INVESTIGATING THE RELATIONS BETWEEN THE WAVE FINITE ELEMENT AND SPECTRAL ELEMENT METHODS USING SIMPLE WAVEGUIDES

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Abstract. *Low order flexural waves can be modeled, at low frequencies, using models such as Euler-Bernoulli and Timoshenko for beams and Kirchhoff and Mindlin for plates. At higher frequencies, higher order theories such as Flügge's and Donnell-Mushtari for cylindrical shells and Mindlin-Herrmann for rods can be used to predict analytically the wave solutions for simple geometries. However, in the case of more complex geometries, modeling can only be achieved practically using numerical methods such as the Finite Element Method (FEM). As frequency goes higher, the size of the numerical model of the complete structure becomes excessive and the computational cost sets a high frequency limit for numerical methods. Aiming at overcoming this limitation in the case of structures that have one dimension much larger than the others and can be treated as a waveguide, waveguide finite element methods (WFEM) have been developed in recent years. They are closely related with spectral element methods (SEM), as both approach the structure as a waveguide. In the WFEM, a slice of the waveguide is modeled by FE and, from this model, a waveguide model can be derived and used to compute the spectral relations, the group and energy velocities, and the forced response.This approach can be usefu, for instance, when simulating structural health monitoring techniques based upon wave propagation.In this paper, spectral elements are used to model waveguide slices of simple beam and plate structures in a WFEM approach. This allows us to investigate the relationships between both approaches and possible synergies to be explored. Numerical aspects of WFEM are also investigated using these simple problems where the SEM solution is exact. Simple WFEM models of a Timoshenko beam are also developed from FEM slices to illustrate similarities and differences of both approaches.*

Keywords*: wave finite elements, spectral element method, waveguides, wave propagation*

1. INTRODUCTION

The modal approach has become overwhelmingly dominant in linear structural dynamic analysis. Most textbooks on linear vibrations do not even mention the wave propagation or the transfer matrix (or state vector) approach. This is mainly due to the abundance of computing power, which allowed the widespread use of numerical methods such as the Finite Element Method (FEM). Analytical methods have been somewhat left aside, although their development continued in theoretical and applied physics and in some advanced engineering fields, mainly associated with acoustics and structural acoustics. More recently, the development of symbolic computation software brought back the interest for analytical and semi-analytical methods that can be more easily implemented with these new tools. In many situations the semi-analytical approaches are much more economical computationally than purely numerical methods such as FEM. Nevertheless, they still require a more skilled user, as the solutions are typically *ad hoc*. Therefore, the possibility combining semi-analytical with standard numerical methods is very promising for solving structural dynamic problems in situations where purely numerical FEM analyses face difficulties, as it is the case with large periodic structures and at high frequencies.

Many structures are made of thin panels and beams. Waves that propagate in waveguides formed by two parallel free surfaces are usually referred to as Lamb waves (Graff, 1991; Doyle, 1997). These include flexural, torsional, longitudinal and other types of symmetric and anti-symmetric waves. Low order flexural waves can be modeled, at low frequencies, using models such as Euler-Bernoulli and Timoshenko for beams and Kirchhoff and Mindlin-Reissner for plates. At higher frequencies, higher order theories such as Flügge's and Donnell-Mushtari for cylindrical shells and Mindlin-Herrmann for rods can be used to predict analytically the wave solution for simple geometries. However, in the case of more complex geometries, modeling can only be achieved practically using numerical methods such as Finite and Boundary Element methods.

As frequency goes higher, given the rule-of-thumb that the characteristic length of the finite element must be one sixth of the wavelength (Petyt, 1996), the size of the numerical model of the complete structure becomes excessive and the computational cost sets a high frequency limit for numerical methods.

Aiming at overcoming this limitation by taking advantage of the periodic nature of many structures, hybrid waveguide-finite element methods have been developed in recent years. The first approach, proposed a decade ago by Gavric (1995) and Finnveden (1994), was called Spectral Finite Elements (SFEM). Its drawback stems from the necessity of developing a solution for each case under investigation. Mode recently, Ichchou and collaborators (Mencik and Ichchou, 2005) developed an approach where a standard FEM code can be used to model a slice of the waveguide and, from this model, a waveguide model can be derived and used to compute the spectral relations, the group and energy velocities, and the forced response. Mace and collaborators (Mace *et al.,* 2005) have also developed this method with a similar approach, and have investigated numerical issues (Wake *et al.,* 2006) based on the work by Zhong and Williams (Zhong and Williams, 1995) and proposed a different method for computing the forced response (Duhamel *et al.,* 2006). These wave approaches based upon a finite element model of a slice of the waveguide is based upon the periodic structures theory developed by Mead (1973) in the early seventies.

