TIMOSHENKO BEAM WITH UNCERTAINTY ON THE BOUNDARY CONDITIONS

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Abstract. *In mechanical modeling, uncertainties are present and they should be taken into account. Particulary, this work wants to discuss uncertainties presented in boundary conditions. The model used is a vibrating Timoshenko beam, free in one end and, in the other, pinned and with rotation constrained by a linear elastic torsional spring. The Finite Element Method is used to discretize the system. Two probabilistic approaches are considered: (1) The stiffness of the torsional spring is taken as uncertain and a random variable is associated to it; (2) The whole stiffness matrix is considered as uncertain and a probability model is constructed for the random matrix associated. In both approaches, the probability density functions are deduced from the Maximum Entropy Principle. In the first approach, only the uncertainties of a parameter is taken into account and in the second approach, the model uncertainties are studied. Both approaches are compared and their capability to improve the predictability of the system response is discussed.*

Keywords: *uncertainties, Timoshenko beam, boundary conditions, stochastic mechanics*

1. INTRODUCTION

Uncertainties in mechanical systems have been studied in order to improve the predictability of the models. In a general way, uncertainties can be classified in two types, data uncertainties and model uncertainties. This paper discusses both types of uncertainties for the case of uncertainties presented on the boundary conditions of a vibration beam.

Probability tools are used to model the uncertain boundary conditions, that is, random variables are associated to the uncertain parameters or matrices and probability density functions are constructed. The strategy used here is the same applied on Cataldo et al. (2007), and Sampaio and Soize (2007).

The process of modeling mechanical systems introduces two types of uncertainties: (1) uncertainties related to the parameters of the models such as geometrical parameters, parameters used in the constitutive equations, boundary conditions and etc, which we call *data uncertainties* and (2) uncertainties due to the introduction of simplifications in order to decrease the complexity of the mean model, which we call *model uncertainties*.

To discuss uncertainties present on the boundary conditions of a beam in vibration, the model used is of a Timonshenko beam free in one end and, in the other, pinned and with rotation constrained by a linear elastic torsional spring. The uncertainties analyzed are those ones related to the torsional spring and, also, related to the stiffness matrix, in the corresponding discretized problem by using the Finite Element Method. First, the torsional spring is considered uncertain and a random variable is associated to it. A parametric approach is used to construct the density probability function associated to that random variable. After, the whole stiffness matrix is considered as uncertain. A nonparametric probabilistic approach is then used to construct a probability density function associated to the stiffness matrix.

The probability density functions are constructed based on the Maximum Entropy Principle (Jaynes, 1957a and 1957b). This strategy considers only the usable information to construct the possible probability density functions, and among all of them, it is chosen that one with the maximum entropy (or uncertainty). The concept of entropy is the one used by Shannon (1948) and some applications of this method can be found in Kapur and Kesavan (1992).

In order to avoid misunderstandings with deterministic and random variables, deterministic stiffness is denoted by k_t and the corresponding random variable is denoted by K_t. Deterministic stiffness matrix is denoted by $[K]$, and the corresponding random matrix is denoted by $[\mathbf{K}]$. Mean value associated to the random stiffness matrix is represented by $[\underline{K}]$.

The organization of this article is as follows: Section 2 presents the mean model, that is, the corresponding deterministic model. In Sec. 3, the procedure to build the corresponding stochastic problem is presented. Section 4 presents the numerical simulations and, finally, in Sec. 5 concluding remarks are outlined.

2. MEAN MODEL

Figure 1 shows the beam modeled as well the beam element

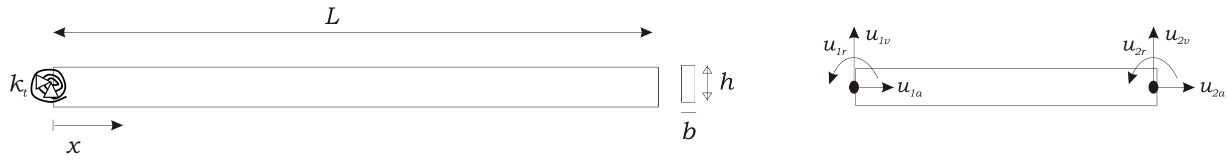

Figure 1. Beam modeled (left) and beam element (right).

