A z Transform State Space Formulation for Flutter Predictions

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Abstract. *The use of numerical tools for the calculation of the unsteady aerodynamic behavior in complex flow situations is every time more common. However, the discrete-time nature of the numerical solutions is not readily applicable to the usual continuous-time state space representations of aeroelastic systems. The present work presents an alternate formulation for the state space representation of aeroelastic systems based on digital control theory that is shown to be more effective and accurate for the coupling of numerical solutions with such systems. The new state space formulation is based on the z transform, which allows for direct frequency domain representations of the aerodynamic solutions without the need for approximating models. This fact makes this new methodology also a more straightforward procedure for aeroelastic analyses. A typical section model of a NACA 0012 airfoil at transonic speed is used as test case in order to assess the correctness and accuracy of the proposed formulation. The present results are compared with data obtained from continuous-time state space formulations and through the direct integration of the structural dynamic and aerodynamic equations.*

Keywords: *z Transform, State Space, Flutter, Aeroelasticity, CFD*

1. INTRODUCTION

The lack of accurate aerodynamic analytical models for complex flow situations makes the solution of realistic aeroelastic problems a very difficult task. Over the years, approximate solutions of the aerodynamic potential theory (Dietze, 1947, Fettis, 1952, Bisplinghoff, Ashley and Halfman, 1955), and, more recently, numerical solutions obtained with CFD solvers (Beam and Warming, 1974, Traci, Albano and Farr, 1975, Ballhaus and Goorjian, 1978, Batina, 1989, Rausch, Batina and Yang, 1990, Oliveira, 1993, Raveh, 2001, Marques and Azevedo, 2006, Marques, 2007) have been used in order to represent the aerodynamic response to generic structural behavior in order to decouple the aerodynamic and structural effects. Such aerodynamic data are generally modeled with rational functions (Vepa, 1977) or interpolating polynomials (Roger, 1977, Abel, 1979, Dunn, 1980) in a convenient fashion so as to represent the aeroelastic system through a continuous-time state space formulation.

The present work presents an alternate approach for the state space representation of aeroelastic systems based on discrete-time control techniques. The main idea behind this approach is to avoid the need for an approximate model of the aerodynamic responses. This goal is achieved by calculating the aerodynamic characteristics in the frequency domain with the direct use of the z transform. The complete formulation of the new state space representation is presented in the next sections. Finally, a typical section model of a NACA 0012 airfoil at transonic flow condition is used as test case for a detailed comparison among the different state space formulations and direct integration results.

The CFD solver employed for evaluating the aerodynamic responses contained in the present work is the same presented by Marques (2007). It is based on the 2-D Euler equations, which represent two-dimensional, compressible, rotational, inviscid and nonlinear flows. Therefore, it is completely capable of capturing the shock waves present in transonic flows. These equations are discretized in space with a cell centered, finite volume scheme, and they are advanced in time using a second-order accurate, 5-stage, explicit, hybrid Runge-Kutta scheme.

2. THEORETICAL FORMULATION

A general aeroelastic system is characterized by aerodynamic, elastic and inertial forces dynamically interacting with structural deformations. It is very common, therefore, to represent the aerodynamic effects exclusively through the resulting forces and moments acting on the structure as a forcing term. The methodology proposed in the present work is focused on the adequate representation the aerodynamic operator for complex flow situations. Hence, it is instructive to apply such methodology with simple structural models in order to avoid further complications that might hide the behavior of the aerodynamic model. The structural model considered in the present work is the typical section, which is widely known and reported in the literature (Bisplinghoff, Ashley and Halfman, 1955, Oliveira, 1993). The dynamic system represented in the typical section is a rigid airfoil with two degrees of freedom, plunge and pitch. The governing equation for the motion of such dynamic system is given by

$$[M]\{\ddot{\eta}(\bar{t})\}+[\bar{K}]\{\eta(\bar{t})\}=\{\bar{Q}a(\bar{t})\}, \tag{1}$$

where the generalized mass and stiffness matrices are, respectively,

$$[M]=\begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix} \qquad \text{and} \qquad [\bar{K}]=\begin{bmatrix} (\omega_h/\omega_r)^2 & 0 \\ 0 & r_\alpha^2(\omega_\alpha/\omega_r)^2 \end{bmatrix}, \tag{2}$$

and the generalized coordinate and force vectors are

$$\{\eta(\bar{t})\}=\begin{bmatrix} \xi(\bar{t}) & \alpha(\bar{t}) \end{bmatrix}^T \qquad \text{and} \qquad \{\bar{Q}a(\bar{t})\}=\begin{bmatrix} \frac{Qa_h(\bar{t})}{mb\omega_r^2} & \frac{Qa_\alpha(\bar{t})}{mb^2\omega_r^2} \end{bmatrix}^T. \tag{3}$$

In the previous equations, α is the pitch mode coordinate, positive in the nose-up direction, and h is the vertical translation, positive downwards. Moreover, x_α is the distance from the elastic axis to the center of mass, a_h is the distance from midchord to the elastic axis, r_α denotes the airfoil radius of gyration about the elastic axis, m is the airfoil mass, and ω_h and ω_α are the uncoupled natural circular frequencies of the plunge and pitch modes, respectively. All length variables are nondimensionalized by the airfoil semi-chord length, b. Furthermore, in Eq. (3), ξ is the plunge mode coordinate, where $\xi = h/b$. Finally, the time variable, t, is also nondimensionalized considering a reference circular frequency ω_r, *i.e.*, $\bar{t}=\omega_r t$.

