KINEMATIC ANALYSIS OF AUTOMOTIVE GEARBOX MECHANISMS USING DAVIES' METHOD

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Abstract. *In this article, a kinematic analysis of automotive gearbox mechanisms using Davies' method is presented. Two known topologies used in automatic transmissions are studied: Ravigneaux, a two degrees of freedom gear train resulting in three gear ratios forward and one backward; and Lepelletier, a three degrees of freedom gear train resulting in six gear ratios forward and one backward. In both cases, Davies' method is employed to automate the Kirchhoff's circuit law by means of a graph theoretic representation. The screw theory is applied to add coupling characteristics to the system of equations and solve kinematics. Once having the kinematic equations system, some actuations are required to be carried out in the mechanisms in order to obtain each gear ratio and its related couplings velocity map. The process is depicted step-by-step and its resulting kinematic solutions are discussed. Overall, this procedure unifies the kinematics calculation for gear train mechanisms.*

Keywords: *kinematic analysis, planetary gear train, couplings graph, motion screw, Davies' method*

1. INTRODUCTION

Automotive gearbox mechanisms are being intensively studied with focusing on raising powertrain efficiency. Nowadays, two distinct types of mechanisms are in evidence: planetary gear trains (PGTs) and continuous variable transmissions (CVTs). In this paper, discussions are limited to kinematic analysis of PGTs through Davies' method. Two well known topologies are studied: Ravigneaux (Ravigneaux, 1940) and Lepelletier (Lepelletier, 1992).

Kinematic analysis of mechanisms performed through a graph theoretic formulation of Kirchhoff's circulation law and motion screw are not new and was firstly presented in Davies (1981), so named Davies' method. Further, that method was applied to solve kinematic analysis of a gripper mechanism (Davies, 1995b) and an simple PGT (Davies, 1995a). Another work, more historic, was presented by Davies (2000) describing in detail the referred method. Campos (2004) applied the method to solve differential kinematics of manipulators using virtual chains concept.

Other methods are based on multibody systems. These methods derive the dynamic equations of the kinematic chain (multibody system) using Euler's or Lagrange approaches. Despite its generality the system is quite complex and computer time consuming, specially with multi closed loops as found in PGTs. Moreover, these approaches depend upon dynamic data such as inertias, masses etc. as inputs. For a simplified kinematic or static analysis they are not recommended. The computational cost is even more significant when we consider optimisation processes when the each evaluation a kinematic or static analysis is done.

In specialized literature it is possible to find works using graph tools to solve gearbox mechanisms problems. Wojnarowski et al. (2006) presents a survey of works connected with the problem of modelling gear pairs by means of versatile graph theory models. Talpasanu et al. (2006) applies matroid method to systemize the generation of kinematic equations of any epicyclic gear trains (EGTs), approach which deal with edge-oriented graph models and excessive matrix manipulation. Other works applies graph theory to model bevel gear trains (Nelson and Cipra, 2005; Tsai, 1988; Uyguroğlu and Demirel, 2005) and to develop a method for detecting degenerate structures in planetary gear trains (Hsu and Lin, 1994; Hsu and Wu, 1997; Salgado and Castillo, 2005). Kahraman et al. (2005) proposed a methodology based on analytical equations to solve automatic transmission planetary gear trains limited to one-degree-of-freedom (1-DOF).

Tian and Li-qiao (1997) exposes an interesting matricial method to analize kinematic of planetary gear trains and its auxiliary components. However, they introduce some specific concepts that require another insight about gear trains mechanisms.

All above cited works have in common the procedure of mathematical modelling mechanisms through graphs to model mechanical systems and to assemble its system equations based on fundamental circuits. Couplings characteristics like angular velocity and linear velocity are introduced into the equations by Willis *fundamental law of gearing* (Willis, 1870). However, none of them uses screw theory to arise kinematic equations and consequently, solutions employing Davies' method were not found in the correspondent literature for multi-DOF gearboxes.

The aimed goal in this paper is to present Davies' method and to depict step-by-step its usage for gear train kinematic analysis.

2. REVIEW OF BASIC CONCEPTS

For convenience, this section was introduced to clarify some concepts and definitions involving the main subjects that constitute Davies' method and its application.

2.1 Planetary Gear Train (PGT)

Planetary gear train (PGT) is the appropriate term for the class of gear trains in which some body describe planetary motion (rotation and revolution) (Arnaudov et al., 2005). A gear that rotates around a central stationary axis is called *sun* or *ring gear* depending on whether it is an external or internal gear, and those gears whose joint axes revolve around the central axis are called the *planet* gears. Each meshing gear pair has a supporting bodie, called *carrier* or *arm*, which keeps distance between the center of the two meshing gears constant (Tsai, 2001).