Each finite element of the Timoshenko beam model has three degrees of freedom (axial, vertical and rotational) at each node. The variational form of strain energy is given by:

$$\delta H = \int_0^L [\delta u_a'(EAu_a') + \delta u_v''(EIu_v'') + \delta u_r'(GIu_r')]dx \tag{1}$$

Where $'$ is the derivative with respect to x (d/dx). The degrees of freedom are: the axial displacement (u_a); the vertical displacement (u_v); and the rotation (u_r). E is the elasticity modulus, G is the shearing modulus, I is the inertia momentum, L is the beam length and A is the cross section area. The virtual work done by inertial forces and damping forces are the following:

$$\{\delta T, \delta D\} = \int_0^L \{\delta u_a(\rho A\ddot{u}_a) + \delta u_v(\rho A\ddot{u}_v) + \delta u_r(\rho I\ddot{u}_r), -\delta u_a c\dot{u}_a - \delta u_v c\dot{u}_v - \delta u_r c\dot{u}_r\}dx \tag{2}$$

The damping forces are taking into account as a Rayleigh damping proportional to the mass, where c is the damping constant. The element matrices are the following ones:

$$[M^{(e)}] = \rho A\frac{h}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22h & 0 & 54 & -13h \\ 0 & 22h & 4h^2 & 0 & 13h & -3h^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13h & 0 & 156 & -22h \\ 0 & -13h & -3h^2 & 0 & -22h & 4h^2 \end{bmatrix} \quad ; \quad [C] = \frac{c}{\rho A}[M] \tag{3}$$

$$[K^{(e)}] = \rho\frac{E}{1+m}\begin{bmatrix} A(1+m)/h & 0 & 0 & -A(1+m)/h & 0 & 0 \\ 0 & 12I/h^3 & 6I/h^2 & 0 & -12I/h^3 & 6I/h^2 \\ 0 & 6I/h^2 & 4I(1+m/4)/h & 0 & -6I/h^2 & 2I(1-m/2)/h \\ -A(1+m)/h & 0 & 0 & A(1+m)/h & 0 & 0 \\ 0 & -12I/h^3 & -6I/h^2 & 0 & 12I/h^3 & -6I/h^2 \\ 0 & 6I/h^2 & 2I(1-m/2)/h & 0 & -6I/h^2 & 4I(1+m/4)/h \end{bmatrix} \tag{4}$$

Where $m = \left(\frac{12}{h^2}\right)\left(\frac{EI}{GAk_s}\right)$, k_s is the shear correction factor and h is the element size.

The essential boundary conditions at $x = 0$ are given by $u_a|_{x=0} = 0$ and $u_v|_{x=0} = 0$. The natural boundary condition at $x = 0$ is $GI\frac{\partial u_r}{\partial x}|_{x=0} = k_t u_r|_{x=0}$. The natural boundary conditions at $x = L$ are given by $EA\frac{\partial u_a}{\partial x}|_{x=L} = 0$, $EI\frac{\partial^3 u_v}{\partial x^3}|_{x=L} = 0$ and $GI\frac{\partial u_r}{\partial x}|_{x=L} = 0$. The system is discretized using the Finite Element Method, and after assembling the matrices, one can write:

$$[M]\ddot{\mathbf{u}}(t) + [C]\dot{\mathbf{u}}(t) + [K]\mathbf{u}(t) = \mathbf{f}(t) \tag{5}$$

Where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices, which are real and positive-definite. The external force is represented by vector $\mathbf{f}(t) = (f_{1a}, f_{1v}, f_{1r}, f_{2a}, f_{2v}, f_{2r}, \ldots, f_{na})$ and the displacements $(u_{1a}, u_{1v}, u_{1r}, u_{2a}, u_{2v}, u_{2r}, \ldots, u_{na})$ are the components of the vector $\mathbf{u}(t)$.

2.1 Frequency Response Function (FRF) calculated from the mean model

Let $\widehat{f}_{Lv}(\omega)$ be the Fourier transform of the force component $f_{Lv}(t)$ applied at $x = L$, and $\widehat{u}_{Lv}(\omega)$ be the Fourier transform of $u_{Lv}(t)$, which is the component vertical of the displacement vector at $x = L$.

