions of the equation $V_{i,\theta\theta}^* = W_{i3}^*$. According with Mindlin, the term: $v/((1-v)\lambda^2)$ has negligible contribution to the results. This term will be ignored in this work.

4. BOUNDARY ELEMENT DISCRETISATION

The analytical solution of the integral Eq.(9) is difficult even for a simple plate problem. Therefore, the numerical solution can be considered. In this work, the boundary has to be discretised into N_e elements, over which the unknows are approximated to vary quadratically using quadratic discontinuos elements. After the discretization, Eq.(9) can be rewritten as:

$$c_{ij}\left(\mathbf{x}'\right) w_i\left(\mathbf{x}'\right) + \sum_{j=1}^{N_e}\sum_{m=1}^{3} w_j^m \int_{-1}^{+1} P_{ij}^{*}\left(\mathbf{x}',\mathbf{x}\left(\xi\right)\right)\Phi^m\left(\xi\right) J_j\left(\xi\right) d\xi = \sum_{j=1}^{N_e}\sum_{m=1}^{3} p_j^m \int_{-1}^{+1} W_{ij}^{*}\left(\mathbf{x}',\mathbf{x}\left(\xi\right)\right)\Phi^m\left(\xi\right) J_j\left(\xi\right) d\xi + q\sum_{j=1}^{N_e}\int_{-1}^{+1} V_{i,\alpha}^{*}\left(\mathbf{x}',\mathbf{x}\left(\xi\right)\right) n_{\alpha}\left(\xi\right) J_j\left(\xi\right) d\xi \tag{11}$$

where J is the jacobian of the transformation and Φ is the element shape function. For a general quadratic element we as shown in Fig.1 we have:

$$\Phi^1\left(\xi\right) = \frac{1}{\bar{\xi}\left(\bar{\xi}-\bar{\bar{\xi}}\right)}\xi\left(\xi-\bar{\bar{\xi}}\right) \tag{12}$$

$$\Phi^2\left(\xi\right) = \frac{1}{\bar{\xi}\bar{\bar{\xi}}}\left(\xi-\bar{\xi}\right)\left(\xi-\bar{\bar{\xi}}\right) \tag{13}$$

$$\Phi^3\left(\xi\right) = \frac{1}{\bar{\bar{\xi}}\left(\bar{\bar{\xi}}-\bar{\xi}\right)}\xi\left(\xi-\bar{\bar{\xi}}\right) \tag{14}$$

and

$$J\left(\xi\right) = \sqrt{\frac{\partial x_{\theta}\left(\xi\right)}{\partial\xi}\frac{\partial x_{\theta}\left(\xi\right)}{\partial\xi}} \tag{15}$$

$$n_\alpha(\xi) = \frac{1}{J(\xi)}\frac{\partial x_\beta(\xi)}{\partial \xi}\epsilon_{\alpha\beta 3} \tag{16}$$

where $\epsilon_{\alpha\beta 3}$ is the permutation symbol. After performing the collocation process, Eq.(11) can be written as follows:

$$\mathbf{Hw} = \mathbf{Gp} + \mathbf{Q} \tag{17}$$

The influence matrix $\mathbf{G}$ and the load vector $\mathbf{Q}$ contains weakly singular kernels, which can be cancelled using a non-linear coordinate transformation (Telles(1987)). In addition, for better numerical accuracy a suitable number of element sub divisions along with the non-linear transformation will be used in this work. On the other hand, The influence matrix $\underline{\mathbf{H}}$, contains a strongly singular kernel, which can be evaluated indirectly be expressing that the free stress problem admits non-trivial solutions which are arbitrary combinations of three basic rigid-body displacements (see Vander Weeën(1982)):

$$\begin{aligned} w_1 &= C, w_2 = 0, w_3 = -Cr_1 \\ w_1 &= 0, w_2 = C, w_3 = -Cr_1 \\ w_1 &= 0, w_2 = 0, w_3 = C \end{aligned} \tag{18}$$

where C is an arbitrary constant. In this way one obtains:

$$\begin{aligned} c_{i\beta}(\mathbf{x}') &= -\int_\Gamma \left[P^*_{i\beta}(\mathbf{x}',\mathbf{x}) + r_\beta P_{i3}(\mathbf{x}',\mathbf{x})\right] d\Gamma \\ c_{i3}(\mathbf{x}') &= -\int_\Gamma P_{i3}(\mathbf{x}',\mathbf{x})\, d\Gamma \end{aligned} \tag{19}$$