Usually, gear trains are overconstrained chains composed by two types of direct couplings: rotative pairs and gear pairs.Rotative pairs, R, are *lower kinematic pairs* since one body envelops the other through surface contact, being characterised by permitting only one relative angular displacement between bodies. Gear pairs, G, are considered *higher kinematic pairs* once one body contacts the other through a point, being characterized by allowing two relative motions between body constrained in a plane (perpendicular to central axis of rotation): one linear displacement and one angular displacement.

However, the relative linear displacement is constrained when gear pairs have its central axis distance fixed by other bodies, resulting in only an angular relative motion between gears. The meshing surfaces must satisfy the *law of gearing*, and the diametric pitch of a pair of gears must be equal to one another (Willis, 1870). In vehicle gearboxes, three types of meshing teeth are usually found: spur, helical and bevel gears.

Direct couplings that permit only one degree of freedom are called *joint* (Davies, 1981).

According to the coupling characteristics described above, it is possible to use the *Grübler/Kutzbach* criterion for planar and spherical mechanisms in Eq. (1) to determine the nett degree of freedom F_N for the gear trains.

$$F_N = d(n - j - 1) + \sum_{i=1}^{e} f_i \tag{1}$$

where d is the motion space; n is the number of bodies; j is the number of joints; and f_i is the degree of freedom in each coupling and its sum $F = \sum f$ is known as the gross degree of freedom of mechanism.

This work is focused in the vehicles main gearboxes (not in differential gear trains), assuming all gear pairs to be spur gears with the same modulus, without axial displacement of gears to simplify the problem. The domain of this research is limited to planar mechanisms with $F_N \geq 2$.

In order to determine each gear ratio of vehicle gearboxes, it is necessary to use actuators to activate (join two bodies) specific couplings. There are two main types: *clutches*, which involve only mobile bodies, and *brakes*, which involve a grounded bodie and any other mobile bodie.

2.2 Screw Representation of Motion State

As a point (geometric element) can be used to represent a mass particle, and a directed line (geometric element) can be used to represent a moment, a screw (geometric element) can also be used to represent mechanical properties (Campos, 2004).

A screw \$ is defined by a straight line with an associated pitch h and is conveniently denoted by six Plücker homogeneous coordinates:

$$\$ = \begin{pmatrix} \vec{S} \\ \text{- - - - - - - - -} \\ \vec{S} \times \vec{S}_0 + h\vec{S} \end{pmatrix} = \begin{pmatrix} L \\ M \\ N \\ \text{- - - - - - - -} \\ P^* = P + hL \\ Q^* = Q + hM \\ R^* = R + hN \end{pmatrix} \tag{2}$$

where $\vec{S}$ denotes direction ratios pointing along the screw axis, $\vec{S}_0$ is the position vector of any point on the screw axis with respect to the coordinate system (Zhao et al., 2006).

The instantaneous motion state of a rigid body relative to an inertial system of coordinates O_{xyz} can be described by a screw, named *motion screw* or *twist* $\$_M$. It is composed by a differential angular velocity ω around the instantaneous screw axis and a differential translational velocity τ through the same axis. The twist $\$_M$ has a scalar pitch $h = \tau/\omega$ relating linear and angular velocities.

In kinematics analysis, Plücker homogeneous coordinates in Eq. (2) can be rewritten as six motion coordinates:

$$\$_M = \begin{pmatrix} \mathcal{L} \\ \mathcal{M} \\ \mathcal{N} \\ \hdashline \mathcal{P}^* = \mathcal{P} + h\mathcal{L} \\ \mathcal{Q}^* = \mathcal{Q} + h\mathcal{M} \\ \mathcal{R}^* = \mathcal{R} + h\mathcal{N} \end{pmatrix} = \begin{pmatrix} \vec{\omega} \\ \hdashline \vec{\omega} \times \vec{S}_0 + h\vec{\omega} \end{pmatrix} = \begin{pmatrix} \vec{\omega} \\ \hdashline \vec{V}_P \end{pmatrix} \tag{3}$$

where the first three components denote the angular velocity $\vec{\omega}$ and are related by $\mathcal{L}^2 + \mathcal{M}^2 + \mathcal{N}^2 = \omega^2$; and last three components denote the linear velocity $\vec{V}_P$ of a point P in the rigid body that is instantaneously coincident with the origin of O_{xyz} and are related by $\mathcal{P}^{*2} + \mathcal{Q}^{*2} + \mathcal{R}^{*2} = v_P^2$.