The FRF (h) used here is defined by

$$h(\omega) = \frac{\widehat{u}_{Lv}(\omega)}{\widehat{f}_{Lv}(\omega)}. \tag{6}$$

An example is performed considering the material homogeneous and isotropic with $E = 2e11\,\mathrm{N/m^2}$, $c = 10\,$Ns/m, $k_t = 10^7\,$N/m, $\rho = 7850\,\mathrm{kg/m^3}$, $\nu = 0.3$, $L = 0.5$m, $b = 1$cm and $h = 5$cm. Allowing a maximum error of 1% on the fourth normal mode (using the Rayleigh coefficient), it is necessary a mesh of sixteen finite elements. The corresponding FRF is showed in Fig. 2.

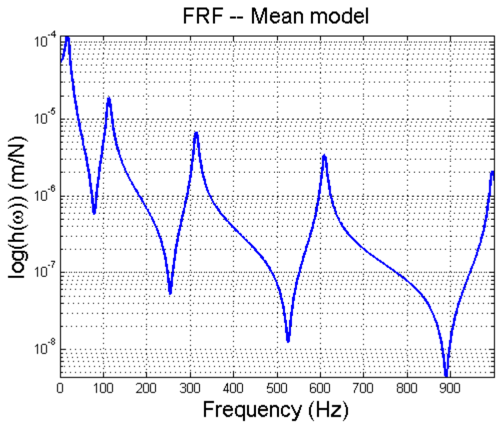

Figure 2. Frequency Response Function

The finite element approximation of the displacement is computed on frequency band $B = [0, 1000]$ Hz, and the first 4 natural frequencies are 18 Hz, 113 Hz, 314 Hz and 609 Hz. Figure 3 shows the four first modes.

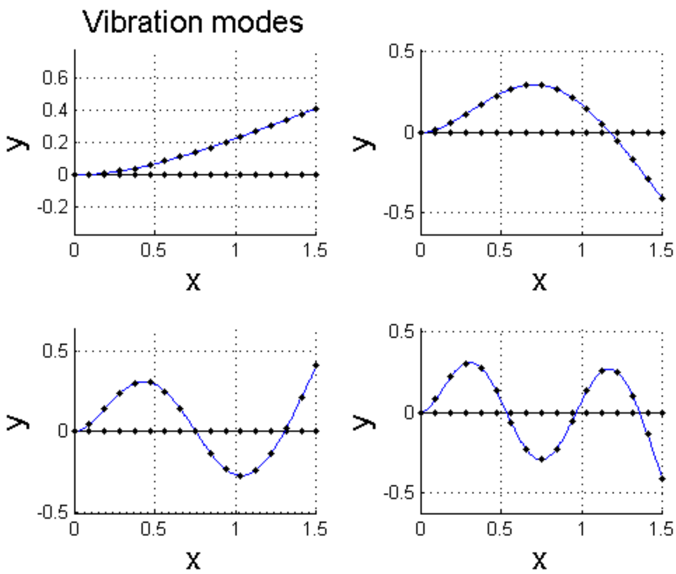

Figure 3. Four first modes.

3. STOCHASTIC MODEL

In order to study the system behavior when uncertainties are present and consequently to improve its predictability, some parameters and matrices will be considered as uncertain and random variables will be associated to them.

As discussed, the problem will be divided considering two probabilistic approaches: (1) parametric, in which only the stiffness of the torsional spring is taken as uncertain and a random variable is associated to it, and (2) nonparametric, in which the whole stiffness matrix in the corresponding system discretized is taken as uncertain and a random matrix is associated to it. In both approaches, the deterministic stiffness matrix $[K]$ will be substituted by a random matrix $[\mathbf{K}]$, in Eq. 5. However, the way of constructing the probability density function will change, depending on the approach used. Then, one can rewrite the dynamics of the system, discretized by Element Finite Method, as

$$[M]\ddot{\mathbf{U}} + [C]\dot{\mathbf{U}} + [\mathbf{K}]\mathbf{U} = \mathbf{f}(t) \tag{7}$$

where $\mathbf{U}$ is the stochastic process associated with the response of the corresponding stochastic system. Writing Eq. 7 in the frequency domain, one has

$$(-\omega^2[M] + i\omega[C] + [\mathbf{K}])\widehat{\mathbf{U}}(\omega) = \widehat{\mathbf{f}}(\omega) \tag{8}$$

3.1 Probabilistic model of the torsional spring stiffness: parametric approach

In the first probabilistic approach used (parametric approach), the uncertain torsional spring stiffness k_t will be modeled by a random variable K_t, for which an appropriate probabilistic model for the random variable must be constructed.