As previously mentioned, the main objective of the present study is to efficiently determine the generalized aerodynamic force vector $\{\bar{Q}a(\bar{t})\}$ for an arbitrary structural behavior. However, due to the nonlinearities of the aerodynamic equations, it is very difficult to obtain a general expression for the aerodynamic response. This problem is simplified by extending linearity concepts present in the formulation of the potential aerodynamic equations. As presented by Bisplinghoff, Ashley and Halfman (1955), and Vepa (1977), the linear aerodynamic responses can be individually determined for each mode, and then superposed for more general responses. Based on these ideas, Oliveira (1993) proposed the assumption of linearity of the aerodynamic response in the transonic regime with regard to the modal motion. As there are no rigorous linearization procedures involved, there are no guarantees that such assumption holds. But, it is natural to expect this sort of linear hypothesis to be valid at least for small amplitudes. Actually, as it is shown in section 4. there is a certain amplitude range in which this hypothesis holds. Furthermore, it is important to emphasize that the aeroelastic phenomena analyzed in the present work are restricted to small amplitude motions. Therefore, when considering flutter, which is characterized by diverging oscillations, it is possible that only the linear onset of this phenomenon can be identified with such approach. There may occur situations in which the aerodynamic nonlinear behavior suppress the diverging oscillations creating limit cycle oscillations, and the present formulation may not correctly capture such cases. Nevertheless, for safety reasons, most designers are interested in the flutter onset point.

As a consequence of the linearity assumptions, it is possible to determine the aerodynamic response to a general structural behavior from the convolution of an impulsive or indicial aerodynamic solution (Bisplinghoff, Ashley and Halfman, 1955, Oppenheim and Schafer, 1989, Marques, 2007). The convolution operation, however, is more easily handled in the frequency domain, in which it is represented by a simple multiplication operation. Hence, the linearized generalized aerodynamic forces can be written, in the frequency domain, as

$$\{\bar{Q}a(k)\}=\frac{(U^*)^2}{\pi\mu}[A(k)]\{\eta(k)\}, \qquad [A(k)]=\begin{bmatrix} -C_{\ell_{Ih}}(k)/2 & -C_{\ell_{I\alpha}}(k) \\ C_{m_{Ih}}(k) & 2C_{m_{I\alpha}}(k) \end{bmatrix}. \tag{4}$$

where the mass ratio is defined as $\mu = m/(\pi\rho_\infty b^2)$, the characteristic speed is $U^* = U_\infty/(b\omega_r)$, and the reduced frequency, $\kappa = \omega b/U_\infty$, represents the frequency domain. Moreover, ρ_∞ and U_∞ designate, respectively, the undisturbed flow density and speed.

There is a number of different approaches used to determine the desired aerodynamic coefficients. As there is no closed-form solution for the compressible subsonic and transonic unsteady aerodynamic forces, this work is aimed at numerical methods where the impulsive solution is known for a determined number of discrete reduced frequency values. However, the calculation of aerodynamic responses in the frequency domain is beyond the scope of the present paper, and it is assumed that this information is known a priori. All the aerodynamic responses presented here are obtained by exciting the aerodynamic system on a frequency range of interest applying a smooth pulse motion in both modes, as described by Rausch, Batina and Yang (1990), Oliveira (1993), Marques and Azevedo(2006), and Marques (2007).

3. DISCRETE-TIME STATE SPACE FORMULATION

A state space representation of a system corresponds to the description of the system dynamics in terms of first-order differential equations, which may be combined into a first-order vector-matrix differential equation (Ogata, 1987). In the present case, this formulation can be achieved by following the ideas of Oliveira (1993) by defining

$$\{x_1(\bar{t})\}=\{\eta(\bar{t})\}, \qquad \{x_2(\bar{t})\}=\{\dot{\eta}(\bar{t})\}=\{\dot{x}_1(\bar{t})\}, \qquad \{x(\bar{t})\}=\begin{bmatrix} \{x_2(\bar{t})\} & \{x_1(\bar{t})\} \end{bmatrix}^T, \tag{5}$$

where $\{x(\bar{t})\}$ is the system state vector. Hence, the governing equation of motion becomes