The vector $\vec{V}_P$ results from two parcels: one is a velocity parallel to the instant axis of screw, represented by $\tau = h\omega$ and the other is a velocity normal to the instant axis of screw, represented by $S_0 \times \omega$. Figure 1 illustrates a twist $\$_M$ with its velocity components in a point P.

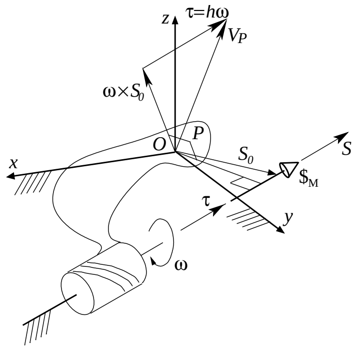

Figure 1. Twist components of a rigid body (Campos, 2004).

Normalizing the twist $\$_M$ allows to separate it into a geometrical element $\hat{\$}$, with no mechanical significance attached, associated to a scalar amplitude (magnitude) ρ being endowed with units of angular speed (Davidson and Hunt, 2004).

$$\$_M \equiv (\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P}^*, \mathcal{Q}^*, \mathcal{R}^*) \equiv \rho\hat{\$} \equiv (\rho L, \rho M, \rho N, \rho P^*, \rho Q^*, \rho R^*)\,. \tag{4}$$

The magnitude of $\$_M$ can assume two particular states according to the pitch value. When the screw has pitch $h = 0$, then $\tau = 0$ and $\vec{V}_P = \vec{\omega} \times \vec{S}_0$ what results in the following twist:

$$\$_M \equiv (\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{R}) \equiv (\vec{\omega}, \vec{\omega} \times \vec{S}_0) \equiv \omega(\vec{S}, \vec{S}_0 \times \vec{S})\,, \tag{5}$$

where $\omega = \rho_0 = \sqrt{\mathcal{L}^2 + \mathcal{M}^2 + \mathcal{N}^2}$ is the associated magnitude. This situation occurs when the motion state of a rigid body is purely rotative.

When $\vec{\omega} = 0$, then the screw has pitch $h = \infty$ resulting in the twist:

$$\$_M \equiv (0, 0, 0, \mathcal{P}^*, \mathcal{Q}^*, \mathcal{R}^*) \equiv (\vec{0}, \vec{V}_P) \equiv v_P(\vec{0}, \vec{S})\,, \tag{6}$$

where $v_P = \rho_\infty = \sqrt{\mathcal{P}^{*2} + \mathcal{Q}^{*2} + \mathcal{R}^{*2}}$ is the associated magnitude. This situation occurs when the motion state of a rigid body is purely translative.

A joint a between the bodies i and j can be represented by a twist $\$_M^a$. In this case, the relative motion state can be obtained by the difference of twists $\$_{Mi}$ and $\$_{Mj}$, measured at the joint axis.

$$\$_M^a = \$_{Mi} - \$_{Mj}\,. \tag{7}$$

In this work, the inertial coordinates system O_{xyz} that will ever be adopted is as follow: *origin* is any point in the gear train central stationary axle; *z-axis* oriented coincidently to the central axle and positive direction denoted from input to output side; *y-axis* oriented belonging to the longitudinal symmetric plane, from input to output, and positive direction from main axle to outwards; *x-axis* is the orthogonal complement of the plane yz. Twists are always relative to this coordinates system.

In a PGT mechanism, each direct coupling is a joint represented by a twist in the form of Eq. (7). Given that the number of twists of a mechanism is a sum of each single degree of freedom of all couplings, the set of F twists can be expressed as the Direct Coupling Motion Matrix $[M_D]_{F,6}$ and decomposed according to Eq. (4):

$$[M_D]_{F,6} = [\psi]_{F,F}[\hat{M}_D]_{F,6}\,, \tag{8}$$

where $[\psi]_{F,F} = diag(\vec{\rho}_F)$ is a diagonal matrix containing F magnitudes and $[\hat{M}_D]_{F,6}$ is the Direct Coupling Unit Motion Matrix containing normalized screws $\hat{\$}$.

2.3 Graphs and Kirchhoff's Circulation Law Adapted

The topology of the mechanisms can be represented by connected graphs, where every pair of vertices has a path of edges connecting them.

Given a kinematic chain of a mechanism, it is represented by a coupling graph G_C where there are v vertices labelled by numbers that represent bodies (0 to grounded), and e edges labelled by lower case letters representing direct couplings. G_C is a directed graph, in which each edge has a positive direction represented by an arrow pointing from minor to major vertex.

Apart from a topological representation, a graph can be expressed in matricial forms. One of them is the *circuit matrix* $[B]_{l,e}$ of G_C containing l rows, representing fundamental circuits, and e columns, representing edges.