The strategy used here is based on the Maximum Entropy Principle and uses only the following information about K_t: (1) It is a positive random variable, so its support equals to $]0, +\infty[$, (2) Its expected value is known and it is given by $E\{K_t\} = \underline{K}_t$, and (3) $E\{1/K_t^2\} = c$, with $c < +\infty$. This constraint is taken into account by requiring that $E\{ln(K_t)\} = c_1$ with $|c| < +\infty$ (Soize, 2001).

The probability density function of K_t, using the Maximum Entropy Principle, yields (Soize, 1999 and 2001):

$$p_{K_t}(K_t) = \mathbf{1}_{]0,+\infty[}(q)\frac{1}{\underline{K}_t}\left(\frac{1}{\delta_{K_t}^2}\right)^{\frac{1}{\delta_{K_t}^2}}\frac{1}{\Gamma\left(1/\delta_{K_t}^2\right)}\left(\frac{q}{\underline{K}_t}\right)^{\frac{1}{\delta_{K_t}^2}-1} exp\left(-\frac{K_t}{\delta_{K_t}^2\underline{K}_t}\right)\ , \tag{9}$$

where δ_{K_t} is the dispersion parameters and $\Gamma(z)$ is the gamma function defined for $z > 0$, $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$. There are limits for the dispersion parameter: $0 < \delta_{K_t} < 1/\sqrt{3}$.

3.2 Probabilistic model for the stiffness matrix: nonparametric approach

Now, the stiffness matrix, in a whole, will be taken as uncertain. The goal is to discuss globally the uncertainties related to the modeling of the stiffness of the discretized system. This strategy will be called nonparametric probabilistic approach and a density probability function will be constructed directly for the corresponding random matrix $[\mathbf{K}]$. This will be done following the recently ideas given by Soize (2001).

Matrix $[\underline{K}]$ is a positive-definite matrix and it can be decomposed (Cholesky Decomposition) as:

$$[\underline{K}] = [\underline{L}_K]^T[\underline{L}_K] \tag{10}$$

where $[L_K]$ is an upper triangular matrix. Consequently, the random matrix $[\mathbf{K}]$ can be written as

$$[\mathbf{K}] = [\underline{L}_K]^T[\mathbf{G}][\underline{L}_K] \tag{11}$$

where $[\mathbf{G}]$ is a random matrix such as: (1) it is positive-definite, (2) its expected value is the identity matrix, so $E\{[\mathbf{G}]\} = [I]$, (3) it is a second order random variable, so $E\{||[\mathbf{G}]^2||\} < +\infty$ and (4) Writing the Eq. 5 in the frequency domain, it can be proved that the corresponding stochastic equation has a unique second-order random solution if and only if $E\{||[\mathbf{K}^{-1}]^2||_F\} < +\infty$ and, consequently, $E\{||[\mathbf{G}^{-1}]^2||_F\} < +\infty$ (Soize, 2001), where $||.||_F$ is the Frobenius norm.

The probability density function of $[\mathbf{G}]$ is constructed using the Maximum Entropy Principle and it is done by (Soize, 2001):

$$p_{[\mathbf{G}]}([G]) = 1_{\mathbb{M}^+(\mathbb{R})}([G]) \times C_{\mathbf{G}} \times det([G])^{(n+1)\frac{(1-\delta^2)}{2\delta^2}} \times \exp\left\{-\frac{(n+1)}{2\delta^2}tr[G]\right\} \tag{12}$$

where n is the size of the matrix $[\mathbf{G}]$ and

$$C_{\mathbf{G}} = \frac{(2\pi)^{-n(n-1)/4}\left(\frac{n+1}{2\delta^2}\right)^{n(n+1)(2\delta^2)^{-1}}}{\left\{\prod_{j=1}^{n}\Gamma\left(\frac{n+1}{2\delta^2}+\frac{1-j}{2}\right)\right\}}\ . \tag{13}$$

The dispersion parameter δ is given by:

$$\delta = \left\{ \frac{1}{n} E\{||[\mathbf{G}] - [I]||_F^2\} \right\}^{\frac{1}{2}} \tag{14}$$

and $0 < \delta < \sqrt{\frac{n+1}{n+5}}$.