In this text, after briefly reviewing the modal approach and the spectral element method, the relation between the dynamic matrix, the transfer matrix, the state transition matrix, and the scattering matrix is shown for a simple case of a simple straight homogeneous rod waveguide. Starting from this basic theory, the Wave Finite Element (WFEM) method is presented and the problem of predicting the propagation modes and corresponding wavenumbers are addressed using simple beam and plate examples. Numerical issues are discussed and numerical examples shown.

2. MODAL FORMULATION

When using FEM, constant parameter stiffness and mass matrices representing the linear dynamic behavior of the structure are obtained. With these two matrices, a generalized eigenvalue problem may be solved, and the system of equations can be de-coupled, yielding independent simple second order ordinary differential equations, which have known closed-form analytical solutions. This modal approach has been extensively developed, and efficient matrix methods that take into account the symmetry and sparsity of the large mass and stiffness matrices can be used to solve the dynamic problem at an affordable computational cost. Internal loss factors (damping) can be easily introduced in the modal approach with modal damping coefficients. However, as frequency goes higher, the size of the numerical model can becomes excessive and the computational cost prohibitive, mainly if the problem must be solved many times for different input parameters, as it is the case in optimization and robustness analyses.

In analytical and semi-analytical methods, the solution to the time-domain partial differential system of equations usually starts by transforming the problem to the frequency domain. Therefore, if a direct stiffness approach is used, a dynamic matrix is obtained. This matrix can be obtained with the FEM simply by making a Fourier transformation of the system of ordinary differential equations:

$$\left[K-\omega^2 M\right]U(\omega)=F(\omega) \quad \text{or} \quad \left[D(\omega)\right]U(\omega)=F(\omega) \tag{1}$$

In this formulation damping can be included by an internal loss factor that makes the stiffness matrix complex:

$$\left[K(1+i\eta)-\omega^2 M\right]U(\omega)=F(\omega) \quad \text{or} \quad \left[D(\omega)\right]U(\omega)=F(\omega) \tag{2}$$

A semi-analytical method that is formulated using a direct stiffness approach is the Spectral Element Method. The Spectral Element Method (SEM) was proposed by Doyle (1997), although its basic formulation was already widely known and currently used in the context of wave propagation solutions. In the SEM, the main idea is to combine all advantages of the spectral analysis with the efficiency and organization of the Finite Element Method (FEM). The major advantage of the SEM in comparison with the FEM is due to the fact that the spectral element dynamic stiffness matrix is computed in the frequency domain, which allows the inertia of the distributed mass to be described exactly. Thus, it is not necessary to refine the mesh as the wavelength becomes smaller. It may be shown that the SEM dynamic stiffness matrix corresponds to an infinite number of finite elements (Doyle, 1997).

3. SPECTRAL ELEMENT FORMULATION

The SEM is formulated based on two types of elements, two-noded and throw-off. The latter are used when the member extends to infinity. The major drawback of SEM is that the elements may only be assembled in one dimension, the solution along the orthogonal dimensions having to be found analytically, which is only possible for simple geometries. Doyle (1997) also proposes a more general approach, which consists of using image sources to enforce arbitrary boundary conditions, but the approach still requires an *ad hoc* solution. Thus, the SEM can be combined with the superposition method proposed by Gorman (1999). In order to illustrate the use of SEM models, the simpler type of spectral element is shown here, namely the low-order rod. The most simple rod theory is described by the following partial differential equation of motion:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right]=\rho A\frac{\partial^2 u}{\partial t^2}-q \quad (3)$$

where EA is the axial stiffness and ρA is the mass density per unit length of the rod. The relation between the traction force and the displacement field may be shown to be:

$$F=EA\frac{\partial u}{\partial x} \quad (4)$$

Following that, spectral analysis can be applied with a solution of the form,

$$\hat{u}\left(x,\omega\right)=Ae^{-ikx}+Be^{ikx} \quad (5)$$

where *A* and *B* are the forward and backward-propagating wave amplitudes at each frequency, respectively, and *k* is the wavenumber, given in this case by $k=\omega\sqrt{\rho/E}$. Now, defining a spectral element of length *L* and using the end displacements as boundary conditions, the following symmetric dynamic stiffness element matrix can be easily obtained (Doyle, 1997):