5. HYPERSINGULAR FORMULATION FOR REISSNER'S PLATES

The stress resultants at boundary point $\mathbf{x}'$ can be evaluated from Eqs.11, by using the resultant stress-displacement relationships (see Rashed, Aliabadi and Brebbia(1998)). For a source point on a smooth boundary, the traction integral equations are obtained as follows:

$$\begin{aligned} \frac{1}{2}p_\alpha(\mathbf{x}') &+ n_\alpha(\mathbf{x}')\int_\Gamma P^*_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})\, w_\gamma(\mathbf{x})\, d\Gamma + n_\alpha(\mathbf{x}')\int_\Gamma P^*_{3\beta 3}(\mathbf{x}',\mathbf{x})\, w_3(\mathbf{x})\, d\Gamma \\ &= n_\alpha(\mathbf{x}')\int_\Gamma W^*_{\alpha\beta\gamma}(\mathbf{x}',\mathbf{x})\, p_\gamma(\mathbf{x})\, d\Gamma + n_\alpha(\mathbf{x}')\int_\Gamma W^*_{\alpha\beta 3}(\mathbf{x}',\mathbf{x})\, p_3(\mathbf{x})\, d\Gamma \\ &+ \frac{1}{h}n_\alpha(\mathbf{x}')\int_A W^*_{\alpha\beta 3}(\mathbf{x}',\mathbf{x})\, q_3 dA \end{aligned} \tag{20}$$

and,

$$\begin{aligned} \frac{1}{2}p_3(\mathbf{x}') &+ n_\alpha(\mathbf{x}')\int_\Gamma P^*_{3\beta\gamma}(\mathbf{x}',\mathbf{x})\, w_\gamma(\mathbf{x})\, d\Gamma + n_\alpha(\mathbf{x}')\int_\Gamma P^*_{3\beta 3}(\mathbf{x}',\mathbf{x})\, w_3(\mathbf{x})\, d\Gamma \\ &= n_\alpha(\mathbf{x}')\int_\Gamma W^*_{3\beta\gamma}(\mathbf{x}',\mathbf{x})\, p_\gamma(\mathbf{x})\, d\Gamma + n_\alpha(\mathbf{x}')\int_\Gamma W^*_{3\beta 3}(\mathbf{x}',\mathbf{x})\, p_3(\mathbf{x})\, d\Gamma \\ &+ \frac{1}{h}n_\alpha(\mathbf{x}')\int_A W^*_{3\beta 3}(\mathbf{x}',\mathbf{x})\, q_3 dA \end{aligned} \tag{21}$$

where n_β is the the outward normal at the source point.

In these equations, the singularity order is higher than the displacement integral equations. In the $[H]$ matrix, the kernels $P^*_{\alpha\beta\gamma}$ and $P^*_{3\beta\gamma}$ are strongly singular, whereas, the kernels $P^*_{\alpha\beta\gamma}$ and $P^*_{3\beta 3}$ are hypersingular. In the off-diagonal sub-matrices, the shape functions will reduce the order of singularity by one. This means that, elements entries in $[H]$ matrix corresponding to the kernels $P^*_{\alpha\beta 3}$ and $P^*_{3\beta\gamma}$ become smooth, whereas, elements of the kernels $P^*_{\alpha\beta\gamma}$ and $P^*_{3\beta 3}$ still remain stringly singular (Dirgantara and Aliabadi(2001)). In $[G]$ matrix, the off-diagonal sub-matrices are smooth again due to the shape functions reducing the order of singularity. The diagonal matrices, on the other hand, contain the kernels $W^*_{\alpha\beta 3}$ and $W^*_{3\beta\gamma}$ which are weakly singular and the $W^*_{\alpha\beta\gamma}$ and $W^*_{3\beta 3}$ which are strongly singular. Weak singularities are treated using a nonlinear coordinate transformation as in Telles(1989). Strong-singular and the hypersingular integrals are evaluated using Taylor series expansion around the singular point.