A set of fundamental circuits can be identified through finding any tree of G_C, which is a connected subgraph with all vertices present but there are no circuits: edges that belongs to the tree are called *branches*, and the omitted edges, which completes the fundamental circuits in the original graph, are called *chords*. The number of chords, i.e. independent circuits, is given by $l = e - v + 1$. Moreover, the circuit directions are adopted following the chord directions.

An element $b_{i\,j}$ of $[B]_{l,e}$ is 1 if an edge j belongs to the circuit i with the same direction; -1 if an edge j belongs to the circuit i with the opposite direction; otherwise it is 0.

Kirchhoff's voltage law states that $[B]_{l,e}\,(V)_e = \vec{0}$. Considering that each edge represents a single degree of freedom, then $e = F$. Now, the constraint law for kinematic analysis can be expressed as an adaptation of Kirchhoff's Circulation Law:

$$[B]_{l,F}\,[M_D]_{F,6} = [0]_{l,6}\ , \tag{9}$$

where $[0]_{l,6}$ is a null matrix.

Assuming that the motion space d varies and combining $[\hat{M}_D]^T_{F,d}$ with each circuit equation $[B_i]_{1,F}$ through doing $[B_i]_{F,F} = diag([B_i]_{1,F})$ for $i = 1, 2, \ldots, l$, the *constraint equation* can be obtained as follow:

$$\begin{bmatrix} \left[\hat{M}_D\right]^T_{F,d} [B_1]_{F,F} \\ \left[\hat{M}_D\right]^T_{F,d} [B_2]_{F,F} \\ \cdots \\ \left[\hat{M}_D\right]^T_{F,d} [B_l]_{F,F} \end{bmatrix}_{dl,F} [\psi]_{F,1} = [\hat{M}_N]_{dl,F}\,[\psi]_{F,1} = [0]_{dl,1}\ , \tag{10}$$

where $[\hat{M}_N]_{dl,F}$ is denoted Network Unit Motion Matrix; $[\psi]_{F,1} = \vec{\rho}$ is a magnitude vector of joint twists; and $[0]_{dl,1}$ is a null vector. The Motion space can be $d = 3$ to planar mechanisms, $d = 6$ to spatial mechanisms and other values according to mechanisms motion characteristics.

3. DAVIES' METHOD TO KINEMATIC ANALYSIS

In this section, Davies' method is depicted step-by-step to gear trains using the information presented in past section.

1. Given a gear train mechanism, draw its kinematic chain identifying all n bodies by numbers (0 to grounded) and k direct couplings by lower case letters. Draw a related coupling graph G_C of mechanism using thin edges to rotative pairs and thick edges to gear pairs.

2. Generate motion graph G_M resulting from G_C through unfolding single motions from direct couplings with freedom greater than one. In this step, each edge of G_C representing a coupling with $f > 1$ is replaced in G_M by f edges in series.

   (a) Assign positive directions to each edge pointing an arrow from minor to major vertex.

   (b) Select a set of $l = e - v + 1$ independent circuits of G_M, assigning the same label and direction of its associated chord. Usually, edges representing passive couplings are chosen as chords to not degenerate the system of equations.

3. Write circuit matrix $[B]_{l,e}$ with suitable signs.

4. Select an inertial coordinates system O_{xyz} according to:

   - *origin* selected in the gear train central stationary axle;
   - *z-axis* oriented coincidently to the central axle and positive direction denoted from input to output side;

- *y-axis* oriented belonging to the longitudinal symmetric plane, from input to output, and positive direction from main axle to outwards;
- *x-axis* is the orthogonal complement of the plane yz, forming a dextral system.

5. Write a twist $\$_M$ in the form of Eq. (4) for each edge from G_M as following:
   (a) Determine $\vec{S}$ that is the direction ratios of the screw axis.
   (b) Determine $\vec{S_0}$ that is the position vector of any point on the screw axis.
   (c) Construct the twist:
      - For rotative and gear pairs, using Eq. (5).
      - For prismatic pairs, using Eq. (6).
6. Write motion matrix $[\hat{M}_D]_{F,d}$ in the form of Eq. (8).
7. Apply the constraint law shown in Eq. (9) to formulate the constraint equation given by Eq. (10).
8. Constraint equation has rank r for dl conditions and F unknowns. Consequently, the nett degree of freedom of mechanism can be evaluated by $F_N = F - r$ and the system of equations should be separated into:

$$\left[\begin{array}{c:c} [\hat{M}_N]_{r,r} & [\hat{M}_N]_{r,F_N} \end{array}\right] \left[\begin{array}{c} [\psi]_{r,1} \\ \hdashline [\psi]_{F_N,1} \end{array}\right] = [0]_{dl,1} \ . \tag{11}$$

   Being rearranged, Eq. (11) becomes the final form to solve to kinematic analysis:

$$[\psi]_{r,1} = -[\hat{M}_N]^{-1}_{r,r}[\hat{M}_N]_{r,F_N}\,[\psi]_{F_N,1} \ , \tag{12}$$

   where the r secondary variables of $[\psi]_{r,1}$ are expressed in terms of the F_N primary variables of $[\psi]_{F_N,1}$.