$$\left[\tilde{M}\right]\{\dot{x}(\bar{t})\}+\left[\tilde{K}\right]\{x(\bar{t})\}=\{\tilde{q}(\bar{t})\}, \tag{6}$$

where

$$\left[\tilde{M}\right]=\begin{bmatrix}[M] & [0_{2\times 2}]\\ [0_{2\times 2}] & [I_{2\times 2}]\end{bmatrix},\qquad \left[\tilde{K}\right]=\begin{bmatrix}[0_{2\times 2}] & [\bar{K}]\\ -[I_{2\times 2}] & [0_{2\times 2}]\end{bmatrix},\qquad \{\tilde{q}(\bar{t})\}=\left\{\begin{array}{c}\{\bar{Q}a(\bar{t})\}\\ \{0_{2\times 1}\}\end{array}\right\}. \tag{7}$$

Instead of approximating the discrete-time aerodynamic responses in order to make them suitable for application in a continuous-time state space formulation, as presented by Oliveira (1993), and Marques and Azevedo (2006), another alternative is to represent every other time-dependent variable in a sampled, or discrete, manner. By doing so, it is possible to convert continuous-time state space equations into discrete-time state space equations and use digital control theory for the stability analysis (Ogata, 1987). The time step required for an accurate solution of the aerodynamic behavior must be small compared with the significant time constants of the system, which guarantees that no considerable error is introduced by the discretization procedure. As indicated by Ogata (1987), the solution of a continuous-time state space system, such as the one represented in Eq. (6), is given by

$$\{x((n+1)\Delta\bar{t})\}=e^{([W]\Delta\bar{t})}\{x(n\Delta\bar{t})\}+e^{([W](n+1)\Delta\bar{t})}\int_{n\Delta\bar{t}}^{(n+1)\Delta\bar{t}}e^{(-[W]\tau)}\{q_x(\tau)\}\,d\tau, \tag{8}$$

where

$$[W]=-\left[\tilde{M}\right]^{-1}\left[\tilde{K}\right]\qquad\text{and}\qquad \{q_x(\bar{t})\}=\left[\tilde{M}\right]^{-1}\{\tilde{q}(\bar{t})\}. \tag{9}$$

It is very important to emphasize that the nondimensionalization of the aeroelastic system time step, $\Delta\bar{t}$, is not necessarily the same used for the time step of the flow solver. For example, according to the nondimensionalization of the flow variables generally used in CFD solvers, the correspondence between the aeroelastic and the aerodynamic time steps is given by

$$\Delta\hat{t}=(\Delta t)\frac{a_\infty}{2b},\qquad \Delta\bar{t}=(\Delta t)\omega_r=\frac{U_\infty 2b}{a_\infty b}\frac{\Delta\hat{t}}{U^*}=\frac{2M_\infty}{U^*}\Delta\hat{t}, \tag{10}$$

where $\Delta\hat{t}$ designates the flow solver time step and a_∞ is the undisturbed flow sound speed. Considering that both time steps are small enough, it is reasonable to assume that $\{\bar{Q}a(\bar{t})\}$, and consequently, $\{\tilde{q}(\bar{t})\}$ and $\{q_x(\bar{t})\}$, are constant along a time step interval. Hence, as demonstrated by Ogata (1987), Eq. (8) can be rewritten as

$$\{x((n+1)\Delta\bar{t})\}=[G(\Delta\bar{t})]\{x(n\Delta\bar{t})\}+[H(\Delta\bar{t})]\{q_x(n\Delta\bar{t})\}, \tag{11}$$

where

$$[G(\Delta\bar{t})]=e^{([W]\Delta\bar{t})},\qquad [H(\Delta\bar{t})]=\int_0^{\Delta\bar{t}}e^{([W]\tau)}d\tau=[W]^{-1}\left[e^{([W]\Delta\bar{t})}-[I_{4\times 4}]\right]. \tag{12}$$

Moreover, the sampling of the state vector $\{x(\bar{t})\}$ and $\{q_x(\bar{t})\}$ results in the following sequences, respectively,

$$\{x[n]\}=\{x((n-1)\Delta\bar{t})\},\qquad \{q_x[n]\}=\{q_x((n-1)\Delta\bar{t})\}. \tag{13}$$