The matrix $[\mathbf{G}]$ is built, for each realization of matrix $[\mathbf{K}]$, first, decomposing (Choelsky decomposition) $[\mathbf{G}]$: $[\mathbf{G}] = [\mathbf{L}]^T[\mathbf{L}]$ and $[\mathbf{L}]$ is an upper triangular real positive-definite random matrix such that:

- The random variables $\{[\mathbf{L}]_{jj'}, j \leq j'\}$ are independents.
- For $j < j'$ the real-valued random variable $[\mathbf{L}]_{jj'} = \sigma V_{jj'}$, in which $\sigma = \delta(n+1)^{-1/2}$ and $V_{jj'}$ is a real-valued gaussian random variable with zero mean and unit variance.
- For $j = j'$ the real-valued random variable $[\mathbf{L}]_{jj'} = \sigma\sqrt{2V_j}$. In which V_j is a real-valued gamma random variable with probability density function:

$$p_{V_j}(v) = \mathbf{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma\left(\frac{n+1}{2\delta^2} + \frac{1-j}{2}\right)} v^{\frac{n+1}{2\delta^2} - \frac{1+j}{2}} \exp(-v) \tag{15}$$

3.3 Convergence of the stochastic solution

Let $[\mathbf{U}(\theta, \omega)]$ be the response of the stochastic system calculated for each realization θ. The mean-square convergence analysis with respect to independent realizations of the random variable $\widehat{\mathbf{U}}$, denoted by $\widehat{\mathbf{U}}_j(\theta, \omega)$, is carried out studying the function $n_s \mapsto \text{conv}(n_s)$ defined by

$$\text{conv}(n_s) = \frac{1}{n_s} \sum_{j=1}^{n_s} \int_B ||\widehat{\mathbf{U}}_j(\theta, \omega)||^2 d\omega \tag{16}$$

For each realization θ, the FRF $\text{H}(\theta, \omega)$, according to Eq. 6, is given by

$$\text{H}(\theta, \omega) = \frac{\widehat{U}_L(\theta, \omega)}{\widehat{f}_L(\omega)} = \widehat{U}_L(\theta, \omega)\,. \tag{17}$$

As the objective is to compare the probabilistic approaches parametric and nonparametric, it is important to know the number n_s (Eq. 16) such that the solution converges. This convergence analysis is performed for different values of the dispersion parameter (that will be discussed later) and it is verified that the solution always converges for $n_s = 500$. Figure 4 shows an example for the function conv, considering the nonparametric probabilistic approach, with $\delta_{[\mathbf{K}]} = 0.1$.

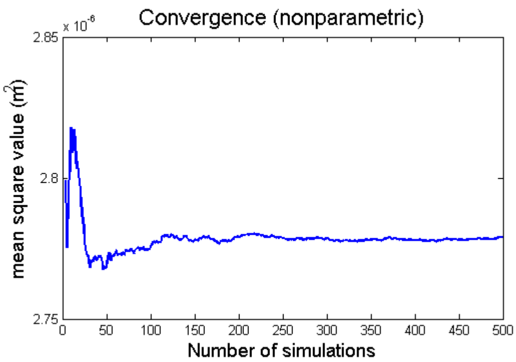

Figure 4. Convergence of the mean square error.

4. NUMERICAL SIMULATIONS

Some values of the coefficients of dispersion are chosen and confidence regions are constructed for the corresponding frequency response functions, using the quantiles (Serfling (1980)) and for a confidence level of 98%.