4. EXAMPLES

4.1 Ravigneaux Gear Train

Ravigneaux (1940) presents an invention of a speed changing device composed by $j = 10$ joints and $n = 7$ bodies: one grounded (0), two sun gears (1, 2), one ring gear (3), one carrier (4) and two planets (5, 6) unbalanced in plane xy.

In order to easy calculation, the mechanism topology was modified by aligning planets (5 and 6) in plane zy. To evaluate this proceeding without loosing kinematic characteristics, it was assumed that Ravigneaux PGT external diameter given by ring (3) and the relations $i_{1,3}$ and $i_{2,3}$ should be as close as possible to the original values.

This PGT is shown in Fig. 2 and is characterized by having two planets (single (5) and double ($6a$,$6b$)) connected to a same carrier and providing 4-speed (3 FWD, 1 RWD). There are four actuators being two input clutches ($C1, C2$) and two brakes ($B1, B2$).

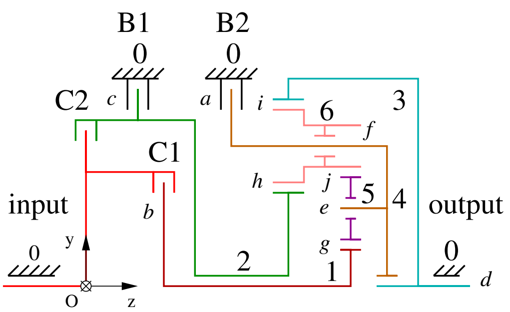

Figure 2. Schematic representation of a Ravigneaux PGT.

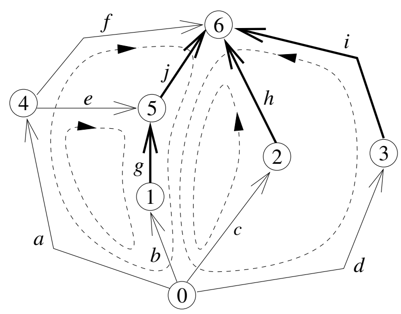

Figure 3. Coupling graph G_C of Ravigneaux PGT.

Davies' method is applied in accordance with Section 3. to analyse the speed ratios of this mechanism.

1. Figure 3 presents coupling graph G_C of mechanism, where thin edges (a, b, c, d, e, f) are rotative pairs and thick edges (g, h, i, j) are gear pairs.
2. Motion graph G_M is equal G_C since all edges from G_C have single degree of freedom.

(a) Positive directions are given in Fig. 3.

(b) There are $l = 4$ independent circuits denoted by dashed loops in Fig. 3 associated to chords e, f, h and i, none of them active coupling.

3. $$[B]_{4,10} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix} \quad (13)$$

4. The inertial coordinates system O_{xyz} is shown in Fig. 2 according to: *origin* belongs to central stationary axle; *z-axis* lies along central axle directed from input to output; *y-axis* forms, with z, a longitudinal symmetric plane of gear train and *x-axis* is the orthogonal complement of plane yz, in accordance with dextral system.

5. (a) All joints have direction ratios of twist $\vec{S} = \{0, 0, 1\}$ parallel to *z-axis*.

   (b) All joints lies on plane yz, consequently, their first components (x) of position vectors S_0 are null. The position vectors are given by:

   $$\vec{S}_0^{a,b,c,d} = \{0, 0, z\}\,, \quad \vec{S}_0^{e,f,g,h,i,j} = \{0, y, z\}\,. \quad (14)$$

   (c) Twists are given by Eq. (5):

   $$\$_M^{a,b,c,d} = \omega\{0, 0, 1, 0, 0, 0\}\,, \quad \$_M^{e,f,g,h,i,j} = \omega\{0, 0, 1, y, 0, 0\}\,, \quad (15)$$

   where must be relevant to emphasize that four twist coordinates $\mathcal{L}$, $\mathcal{M}$, $\mathcal{Q}$ and $\mathcal{R}$ are null, existing only angular velocities with axes of rotation parallel with *z-axis* in plane x= 0. Collectively, these motions belong to a second special 2-system of motions where $d = 2$ requiring only $\{\mathcal{N}, \mathcal{P}\}$ as motion coordinates. This resulted from mechanism symmetry in plane xy.