Therefore, the discrete version of Eq. (11) is

$$\{x[n+1]\}=[G(\Delta\bar{t})]\{x[n]\}+[H(\Delta\bar{t})]\{q_x[n]\}. \tag{14}$$

The z transform stands for discrete-time sequences as an operation similar to the Laplace transform for the continuous-time functions (Ogata, 1987). Therefore, instead of using approximating polynomials for the discrete aerodynamic response, it may be more convenient to perform the frequency domain analysis with the application of the z transform, or z domain analysis. The present formulation is based on the one-sided z transform, defined as

$$Y(z)=\mathcal{Z}\{y[n]\}=\sum_{n=0}^{\infty}y[n]z^{-n}, \tag{15}$$

where $Y(z)$ is the z transform of the discrete sequence $y[n]$. Furthermore, it is important to notice that the definition of the z transform itself results in a polynomial in the z domain. This fact facilitates the construction of the state space model, avoiding the need for a polynomial fitting process. One very important theorem concerning the z transform is the shifting theorem (Ogata, 1987). Such theorem states that, if $y[n]=0$ for $n<0$, then

$$\mathcal{Z}\{y[n+j]\}=z^jY(z). \tag{16}$$

Consequently, the z transform of Eq. (14) yields

$$z\{X(z)\} = [G(\Delta\bar{t})]\{X(z)\} + [H(\Delta\bar{t})]\{Q_x(z)\} \tag{17}$$

Thus, the formulation of the discrete aeroelastic problem depends on the determination of the forcing term $\{Q_x(z)\}$ in the z domain. Moreover, Oppenheim and Schafer (1989) show that the z transform also presents the convolution theorem property, which states that the z transform of the convolution sum of two sequences is identical to the multiplication of their individual z transforms. Therefore, the discrete-time, z domain, equivalent statement of the linearization hypothesis adopted in Eq. (4) is

$$\{\bar{Q}a(z)\} = \frac{(U^*)^2}{\pi\mu}[A(z)]\{\eta(z)\}, \qquad [A(z)] = \begin{bmatrix} -C_{\ell_{Ih}}(z)/2 & -C_{\ell_{I\alpha}}(z) \\ C_{m_{Ih}}(z) & 2C_{m_{I\alpha}}(z) \end{bmatrix}. \tag{18}$$

Hence,

$$\{Q_x(z)\} = \left[\tilde{M}\right]^{-1}\left\{\begin{array}{c} \{\bar{Q}a(z)\} \\ \{0_{2\times1}\} \end{array}\right\} = \frac{(U^*)^2}{\pi\mu}\left[\tilde{M}\right]^{-1}\left\{\begin{array}{c} [A(z)]\{\eta(z)\} \\ \{0_{2\times1}\} \end{array}\right\}. \tag{19}$$

As explained in detail by Marques (2007), according to the suggestions offered by Ogata (1987), by defining the auxiliary function

$$\{Ya(z)\} = \begin{bmatrix} \frac{1}{a_h(z)} & 0 \\ 0 & \frac{1}{a_\alpha(z)} \end{bmatrix}\{\eta(z)\}, \tag{20}$$

where

$$a_h(z) = 1 + a_{h_1}z^{-1} + a_{h_2}z^{-2} + \ldots + a_{h_{(n_T-1)}}z^{-(n_T-1)}, \qquad a_{h_n} = \frac{h_{IN}[n+1]}{h_{IN}[1]}, \tag{21}$$

$$a_\alpha = 1 + a_{\alpha_1}z^{-1} + a_{\alpha_2}z^{-2} + \ldots + a_{\alpha_{n_T-1}}z^{-(n_T-1)}, \qquad a_{\alpha_n} = \frac{\alpha_{IN}[n+1]}{\alpha_{IN}[1]}, \tag{22}$$

IN designates the input used for exciting the aerodynamic system in order to obtain the impulsive responses, and n_T is the total number of points that constitute the aerodynamic response sequence, it is possible to construct the following aerodynamic state variables

$$\{Xa_1(z)\} = z^{-(n_T-1)}\{Ya(z)\}, \quad \{Xa_2(z)\} = z^{-(n_T-2)}\{Ya(z)\}, \ldots \left\{Xa_{(n_T-1)}(z)\right\}, = z^{-1}\{Ya(z)\}. \tag{23}$$

These aerodynamic states relate to each other by

$$z\{Xa_1(z)\} = \{Xa_2(z)\}, \quad z\{Xa_2(z)\} = \{Xa_3(z)\}, \ldots z\left\{Xa_{(n_T-2)}(z)\right\} = \left\{Xa_{(n_T-1)}(z)\right\}, \tag{24}$$

and

$$z\left\{Xa_{(n_T-1)}(z)\right\} = \begin{bmatrix} -\left[\bar{A}_{(n_T-1)}\right] & -\left[\bar{A}_{(n_T-2)}\right] & \ldots & -\left[\bar{A}_1\right] \end{bmatrix}\{Xa(z)\} + [I_{2\times2}]\{\eta(z)\}, \tag{25}$$

where the z transform of the aerodynamic state vector is given by

$$\{Xa(z)\} = \begin{bmatrix} \{Xa_1(z)\} & \{Xa_2(z)\} & \ldots & \left\{Xa_{(n_T-1)}(z)\right\} \end{bmatrix}^T, \quad \text{and} \quad \left[\bar{A}_n\right] = \begin{bmatrix} a_{h_n} & 0 \\ 0 & a_{\alpha_n} \end{bmatrix}. \tag{26}$$