Figure 5 shows confidence limits for both parametric and nonparametric approaches, taking $\delta_{K_t} = \delta_{[\mathbf{K}]} = 0.1$ and also the corresponding FRF calculated for the mean model. One can note that the confidence region is larger for the nonparametric approach and it englobes the limits for the parametric approach. One can also observe that, as the frequency increases, the confidence region becomes larger, showing the great influence of uncertainties at high frequencies.

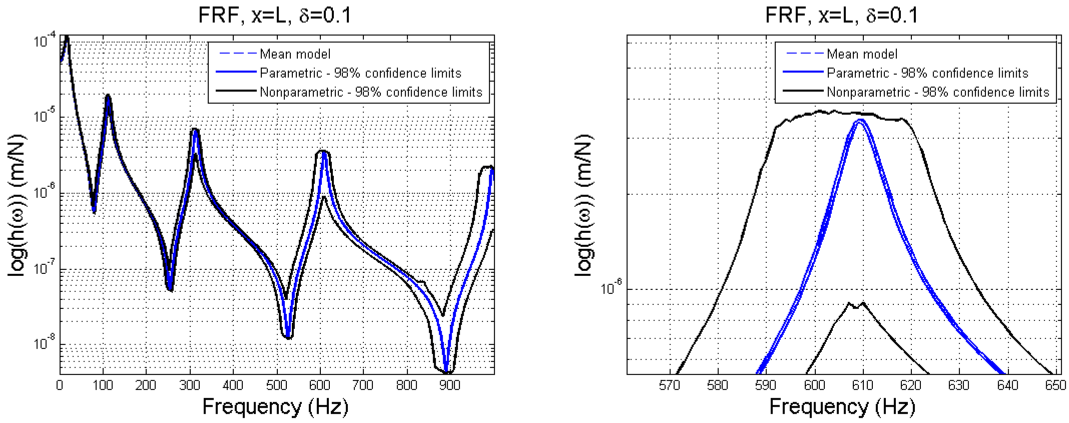

Figure 5. FRF of the mean model with probability level of 98%, for both parametric and nonparametric probabilistic approaches, taking $\delta_{K_t} = \delta_{[\mathbf{K}]} = 0.1$ (left). Zoom image (right).

The coefficient of dispersion is then increased: $\delta_{K_t} = \delta_{[\mathbf{K}]} = 0.2$. Figure 6 shows the confidence region of the FRFs obtained and also the corresponding FRF obtained for the mean model. It is also shown the corresponding FRF at $x = L/2$. One can note that, in this case, the confidence region is larger. The limits for the nonparametric approach englobes the limits for the parametric approach, and, as the frequency increases, the confidence region gets larger, as noted in the previous case.

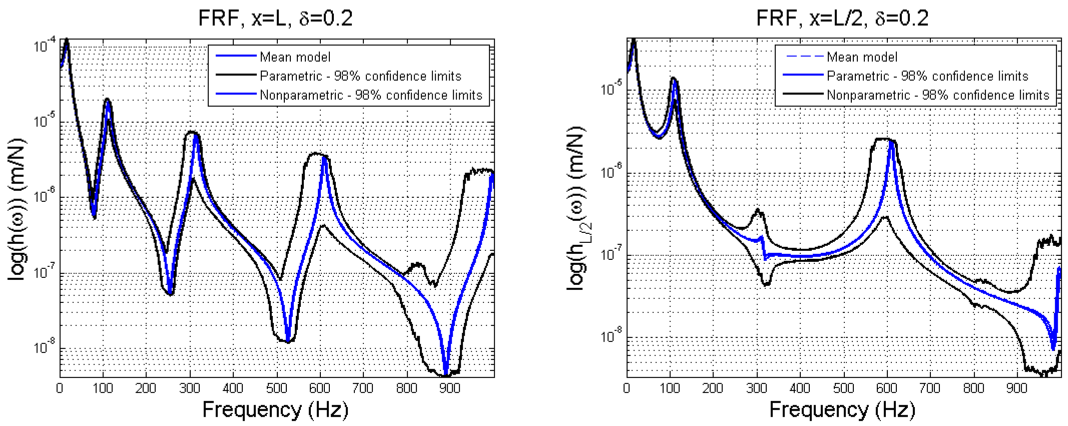

Figure 6. Mean model responses and confidence region for both parametric and nonparametric probabilistic approaches, taking $\delta_{K_t} = \delta_{[\mathbf{K}]} = 0.2$: response at $x = L$ (left) and response at $x = L/2$ (right).