   For this case, Kutzbach-Grübler criterion in Eq. (1) can be correctly employed to evaluate F_N:

   $$F_N = 2(7 - 10 - 1) + 10 = 2\,. \quad (16)$$

6. $$[M_D]_{F,d} = [\psi]_{F,F} \left[\hat{M}_D\right]_{F,d} = \begin{bmatrix} \omega_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_j \end{bmatrix}_{10,10} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & y_e \\ 1 & y_f \\ 1 & y_g \\ 1 & y_h \\ 1 & y_i \\ 1 & y_j \end{bmatrix}_{10,2} \quad (17)$$

7. $$[\hat{M}_N]_{dl,F}\,[\psi]_{F,1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_e & 0 & -y_g & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & y_f & -y_g & 0 & 0 & -y_j \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -y_g & y_h & 0 & -y_j \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -y_g & 0 & y_i & -y_j \end{bmatrix}_{8,10} \begin{bmatrix} \omega_a \\ \omega_b \\ \omega_c \\ \omega_d \\ \omega_e \\ \omega_f \\ \omega_g \\ \omega_h \\ \omega_i \\ \omega_j \end{bmatrix}_{10,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{8,1} \quad (18)$$

8. Now, the solution of Eq. (18) is determined by performing suitable actuations. There are four joints ($C1, C2, B1, B2$) capable of actuation and it is necessary to combine them into $F_N = 2$ actuations. Table 1 shows the $C_4^2 = 6$ possibilities to solve the equation system.

   It is concluded from Eq. (15), that only the y components of position vectors $\vec{S}_0$ are needed to specify couplings characteristics as twists for Ravigneaux PGT. These values were given respecting topological constraints and are expressed as number of teeth, being presented in Tab. 2.

   Using the input magnitude as $\omega_{in} = 1$ and the values from Tab. 2, a summary of the actuations results is presented in Tab. 3. The unknown magnitudes were obtained through Eq. (12), as well as the output was obtained by the magnitude ω_d of the joint d and is used to calculate gear ratios $i = \omega_{in}/\omega_{out}$.

Table 1. Actuation possibilities for Ravigneaux PGT.

	Primary Variables			
A	Joint 1		Joint 2	
1	C_1	$\omega_b = 1$	B_2	$\omega_a = 0$
2	C_1	$\omega_b = 1$	B_1	$\omega_c = 0$
3	C_1	$\omega_b = 1$	C_2	$\omega_c = 1$
4	C_2	$\omega_c = 1$	B_2	$\omega_a = 0$
5	C_2	$\omega_c = 1$	B_1	$\omega_c = 0$
6	B_1	$\omega_c = 0$	B_2	$\omega_a = 0$

Table 2. Number of teeth Z of each body and y coordinates of each joint position vector of the twists.

Body	1	2	3	4	5	6a	6b
Teeth (Z)	19	41	85	0	15	14	22

Joint	a	b	c	d	e	f	g	h	i	j
$S_0(y)$	0	0	0	0	17	31.5	9.5	20.5	42.5	24.5

By inspection, two possibilities can be eliminated. One is Act. 5 because of input actuation in $C2$ is given in the grounded body by brake $B1$. The other is Act. 6, once both actuations ($B1, B2$) are given in brakes resulting no input actuation.

Table 3. Actuation possibilities and results for Ravigneaux PGT.

	Results										Ratios	Dir
A	ω_a	ω_b	ω_c	ω_d	ω_e	ω_f	ω_g	ω_h	ω_i	ω_j	$i = \frac{\omega_{in}}{\omega_d}$	
1	$B2$	$C1$	-1.373	2.847	-0.789	0.737	-0.441	0.480	0.994	0.381	2.8469	FWD
2	2.373	$C1$	$B1$	1.601	-1.364	1.273	-0.762	0.829	1.718	0.659	1.6010	FWD
3	1	$C1$	$C2$	1	0	0	0	0	0	0	1.0000	FWD
4	$B2$	-0.728	$C2$	-2.073	0.575	-0.537	0.321	-0.349	-0.724	-0.278	-2.0732	RWD
5	0	0	$C2, B1$	0	0	0	0	0	0	0	0	
6	$B2$	0	$C2$	0	0	0	0	0	0	0	0	

4.2 Lepelletier Gear Train

Lepelletier (1992) presents an invention of a multispeed automatic transmission for vehicles comprising a simple PGT connected to a Ravigneaux gear as shown in Fig. 4. The sun of planetary is grounded (0), while the carrier (8) is connected, through clutches $C1, C3$, to the large (2) and small (1) sun gears of the Ravigneaux gearset, respectively. The input body (7) is a ring gear connected to planet (9) of PGT and can connect to Ravigneaux's carrier (4) by a clutch ($C2$). The output body is the ring gear (3) permanently connected to the long planet 6 of Ravigneaux. The short planet (5) completes the compound.