Finally, it is possible to describe the forcing term $\{Q_x(z)\}$ through the new state space representation, as follows,

$$\{Q_x(z)\} = [Aa]\{Xa(z)\} + [Ax]\{X(z)\}, \tag{27}$$

where

$$[Aa] = \frac{(U^*)^2}{\pi\mu}\left[\tilde{M}\right]^{-1}\begin{bmatrix} [B] \\ \left[0_{2\times(2n_T-2)}\right] \end{bmatrix}, \qquad [Ax] = \frac{(U^*)^2}{\pi\mu}\left[\tilde{M}\right]^{-1}\begin{bmatrix} [0_{2\times2}] & [B_0] \\ [0_{2\times2}] & [0_{2\times2}] \end{bmatrix}, \tag{28}$$

$$[B] = \begin{bmatrix} \left[B_{C\ell_{(n_T-1)}}\right] & \left[B_{C\ell_{(n_T-2)}}\right] & \ldots & [B_{Cl_1}] \\ \left[B_{Cm_{(n_T-1)}}\right] & \left[B_{Cm_{(n_T-2)}}\right] & \ldots & [B_{Cm_1}] \end{bmatrix}, \qquad [B_0] = \begin{bmatrix} b_{C\ell h_0} & b_{C\ell\alpha_0} \\ b_{Cmh_0} & b_{Cm\alpha_0} \end{bmatrix}, \tag{29}$$

$$[B_{C\ell_n}] = \begin{bmatrix} b_{C\ell h_n} - a_{h_n}b_{C\ell h_0} & b_{C\ell\alpha_n} - a_{\alpha_n}b_{C\ell\alpha_0} \end{bmatrix}, \tag{30}$$

$$[B_{Cm_n}] = \begin{bmatrix} b_{Cmh_n} - a_{h_n}b_{Cmh_0} & b_{Cm\alpha_n} - a_{\alpha_n}b_{Cm\alpha_0} \end{bmatrix}, \tag{31}$$

$$b_{C\ell h_n} = -\frac{C_{\ell_{hOUT}}[n+1]}{2h_{IN}[1]}, \qquad b_{C\ell\alpha_n} = -\frac{C_{\ell_{\alpha OUT}}[n+1]}{\alpha_{IN}[1]}, \tag{32}$$

$$b_{Cm\alpha_n} = \frac{2C_{m_{\alpha OUT}}[n+1]}{\alpha_{IN}[1]}, \qquad b_{Cmh_n} = \frac{C_{m_{hOUT}}[n+1]}{h_{IN}[1]}, \tag{33}$$

and OUT represents the aerodynamic response to the excitaion input designated by IN.

Hence, Eq. (27) describes a state space representation in the z domain that is adequate for application in Eq. (17). The final expression for the state space problem is, then, given by

$$\left(\left[\bar{D}\right] - z\left[I_{N\times N}\right]\right)\left\{\bar{\chi}(z)\right\} = \left\{0_{N\times 1}\right\}, \tag{34}$$

where

$$\left[\bar{D}\right] = \begin{bmatrix} \left[\bar{D}xx\right] & \left[\bar{D}xa\right] \\ \left[\bar{D}ax\right] & \left[\bar{D}aa\right] \end{bmatrix}, \qquad \left[\bar{D}xx\right] = \left[G(\Delta\bar{t})\right] + \left[H(\Delta\bar{t})\right]\left[Ax\right], \tag{35}$$

$$\left[\bar{D}xa\right] = \left[H(\Delta\bar{t})\right]\left[Aa\right], \qquad \left[\bar{D}ax\right] = \begin{bmatrix} \left[0_{(2n_T-4)\times 2}\right] & \left[0_{(2n_T-4)\times 2}\right] \\ \left[0_{2\times 2}\right] & \left[I_{2\times 2}\right] \end{bmatrix}, \tag{36}$$

and

$$\left[\bar{D}aa\right] = \begin{bmatrix} \left[0_{2\times 2}\right] & \left[I_{2\times 2}\right] & \left[0_{2\times 2}\right] & \cdots & \left[0_{2\times 2}\right] \\ \left[0_{2\times 2}\right] & \left[0_{2\times 2}\right] & \left[I_{2\times 2}\right] & \cdots & \left[0_{2\times 2}\right] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left[0_{2\times 2}\right] & \left[0_{2\times 2}\right] & \left[0_{2\times 2}\right] & \cdots & \left[I_{2\times 2}\right] \\ -\left[\bar{A}_{(n_T-1)}\right] & -\left[\bar{A}_{(n_T-2)}\right] & -\left[\bar{A}_{(n_T-3)}\right] & \cdots & -\left[\bar{A}_1\right] \end{bmatrix}. \tag{37}$$