$$\left\{\hat{F}\right\}=\begin{Bmatrix}\hat{F}_1\\ \hat{F}_2\end{Bmatrix}=\frac{EA}{L}\frac{ikL}{\left(1-e^{-i2kL}\right)}\begin{bmatrix}1+e^{-i2kL} & -2e^{-ikL}\\ -2e^{-ikL} & 1+e^{-i2kL}\end{bmatrix}\begin{Bmatrix}\hat{u}_1\\ \hat{u}_2\end{Bmatrix}=\left[\hat{D}_e\right]\left\{\hat{u}\right\} \quad (6)$$

where $\left[\hat{D}_e\right]$ is the complex dynamic stiffness matrix for the rod element, $\left\{\hat{F}\right\}$ is the vector of complex amplitudes of the nodal forces, and $\left\{\hat{u}\right\}$ is the vector of the complex nodal displacement amplitudes. In order to account for structural damping, an internal loss factor η can be applied by using a complex Young's modulus $E\left(1+i\eta\right)$.

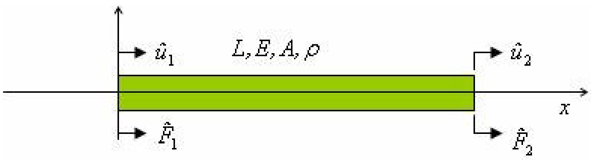

Figure 1. The elementary straight rod element.

With the dynamic stiffness matrix of the elements, it is straightforward to assemble a global stiffness matrix using the direct stiffness method (Craig, 1981). The structural responses can be found by solving, for each frequency, a linear system of equations of the type:

$$\left\{\hat{F}\right\}=\left[\hat{D}\right]\left\{\hat{U}\right\} \quad (7)$$

Boundary conditions can be applied in a standard way to the global system matrix. In the case of a fixed degrees-of-freedom, the corresponding line and column of the global dynamic stiffness matrix are suppressed (they can be used to compute the reaction forces later on). Similar element matrices can be found for beams and shafts, and a three-dimensional frame spectral element can be formulated and used to solve any frame structure problem exactly within the framework of the rod, beam and shafts theories used in the element formulation (Ahmida and Arruda, 2001). Spectral elements have also been derived for plates and shells assembled along one dimension.

4. TRANSFER MATRIX FORMULATION

The transfer matrix method has been extensively used to solve frame structures and rotor system dynamic problems (Pilkey, 2002). Instead of relating forces and displacements at the extremities of an element, the transfer matrix relates forces and displacements at a node with forces and displacements at a neighboring node. With transfer matrices, instead of using a direct stiffness assembling, the state vector (efforts and displacements) is propagated from one extremity of the assembled structure to the other and the boundary conditions are applied, thus generating the solution to the problem.

In the case of the rod treated in the previous section, the element transfer matrix is given by:

$$\begin{Bmatrix} \hat{u}_2 \\ \hat{F}_2 \end{Bmatrix} = \left[T_{21}(\omega)\right] \begin{Bmatrix} \hat{u}_1 \\ \hat{F}_1 \end{Bmatrix} \tag{8}$$

By rearranging the equations and changing the sign of $\left\{\hat{F}_2\right\}$, as now these are internal forces (traction is positive and compression is negative, or vice-versa depending on the sign convention used), whereas in the dynamic stiffness matrix they are external forces, one can write:

$$\left[T_{21}(\omega)\right]_{e21} = \begin{bmatrix} -\hat{D}_{12}^{-1}\hat{D}_{11} & \hat{D}_{12}^{-1} \\ -\hat{D}_{21} + \hat{D}_{22}\hat{D}_{12}^{-1}K_{11} & -\hat{D}_{22}\hat{D}_{12}^{-1} \end{bmatrix} \tag{9}$$

There exist alternative formulations for the transfer matrix approach, where some ill-conditioning problems can be overcome (Zhong and Williams, 1995). Assembling the global matrix is done by propagating the transfer matrices from element 1 to N:

$$\left[T(\omega)\right] = \left[T_{N(N-1)}\right]\cdots\left[T_{32}\right]\left[T_{21}\right] \tag{10}$$