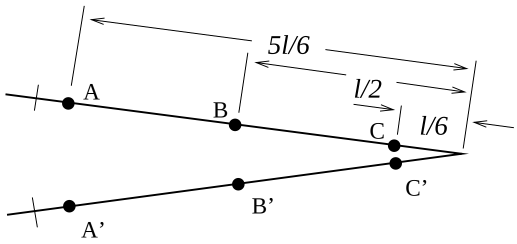

Figure 2. Crack tip element

6. CRACK MODELLING STRATEGY

The finite-part integral of first order, in the displacement equations, requires continuity of the displacement components at the nodes: any continuos or discontinuos boundary element satisfies this requirement. In the tractions, the finite-part integral of second order requires continuity of the displacement derivatives at the nodes, on a smooth boundary. Then, discontinuous quadratic boundary elements can be used. The general modelling strategy implemented in this work is based closely to that used by (Dirgantara and Aliabadi(2001)), and can be summarized as follows:

- (i) The crack boundaries are modelled with discontinuos quadratic flat elements.
- (ii) The displacement equations is applied for collocation on one of the crack surfaces
- (iii) The traction equations is applied for collocation on the opposite surface.
- (iv) discontinuos quadratic flat elements are used along the remaining boundary of the body.

7. STRESS INTENSITY FACTOR EVALUATION

For plate problems, considering bending and plane tension, the stress intensity factors can be represented by three stress intensity factors (SIF's), due to bending and shear loads. In terms of displacements on the crack surfaces they can be written as:

$$\{K\} = \frac{1}{\sqrt{r}}\mathbf{C}\{\Delta w\} \tag{22}$$

where K is a vector containing the stress intensity factors and Δw contains displacements and rotatios. Using the extrapolation technique and discontinuos quadratic boundary elements for modeling crack surfaces, SIF can be calculated as (see Fig. 2):

$$\{K\}^{tip} = \frac{r_{AA'}}{r_{AA'} - r_{BB'}}\left(\{K\}^{BB} - \frac{r_{BB'}}{r_{AA'}}\{K\}^{AA'}\right) \tag{23}$$

8. NUMERICAL EXAMPLES

8.1 Plate with a central crack loaded by bending and tension

A rectangular plate with a central crack loaded by edge bending and tension is analyzed (see Fig.3. The properties of the plate are: $b/h = 2$; $c/b = 2$; $M_o = 1.0 N.m$; $t = b/10$, $E = 210000MPa$ and $\nu = 0.3$. For DBEM analysis, 32 boundary elements for plate border and 16 discontinuos quadratic elements for each faces of the crack has been used (see Fig.3). Table 1 shows SIF for K_{1b} factor for differents a/b relations. The DBEM results show good agreement when compared with those obtained by Dirgantara and Aliabadi(2001).

Table 1. K_{1b} stress intensity factor for rectangular plate with central crack

a/b	$K_{1b}/M_o\sqrt{\pi a}$ **- BEM**	$K_{1b}/M_o\sqrt{\pi a}$ **- Ref.[2]**	**% error**
0.1	0.993	0.995	0.20
0.2	0.992	0.990	0.20
0.4	0.845	0.850	0.59
0.6	0.095	0.100	0.50
0.8	0.134	0.135	0.74

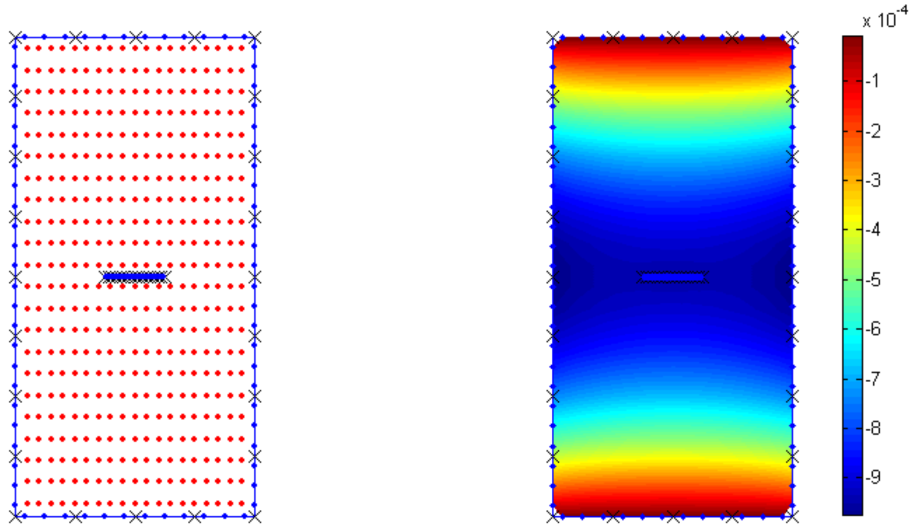

Figure 3. Left: DBEM model for rectangular plate with central crack right: Displacement distribution