Here, N designates the total number of states, which is given by $N = (2n_T + 2)$. The state vector, $\{\bar{\chi}\}$, is formed by all the structural and aerodynamic state variables, and its z transform is defined as

$$\left\{\bar{\chi}(z)\right\} = \begin{bmatrix} \{X(z)\} & \{Xa(z)\} \end{bmatrix}^T. \tag{38}$$

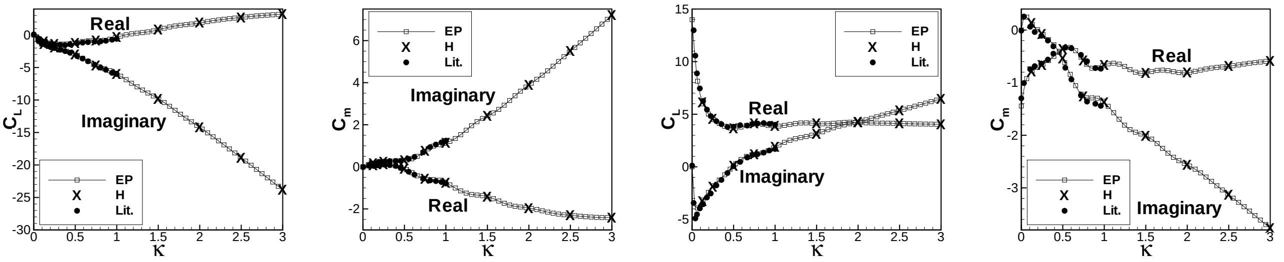
Although this discrete-time formulation does not require the construction of approximating polynomials, it has one drawback. The above formulation indicates that the number of augmented aerodynamic state vectors is the same as the number of the time steps needed to accurately represent the unsteady aerodynamic response. Therefore, the size of the eigenvalue problem is much larger than in the continuous-time formulation. The number of state variables may be hundreds or thousands of times larger in the discrete-time case. However, since in well-constructed root locus analyses the eigenvalues vary little from each other for two consecutive values of root locus parameter, there are numerical schemes that can solve for the desired eigenvalues quite rapidly. The greatest limitation, then, is the memory size. In any event, this difficulty can be overcome by choosing an appropriate number of points in order to represent the aerodynamic response for a certain frequency range of interest. Moreover, since matrix $\left[\bar{D}\right]$ has very few nonzero elements, memory usage can be also reduced by employing techniques for construction of sparse matrices.

4. RESULTS AND DISCUSSION

The test case included in the present paper considers the aeroelastic stability of a NACA 0012 airfoil at $M_\infty = 0.80$ and zero initial angle of attack. The structural parameters which define the problem are $a_h = -2.0$, $x_\alpha = 1.8$, $r_\alpha = 1.865$, $\mu = 60$, $\omega_\alpha = 100$rad/s, $\omega_h = 100$rad/s, and $\omega_r = \omega_\alpha$ is used as reference. This case is also reported by Rausch, Batina and Yang (1990), who performed the same sort of stability analysis with aerodynamic data obtained through a numerical scheme very similar to the one used in the present work. The impulsive aerodynamic response of both modes, in terms of the generalized force coefficients, and already in the frequency domain, are presented in Fig. 1. It is important to notice that these results are based on a pitching motion that occurs around the quarter-chord point of the airfoil, and that the moment coefficients are given with respect to that point. Since the elastic axis is located at another point, before proceeding to the aeroelastic stability analyses, the transformations indicated by Yang, Guruswamy and Striz (1980), and Marques (2007) are necessary.

The results of Fig. 1 present the frequency content of the response up to $k = 3.00$ because this is about the highest reduced frequency involved in the cases considered for the construction of the root locus stability analyses subsequently shown. The results obtained by the authors are those represented by EP, which stand for exponentially-shaped pulse, since this is the type of excitation imposed to the aerodynamic system. Figure 1 also includes the aerodynamic responses to harmonic (H) motions at different reduced frequencies in both modes. Such responses are shown in order to address the linearity question brought up in a previous section of this paper. Oppenheim and Schefer (1989) shows that obtaining impulsive responses from smooth excitations is only possible if the linearity assumptions, that are made in this work, hold. Therefore, the agreement between EP and H data is a proof that such hypotheses are valid for small amplitudes. Furthermore, although Rausch, Batina and Yang (1990) offer comparison data only to reduced frequencies up to $k = 1.00$, the present results match the literature (Lit.) values very closely in that range. This agreement is another assessment of the correctness of the present results.