5. WAVE AMPLITUDE FORMULATION AND THE SCATTERING MATRIX

As discussed before, the displacement field of the rod may be represented by the sum of a forward and a backward propagating wave with amplitudes *A* and *B*, respectively. Using Eq. (5), it is possible to express the relation between displacement at nodes and wave amplitudes by the matrix equation:

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{F}_1 \end{Bmatrix} = \left[C_{21}\right] \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \hat{u}_2 \\ \hat{F}_2 \end{Bmatrix} = \left[C_{21}\right] \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} \quad \text{with} \quad \left[C_{21}\right] = \begin{bmatrix} 1 & 1 \\ -iEAk & iEAk \end{bmatrix} \tag{11}$$

where $A_2 = A_1 e^{-ikL}$; $B_2 = B_1 e^{ikL}$. Substituting the elements of the dynamic stiffness matrix in Eq. (9) we can write:

$$\left[T_{21}\right] = \begin{bmatrix} \cos(kL) & -\sin(kL)/EAk \\ EAk\sin(kL) & \cos(kL) \end{bmatrix} \tag{12}$$

and it is easy to show that the eigenvectors of $\left[T(\omega)\right]$ are the columns of $\left[C\right]$ and the eigenvalues are:

$$\left[C_{21}\right]^{-1}\left[T_{21}\right]\left[C_{21}\right] = \begin{bmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{bmatrix} \tag{13}$$

It may be shown that the structure of this matrix is always of the form (Moulet, 2003): $\left[C\right]^{-1}\left[T\right]\left[C\right] = diag\left[\Lambda \quad \Lambda^{-1}\right]$.

In the general case, with *l* and *r* denoting the left and right side sections of the waveguide, the eigenvalue problem with the transfer matrix can be written as:

$$\left[T\right] \begin{Bmatrix} \hat{u}_l \\ \hat{F}_l \end{Bmatrix} = \begin{Bmatrix} \hat{u}_r \\ \hat{F}_r \end{Bmatrix} = \lambda \begin{Bmatrix} \hat{u}_l \\ \hat{F}_l \end{Bmatrix} \tag{14}$$

Given Eqs. (13) and (11) we can write:

$$\begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \begin{bmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} \tag{15}$$

When two transfer matrices are associated, one can write:

$$\begin{Bmatrix} \hat{u}_3 \\ \hat{F}_3 \end{Bmatrix} = [T_{32}][T_{21}] \begin{Bmatrix} \hat{u}_1 \\ \hat{F}_1 \end{Bmatrix} \tag{16}$$

Transforming to the wave amplitude relation:

$$\begin{Bmatrix} A_3 \\ B_3 \end{Bmatrix} = [C_{32}]^{-1}[T_{32}][T_{21}][C_{21}] \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = [Q] \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} \tag{17}$$

When two homogeneous elements are coupled, a discontinuity can be introduced, as the two rod elements may have different properties, causing waves to reflect and/or be transmitted at the discontinuous junction. A scattering matrix can be defined, which relates waves incoming at the junction to waves out coming from the junction (see Fig. 2), by rearranging Eq. (17):

$$\begin{Bmatrix} B_1 \\ A_3 \end{Bmatrix} = \begin{bmatrix} R_{13} & T_{31} \\ T_{13} & R_{31} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_3 \end{Bmatrix} \tag{18}$$

where the following relations apply for the reflection and transmission coefficients:

$$\begin{aligned} R_{13} &= -Q_{22}^{-1}Q_{21} \qquad T_{31} = Q_{22}^{-1} \\ T_{13} &= Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \quad R_{31} = Q_{12}Q_{22}^{-1} \end{aligned} \tag{19}$$

Replacing the values for two-rod elements of the same material and of equal length L, but different cross-section areas (now denoted S not to mix with the wave components), one can obtain:

$$\begin{aligned} R_{13} &= \frac{S_1 - S_2}{S_1 + S_2} e^{i2kL} \quad T_{31} = \frac{2S_1}{S_1 + S_2} e^{i2kL} \\ T_{13} &= \frac{2S_2}{S_1 + S_2} e^{i2kL} \quad R_{31} = \frac{S_2 - S_1}{S_1 + S_2} e^{i2kL} \end{aligned} \tag{20}$$

These values are equal to the theoretical values that can be obtained by standard wave equation solutions for the same discontinuity multiplied by e^{i2kL}, which is the phase delay caused by the crossing of the element twice for each wave component. Therefore, the scattering matrix for this area discontinuity in a rod is expressed as:

$$[S] = \begin{bmatrix} \dfrac{S_1 - S_2}{S_1 + S_2} & \dfrac{2S_1}{S_1 + S_2} \\ \dfrac{2S_2}{S_1 + S_2} & \dfrac{S_2 - S_1}{S_1 + S_2} \end{bmatrix} \tag{21}$$

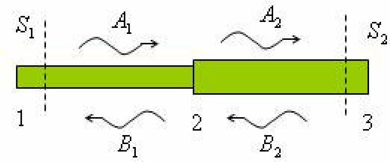

Figure 2. Incoming and out coming waves at a junction.