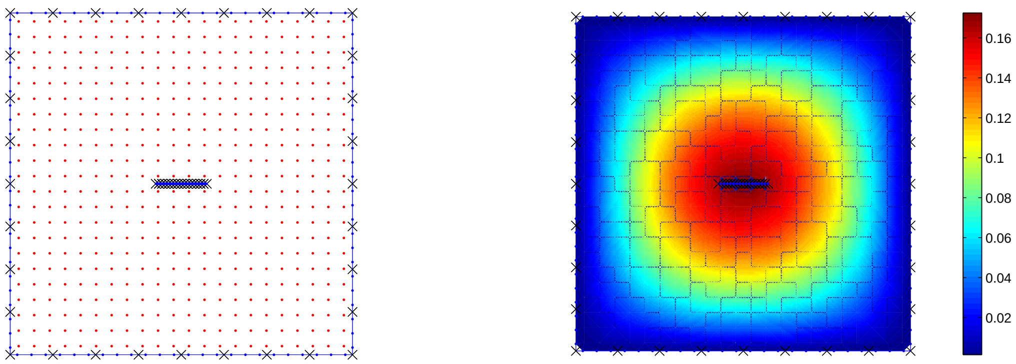

Figure 4. Left: DBEM for Simply supported square plate with a central crack right: Displacement distribution

8.2 Square plate with a center crack: uniform pressure

A simply supported square plate with a central crack loaded by uniform pressure $p_o = 1.0$ is analyzed. The properties of the plate are: b = 1; $b/h = 2$; $E/p_o = 1000$ and $\nu = 0.3$. DBEM contains 4 elements per side of the plate and 16 elements for each crack surface as shown in Fig. 4. Table 2 shows SIF for K_{1b} factor for differents a/b relations. DBEM results show good agreement when compared with those obtained in the literature. Bending deflection distribution is showed in Fig.4.

Table 2. K_{1b} stress intensity factor for square plate with central crack

a/b	$K_{1b}/p_o b^2\sqrt{\pi a}$ **- BEM**	$K_{1b}/p_o b^2\sqrt{\pi a}$ **- Ref.[2]**	**% error**
0.1	0.149	0.150	0.67
0.2	0.139	0.138	0.72
0.4	0.120	0.119	0.84
0.6	0.099	0.098	1.02
0.8	0.061	0.060	1.67

9. CONCLUSIONS

The Dual Boundary Element Method applied to plate fracture analysis considering bending moments and shear forces was presented. The hypersingular for Reissner's plates were stablish. Different types of singularities arising in the displacement and traction equations has been identify and the Taylor's expansion technique was used to threated it. A general metodology for application of the dual boundary element method was presented. Numerical examples shows a good agreement for SIF's calculated with those reported in the literature.

10. REFERENCES

Brebbia, C. A. and Dominguez, J., 1989, "Boundary Element - An Introductory Course"', C.M.P., 2 ed., Southamptom.

Dirgantara, T., Aliabadi, M. H., 2001, "Dual boundary element formulation for fracture mechanics analysis of shear deformable shells", International Journal of Solid and Structures, Vol. 38, pp. 7769-7800.

Guimaraẽ s, S., Telles, J.C.F.,2007, "On the numerical GreenŠs function technique for cracks in ReissnerŠs plates", Computer methods in applied mechanics and engineering, Vol. 196, pp. 2478Ű2485.

Rashed, F., Aliabadi, M. H. and Brebbia, C. A, 1998, "Hypersingular boundary element formulation for Reissner plates", International Journal of Solids and Structures, Vol. 35, pp 2229-2249.

Rashed, Y. F., 1999, "Boundary Element Formulation for thick plates",Topics in Engineering, Vol. 35, W.I.T. Press Inc.

Reissner, E., 1947, "Bending of elastic plates", Quarterly of applied mathematics, Vol. 5, pp. 55-68.

Telles, J.C.F., 1987, "A self adaptative coordinate transformation for efficient numerical evaluation of general boundary element integrals", International Journal for Numerical Methods in Engineering, pp. 959-973.

Vander Weeën, F., 1982, "Application of the boundary integral equation method to Reissner's plate model", International Journal for Numerical Methods In Engineering, Vol. 18, pp. 1-10.

Wen, P. H., Aliabadi, M. H., Young, A., 2003, "Boundary element analysis of curved cracked panels with adhesively bonded patches", International Journal for Numerical Methods In Engineering, Vol. 58, pp. 43-61.

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