Now, the values of the coefficients of dispersion are taken at their maxima values, that is, $\delta_{K_t} = \dfrac{1}{\sqrt{3}} \simeq 0.58$ and $\delta_{[\mathbf{K}]} = \dfrac{49+1}{49+5} \simeq 0.96$. The results are shown in Fig. 7. On the left, it is shown the mean model FRF and the confidence region for both parametric and nonparametric approaches. On the right, the expected values for both probabilistic approaches are plotted. For the parametric approach, the behavior of the limits does not change much from what was seen in Figs. 5 and 6. However, for the nonparametric approach, one can note a different behavior. It happens because the confidence region is very wide.

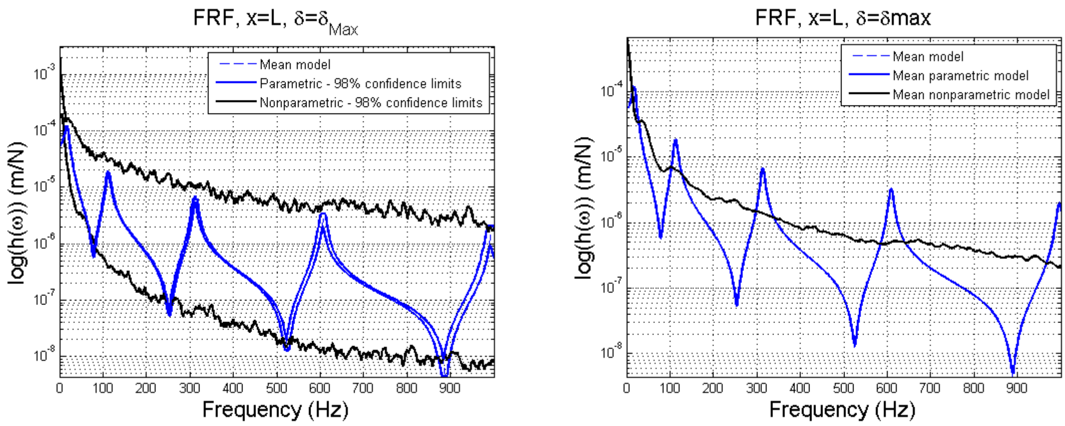

Figure 7. Mean model FRF with probability level of 98%, confidence region for both parametric and nonparametric probabilistic approach with maxima values of the coefficients of dispersion: $\delta_{K_t} = 0.58$ and $\delta_{[\mathbf{K}]} = 0.96$ (left) and expected values for the FRFs, with maxima values of the coefficients of dispersion: $\delta_{K_t} = 0.58$ and $\delta_{[\mathbf{K}]} = 0.96$ (right).

The idea of using nonparametric probabilistic approach is to take into account model uncertainties, including boundary conditions uncertainties. Then, three different (deterministic) boundary conditions are considered and their results compared with those ones previously obtained with the stochastic system discussed. The three cases considered are: (Fig. 8).

1. Essential boundary conditions at $x = 0$: $u_a|_{x=0} = 0$ and $u_v|_{x=0} = 0$;
2. Essential boundary conditions at $x = 0$: $u_a|_{x=0} = 0$, $u_v|_{x=0} = 0$ and $u_r|_{x=0} = 0$;
3. Essential boundary conditions at $x = 0$: $u_a|_{x=0} = 0$, $u_v|_{x=0} = 0$ and $u_r|_{x=0} = 0$. Essential boundary conditions at $x = L$: $u_a|_{x=L} = 0$ and $u_r|_{x=L} = 0$

Figure 8. New boundary conditions. Cases (1), (2) and (3).

Clearly, the idea is to discuss uncertainties present in a model, and it is not to change completely the model. However, it is important to investigate the limits of the nonparametric approach in predicting possible responses of a system.

It should be noted that matrix $[K]$ will have different dimensions for each case because of the restricted degrees of freedom.