This mechanism was also simplified by aligning planets of Ravigneaux gearset becoming symmetric to plane xy.

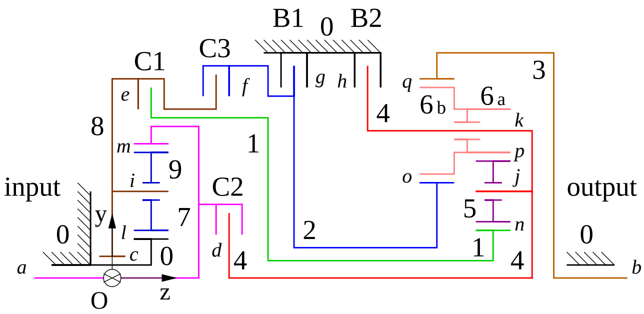

Figure 4. Schematic representation of Lepelletier PGT.

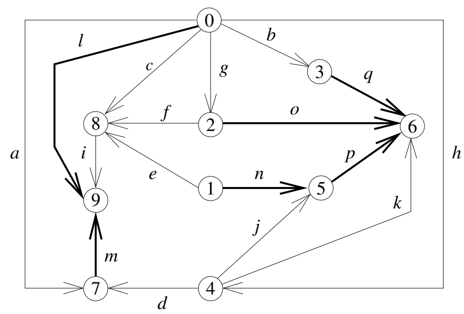

Figure 5. Coupling graph G_C of Lepelletier PGT.

Lepelletier PGT can be analysed as a compound of two subsystems. The first subsystem is an input set of one degree of freedom given by a simple PGT supplying two angular velocities ω_{in}, being one directly in the input shaft (7) and the other given by the carrier (8). The second subsystem is a Ravigneaux PGT which has two degrees of freedom, as presented in the previous example.Combining this two known topologies it is possible to achieve 7 speed ratios through given an input velocity and actuating two active couplings.

Applying Davies method to this mechanism results in:

1. Figure 5 presents coupling graph G_C of mechanism having 11 thin edges $(a, b, c, d, e, f, g, h, i, j, k)$ for rotative pairs and 6 thick edges (l, m, n, o, p, q) for gear pairs.

2. Motion graph G_M is equal G_C since all edges from G_C have only one degree of freedom.

   (a) Positive directions are given in Fig. 5.

   (b) Graph tree is given by branches $\{a, f, g, d, e, o, q, l, j\}$. Thus, there are $l = 8$ independent circuits associated to chords $\{b, c, h, i, k, m, n, p\}$.

3. $$[B]_{8,17} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad (19)$$

4. The inertial coordinates system O_{xyz} is shown in Fig. 4.

5. (a) All joints have direction ratios of twist $\vec{S} = \{0, 0, 1\}$ parallel to *z-axis*.

   (b) All joints lies on plane yz, consequently, their first components (x) of position vectors S_0 are null. The position vectors are given by:

   $$\vec{S}_0^{a,b,c,d,e,f,g,h} = \{0, 0, z\}\,, \quad \vec{S}_0^{i,j,k,l,m,n,o,p,q} = \{0, y, z\}\,. \quad (20)$$

   (c) Twists are given by Eq. (5):

   $$\$_M^{a,b,c,d,e,f,g,h} = \omega\{0, 0, 1, 0, 0, 0\}\,, \quad \$_M^{i,j,k,l,m,n,o,p,q} = \omega\{0, 0, 1, y, 0, 0\}\,, \quad (21)$$

   where must be relevant to emphasize that Lepelletier PGT has motion space $d = 2$ requiring only $\{\mathcal{N}, \mathcal{P}\}$ as motion coordinates, resulted from mechanism symmetry in plane xy.

6. $$[M_D]_{17,2} = [\psi]_{17,17} \left[\hat{M}_D\right]_{17,2}\,. \quad (22)$$

7. $$[\hat{M}_N]_{16,17}\,[\psi]_{17,1} = [0]_{16,1}\,, \quad (23)$$

   where $[\hat{M}_N]_{16,17}$ has rank $m = 14$.

8. Now, the solution of Eq. (23) is determined by performing suitable $F_N = F - m = 3$ actuations. One is the input in a, given by $\omega_a = 1$, and other two are given by a combination of five joints $(C1, C2, C3, B1, B2)$ capable of actuation. Table 4 shows the $C_5^2 = 10$ possibilities to solve the equation system.