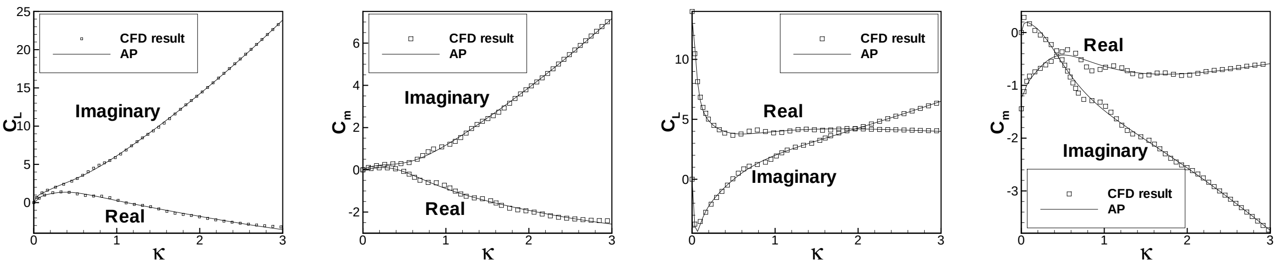
The construction of a continuous-time state space representation of the aeroelastic systems requires the approximation of the frequency domain impulsive aerodynamic loads through the use of approximating polynomials or rational functions.



(a) C_ℓ frequency domain response. (b) C_m frequency domain response. (c) C_ℓ frequency domain response. (d) C_m frequency domain response.

Figure 1. Low reduced frequency response of a NACA 0012 airfoil at $M_\infty = 0.80$ and $\alpha_0 = 0$ to an impulsive input.

In the present work, the authors chose the polynomial suggested by Eversman and Tewari (1991) with the use of six poles. Figure 2 shows comparisons between the discrete CFD results and the chosen approximating (AP) model for the aerodynamic coefficients. The agreement between these data is extremely good, except for some small regions in which abrupt oscillations of the CFD results are smoothed by the polynomial. However, even in these regions, the fitting error is relatively small.



(a) C_ℓ frequency domain response. (b) C_m frequency domain response. (c) C_ℓ frequency domain response. (d) C_m frequency domain response.

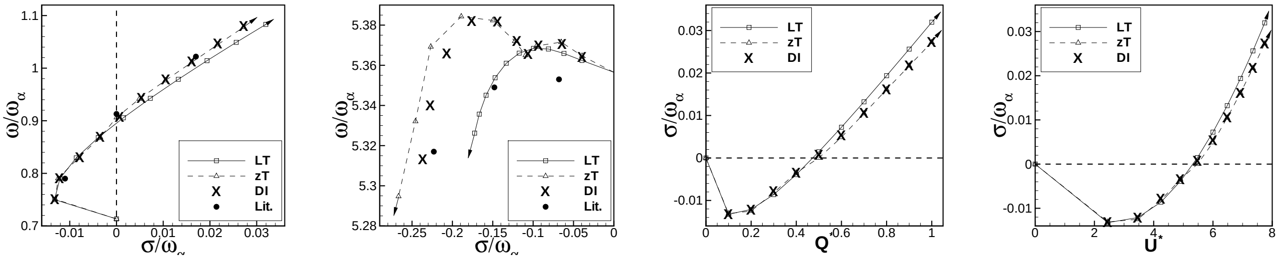
Figure 2. Low reduced frequency response of a NACA 0012 airfoil at $M_\infty = 0.80.$ and $\alpha_0 = 0$ to an impulsive input. Comparisons between CFD results and approximated data.

Finally, the stability analysis is concluded with the construction of a root locus graph, defined by the solution of eigenvalue problem that results form the continuous-time or discrete-time formulation. It is relevant to notice that, in the discrete-time case, the eigenvalue problem is written in terms of the z variable, which is defined in a different manner than the Laplace variable, s. As the root locus is presented in terms of the nondimensional $\bar{s}$ variable, the real relationship between these variables is

$$\bar{s} = \frac{s}{\omega_r} = \frac{1}{\Delta \bar{t}} \ln(z) = \frac{U^*}{2M_\infty \Delta \hat{t}} \ln(z). \tag{39}$$

Figures 3(a) and 3(b) present the root locus of the aeroelastic problem in question. The characteristic dynamic pressure parameter varies from $Q^* = 0.0$ up to 1.0 in $\Delta Q^* = 0.1$ intervals. Each plotted point corresponds to one of these values. The figure includes the solutions obtained using the continuous-time Laplace transform (LT), discrete-time z transform (zT), and direct integration (DI) approaches, as well as the numerical direct integration results given by Rausch, Batina and Yang (1990) (Lit.). The literature data, however, are only available for $Q^* = 0.2$, 0.5, and 0.8. Additionally, the results for the first two approaches are determined from the eigenvalue problems previously mentioned, while damping and frequency characteristics of the direct integration responses are estimated using the modal identification technique of Bennet and Desmarais (1975).