Figure 9 shows the FRFs of the three cases of boundary conditions and also the confidence limits of the probabilistic parametric and nonparametric approaches previously discussed. On the left side, the values of the coefficient of dispersion are taken as 0.4, in both parametric and nonparametric approaches and on the right side, the values of the coefficient of dispersion as taken in their maxima values.

In both plots (left and right) one notes that the limits for the parametric approach are far away from the response perceived. However, it does not happen when the nonparametric approach is considered. For a coefficient of dispersion of 0.4, the confidence region almost englobes the three responses of the deterministic problems simulated. It does not mean that the result is satisfactory, indeed it is not. When the value of the coefficient of dispersion is maximum, the limits are so wide that all possible responses could fit in.

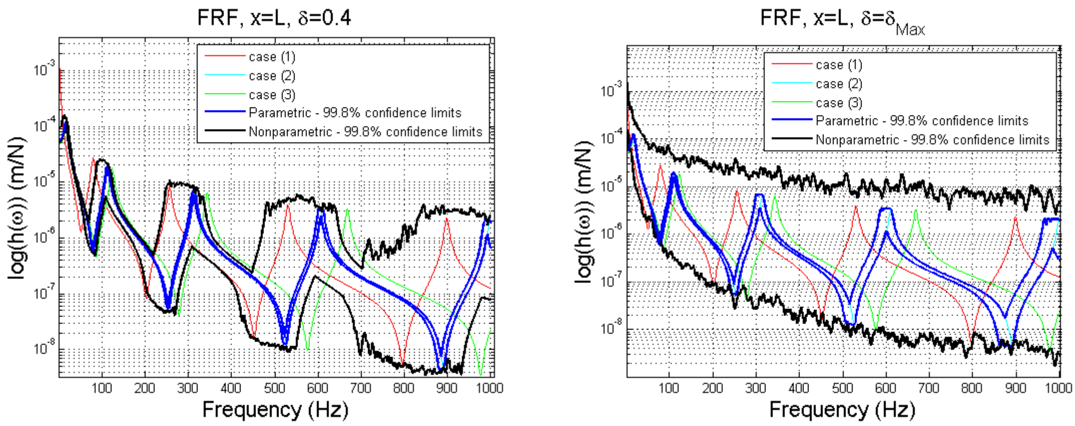

Figure 9. FRFs for three deterministic cases: $\underline{K}_t = 0$, clamped-free and clamped-locked, FRF for parametric and nonparametric approaches with probability level of 99.8%. $\delta_{K_t} = \delta_{[\mathbf{K}]} = 0.4$ (left) and maxima values $\delta_{K_t} = 0.58$ and $\delta_{[\mathbf{K}]} = 0.96$ (right).

5. CONCLUDING REMARKS

A Timoshenko beam with uncertain boundary conditions was analyzed in order to discuss uncertainties on the boundary conditions. First, a parametric probabilistic approach was formulated: the stiffness of a torsional spring inserted in one end of the beam was modeled as uncertain. Then, a nonparametric probabilistic approach was formulated: the stiffness matrix was modeled as uncertain. Each one of the approaches leaded to different results. Considering same values of the coefficients of dispersion, the confidence region of the response for the nonparametric approach was larger than the confidence region for the parametric approach. Concerning the differences, some points should be remarked:

- The numerical simulations showed that, for the problem analyzed, using a 98% confidence limits, the nonparametric strategy englobes the parametric strategy. Indeed, the possible outcomes of the nonparametric approach is higher than the possible outcomes of the parametric approach.
- In the parametric approach one entry of matrix $[\mathbf{K}]$ is perturbed and in the nonparametric approach *all* entries of matrix $[\mathbf{K}]$ are perturbed.
- The event space is one-dimensional manifold for the parametric approach, and, for the nonparametric approach, the event space is multi-dimensional (it depends on the size of $[\mathbf{K}]$).
- It is possible to take into account model uncertainties with the nonparametric strategy, but with the parametric strategy it is not possible.
- For both approaches, as the frequency increases, the predictability decreases. So, one must be careful when analyzing high frequency problems.

6. ACKNOWLEDGEMENTS

The authors acknowledge CNPQ, CAPES and FAPERJ for the financial support.

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8. Responsibility notice

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