Table 4. Actuation possibilities for Lepelletier PGT.

	Primary Variables					
A	Joint 1		Joint 2		input	
1	C_1	$\omega_e = 0$	B_2	$\omega_h = 0$	a	$\omega_a = 1$
2	C_1	$\omega_e = 0$	B_1	$\omega_g = 0$	a	$\omega_a = 1$
3	C_1	$\omega_e = 0$	C_3	$\omega_f = 0$	a	$\omega_a = 1$
4	C_1	$\omega_e = 0$	C_2	$\omega_d = 0$	a	$\omega_a = 1$
5	C_2	$\omega_d = 0$	C_3	$\omega_f = 0$	a	$\omega_a = 1$
6	C_2	$\omega_d = 0$	B_1	$\omega_g = 0$	a	$\omega_a = 1$
7	C_3	$\omega_f = 0$	B_2	$\omega_h = 0$	a	$\omega_a = 1$
8	C_2	$\omega_d = 0$	B_2	$\omega_h = 0$	a	$\omega_a = 1$
9	C_3	$\omega_f = 0$	B_1	$\omega_g = 0$	a	$\omega_a = 1$
10	B_1	$\omega_g = 0$	B_2	$\omega_h = 0$	a	$\omega_a = 1$

By inspection, three possibilities can be eliminated. Two are Act. 8 and Act.9 because of the input actuation is given in a body grounded by brake actuation. The other is Act. 10, once both actuations $(B1, B2)$ are given in brakes implying no movement in Ravigneaux gearset.

Values for the number of teeth (Z) of bodies based on Lepelletier (1992) and Tab. 2 are presented in Tab. 5, respecting topological constraints. Furthermore, the correspondent y coordinates of each joint position vector of the twists are expressed as number of teeth.

Table 5. Number of teeth Z of each body and y coordinates of each twists.

Body	1	2	3	4	5	6a	6b	7	8	9	0						
Teeth (Z)	31	41	85	0	15	14	22	61	0	15	31						
Joint	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
$S_0(y)$	0	0	0	0	0	0	0	0	23	17	31.5	15.5	30.5	9.5	20.5	24.5	42.5

Using the actuations from Tab. 4 and assuming values form Tab. 5, a summary of the actuations results is presented in Tab. 6. The unknown magnitudes were obtained through Eq. (12), as well as the output was obtained by the magnitude ω_b of the joint b and is used to calculate gear ratios $i = \omega_{in}/\omega_{out}$.

Table 6. Actuations and results for Lepelletier PGT.

A	Coupling actuations					Ratio	Step	Direction
	$C1$	$C2$	$C3$	$B1$	$B2$			
1	⊗				⊗	4.2937		FWD
2	⊗			⊗		2.4146	1.7782	FWD
3	⊗		⊗			1.5082	1.6010	FWD
4	⊗	⊗				1.1342	1.3297	FWD
5		⊗	⊗			0.8602	1.3185	FWD
6		⊗		⊗		0.6746	1.2751	FWD
7			⊗		⊗	-3.1267		RWD

In Fig. 6 it is possible to visualize a kinematic map of the angular velocities of bodies $8, 2, 3, 4$ and 1 in Lepelletier PGT, considering $\omega_{in} = 3000$ [rpm].

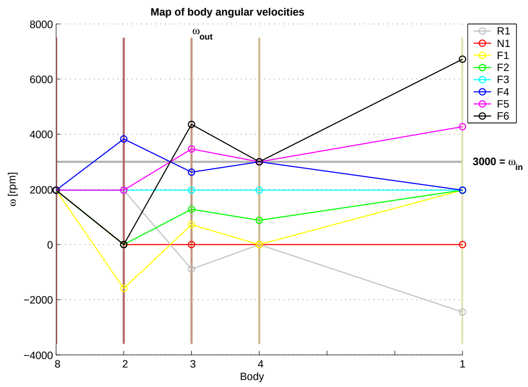

Figure 6. Kinematic map of the angular velocities of the Lepelletier PGT.

5. CONCLUSION

This work applies Davies' method analyze the kinematics of automotive gearboxes, especially those based on planetary gear trains. The method is based on graphs and is general for all gear trains. The use of screw theory enhances the geometrical insight (visualization) of the instantaneous motion state of the mechanisms. The method was applied to two widely spread topologies: Ravigneaux and Lepelletier. It is important to highlight that this method can be applied to any mechanism without constraints. By the way, it may also be analogously employed in order to determine the action state of the mechanisms. As future perspectives, we intend to solve integrated powertrain, as well as applying the method to analyze power losses and, also, to develop techniques to mechanisms synthesis and optimization based on this kind of analysis.

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