6. STATE WAVE FORMULATION

The transfer matrix formulation is a discrete version of a more general formulation for waveguides known as state equation formulation. The idea is to transform the structural equilibrium equations, which are partial differential equations, into a set of ordinary, first order differential equations whose closed-form solution is known.

For the rod we can write, from Eqs. (3) and (4), in the homogeneous case $(q=0)$ transforming to the frequency domain (spectral solution):

$$\begin{cases} \dfrac{\partial \hat{F}}{\partial x} = -\rho A \omega^2 \hat{u} \\ \dfrac{\partial \hat{u}}{\partial x} = \dfrac{\hat{F}}{EA} \end{cases} \tag{22}$$

This is a linear system of first order differential equations and can be written in matrix form as:

$$\frac{\partial \{\hat{X}\}}{\partial x} = [N]\{\hat{X}\} \tag{23}$$

where

$$\{\hat{X}\} = \begin{Bmatrix} \hat{u} \\ \hat{F} \end{Bmatrix} \quad ; \quad [N] = \begin{bmatrix} 0 & 1/EA \\ -\rho A \omega^2 & 0 \end{bmatrix}$$

For this first order system, the homogeneous solution can be written in terms of a transition matrix:

$$\{\hat{X}(x)\} = [\Phi(x,0)]\{\hat{X}(0)\} \tag{24}$$

where

$$[\Phi(x,0)] = e^{[N]x} \tag{25}$$

An eigenvalue decomposition of matrix $[\Phi(x,0)]$ can be made by using the property that the eigenvalues of $e^{[N]}$ are the exponentials of the eigenvalues of $[N]$ and the eigenvectors are the same. This is due to the Cayley-Hamilton theorem. The eigenvalues of $[N]$ can be easily shown to be $\pm ik$, so that:

$$[N] = [\Psi][\Lambda][\Psi]^{-1} \quad \text{where} \quad [\Psi] = \begin{bmatrix} 1 & 1 \\ -ikEA & ikEA \end{bmatrix} \quad \text{and} \quad [\Lambda] = \begin{bmatrix} -ik & 0 \\ 0 & ik \end{bmatrix} \tag{26}$$

Thus, the transition matrix, which in this case is the transfer matrix, can be written as:

$$e^{[N]L} = [\Psi] e^{[\Lambda]L} [\Psi]^{-1} = \begin{bmatrix} \cos(kL) & \dfrac{\sin(kL)}{kEA} \\ -kEA\sin(kL) & \cos(kL) \end{bmatrix} \tag{27}$$

There is a sign change with respect to Eq. (12). This is due to the sense of transition: $[\Phi(L,0)]$ or $[\Phi(0,L)]$, which causes a sign change in the eigenvalues. Finally, it should be noted that, usually, a matrix $[N] = i[\bar{N}]$ is used so that the eigenvalues of $[\bar{N}]$ become simply the wave-number $\pm k$ (Moulet, 2003).

7 CHARACTERISTIC EQUATIONS FOR WAVENUMBER SOLUTIONS

For the general waveguide case, denoting by subscripts l and r the left and right section displacements and forces, respectively, the dynamic matrix condensed (see next section) for the left and right section degrees-of-freedom can be written as:

$$\begin{bmatrix} \hat{D}_{ll} & \hat{D}_{lr} \\ \hat{D}_{rl} & \hat{D}_{rr} \end{bmatrix} \begin{Bmatrix} \hat{u}_l \\ \hat{u}_r \end{Bmatrix} = \begin{Bmatrix} \hat{F}_l \\ \hat{F}_r \end{Bmatrix} \tag{28}$$