Furthermore, it is evident that the first aeroelastic mode curves and points obtained with the different methodologies agree much better than in the second aeroelastic mode. The authors believe that this occurs for two main reasons. First of all, the modal identification technique used in the evaluation of damping and frequency characteristics of the direct integration solutions (DI and Lit.) is based on a curve fitting procedure applied simultaneously for both aeroelastic modes. It turns out that the time domain response of the most damped aeroelastic mode dies out much more rapidly than the response of the least damped aeroelastic mode. Hence, the least damped mode, which in the present case is the first aeroelastic mode, is favored in the curve fitting. The authors attempt to reduce this effect by using different sets of direct integration solutions for the determination of the frequency and damping characteristics of each aeroelastic mode. According to this approach, the most damped aeroelastic mode characteristics are evaluated with the initial portion of the direct integration solution, while the least damped aeroelastic mode characteristics are calculated using the whole solution. This



(a) First aeroelastic mode. (b) Second aeroelastic mode. (c) Characteristic dynamic pressure. (d) Characteristic speed.

Figure 3. Aeroelastic stability analysis.

procedure attenuates the damping difference effect, but it does not eliminate it. Moreover, the polynomial approximation for the aerodynamic coefficients produces the largest errors in the pitch mode responses for reduced frequency values between $k = 0.5$ and 1.0. According to Marques (2007), this means that these errors affect the continuous-time (LT) pitch mode root locus estimates mainly for in the range $0.5 \leq Q^* \leq 1.8$. As the plunge mode natural frequency is smaller, the influence of such errors on this mode is restricted to the range $Q^* < 0.1$.

It is also important to notice that the discrete-time (zT) root locus is the one which is closer to the DI results. This means that the z domain state space formulation suggested by the present authors is capable of adequately representing the aeroelastic system and provides an effective tool for determining flutter instability points. The larger errors resultant from the continuous-time formulation are probably caused by inaccuracies that are intrinsic to the polynomial interpolation procedure. Moreover, solving the large eigenvalue problem given by the z transform formulation has shown to be more practical, efficient, and accurate than using the continuous-time approach with its requirement for data approximation. Additionally, the authors' experience has shown that the large size of the aeroelastic stability matrix in the discrete-time case does not lead to ill-conditioning problems. Finally, as it is shown in Fig. 3, the literature data corroborate the quality of the results contained in the present paper. Even in the second aeroelastic mode case, in which the discrepancies are the largest, the relative differences are smaller than 1%, although the provided axis scales may seem to indicate otherwise.

At last, the flutter condition can be identified as the point at which one of the curves crosses the imaginary axis. A point-to-point linear interpolation of the results presented in Figs. 3(a) and 3(b) indicates the flutter frequency. Furthermore, Figs. 3(c) and 3(d) show the damping behavior of the first aeroelastic mode in relation to the characteristic dynamic pressure and characteristic speed, respectevely. The same sort of interpolation of such results yield the flutter characteristic dynamic pressure and characteristic speed. The flutter points acquired with the different approaches are shown in Table 1, in which the flutter prediction of Rausch, Batina and Yang (1990) is also included. Actually, the literature value is given by a quadratic interpolation using the offered points. As can be seen, despite the mentioned discrepancies, the flutter predictions given by all methods are very similar, and the differences are not greater than 3% in any of the considered parameters.

Table 1. Flutter points.

Method	Q_f^*	U_f^*	ω/ω_α
DI	0.48	5.38	0.90
LT	0.47	5.32	0.90
zT	0.50	5.47	0.91
Lit.	0.50	5.48	0.91

5. CONCLUSIONS

The authors present a complete description of a new discrete-time, z domain state space formulation of the aeroelastic system adequate for aerodynamic unsteady responses obtained with numerical methods. Comparisons among the results obtained with this new model, usual continuous-time formulations, and direct integrations of the structural dynamic and aerodynamic equations, demonstrate that the discrete-time formulation avoids the errors introduced by the approximation of the aerodynamic responses with the direct application of the z transform to the time domain discrete aerodynamic data numerically obtained. Hence, this method has shown to be more accurate than the more usual state space representations. Therefore, the paper shows that, not only a z transform discrete-time state space representation of aeroelastic systems is possible, but it also is a very effective and accurate formulation. Furthermore, since the representation of the aerodynamic response in the frequency domain can be evaluated with the application of the z transform, this new approach also seems to be a more straightforward method for large scale applications.

Furthermore, the large number of aerodynamic state variables, which result from the z domain state space representation, does not affect the eigenvalue problem conditioning. Moreover, although this method leads to unreasonably large stability matrices, only some of their terms present nonzero values. Thus, memory requirements can be lessened with the use of sparse matrix construction techniques.

Finally, it is relevant to notice that the aeroelastic analysis methodologies are successfully applied to a typical section model at transonic speed. The flutter points identified using the differen state space formulations are generally very similar to each other, and also to values predicted by the use of direct integrations. Hence, this demonstrates that the model proposed in the present work for efficient aeroelastic analysis is perfectly capable of offering good results in the transonic regime.

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8. Responsibility notice

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