From the waveguide (periodic) assumption, we have:

$$u(x,y,z,\omega)=\sum_n \hat{u}_n(x,y,\omega)e^{k_n z} \tag{29}$$

from which comes the relations $\hat{u}_{rn}=e^{k_n\Delta}\hat{u}_{ln}=\lambda_n\hat{u}_{ln}$ and $\hat{F}_{rn}=e^{k_n\Delta}\hat{F}_{ln}=\lambda_n\hat{F}_{ln}$. Now, using the first equation in Eq. (14) and the above relations gives:

$$\left(\hat{D}_{ll}+\lambda\hat{D}_{lr}\right)\hat{u}_l=\hat{F}_l \tag{31}$$

and combing with the second equation in Eq. (14) yields $\left[\hat{D}_{ll}+\hat{D}_{rr}+\lambda\hat{D}_{lr}+\frac{1}{\lambda}\hat{D}_{rl}\right]\hat{u}_l=0$, which can be rearranged as:

$$\left[\hat{D}_{rl}+\lambda\left(\hat{D}_{ll}+\hat{D}_{rr}\right)+\lambda^2\hat{D}_{lr}\right]\hat{u}_l=0 \tag{31}$$

To this characteristic equation can be associated the following companion eigenvalue problem:

$$\begin{bmatrix}-\hat{D}_{rl} & 0\\ 0 & \hat{D}_{lr}^T\end{bmatrix}\begin{Bmatrix}\hat{u}_n\\ \lambda_n\hat{u}_n\end{Bmatrix}=\lambda_n\begin{bmatrix}\left(\hat{D}_{ll}+\hat{D}_{rr}\right) & \hat{D}_{lr}\\ \hat{D}_{lr}^T & 0\end{bmatrix}\begin{Bmatrix}\hat{u}_n\\ \lambda_n\hat{u}_n\end{Bmatrix} \tag{32}$$

Solving this eigenvalue problem yields the wavenumbers $k_n(\omega)=\ln\left(\lambda_n(\omega)\right)/\Delta$ and the propagation modes, $\hat{u}_n(x,y,\omega)$. The dependence upon the frequency is shown to emphasize this feature of the wavenumbers and propagation modes. The eigenvalue problem must be solved for each frequency and the pairing of the wavenumbers can be done by the correlation between the corresponding eigenvectors.

8. CONDENSATION OF THE INTERNAL DOFS IN WFEM

When computing the transfer matrix for waveguides with arbitrary shape with a FEM model, the degrees-of-freedom (DOF) must be separated into left surface, right surface and internal DOF. Furthermore, they must be paired by finding at each surface the corresponding DOFs. The dynamic matrix can then be partitioned as (Mace *et al.*, 2005):

$$\begin{bmatrix}\hat{D}_{ii} & \hat{D}_{il} & \hat{D}_{ir}\\ \hat{D}_{li} & \hat{D}_{ll} & \hat{D}_{lr}\\ \hat{D}_{ri} & \hat{D}_{rl} & \hat{D}_{rr}\end{bmatrix}\begin{Bmatrix}\hat{u}_i\\ \hat{u}_l\\ \hat{u}_r\end{Bmatrix}=\begin{Bmatrix}0\\ \hat{F}_l\\ \hat{F}_r\end{Bmatrix} \tag{33}$$

The internal degrees-of freedom can be condensed with:

$$u_i=-D_{ii}^{-1}\left(D_{il}u_l+D_{ir}u_R\right) \tag{34}$$

Thus, it is possible to write the condensed dynamic matrix as:

$$\begin{bmatrix}\hat{D}_{ll}-\hat{D}_{li}\hat{D}_{ii}^{-1}\hat{D}_{il} & \hat{D}_{lr}-\hat{D}_{li}\hat{D}_{ii}^{-1}\hat{D}_{ir}\\ \hat{D}_{rl}-\hat{D}_{ri}\hat{D}_{ii}^{-1}\hat{D}_{il} & \hat{D}_{rr}-\hat{D}_{ri}\hat{D}_{ii}^{-1}\hat{D}_{ir}\end{bmatrix}\begin{Bmatrix}\hat{u}_l\\ \hat{u}_r\end{Bmatrix}=\begin{Bmatrix}\hat{F}_l\\ \hat{F}_r\end{Bmatrix} \tag{35}$$

The transfer matrix $[T]$ can then be written as in Eq. (9). This matrix depends only on the dynamic stiffness of one section of the waveguide. Its eigenvalues can be found by solving the companion matrix Eq. (32) and then be processed to yield the wavenumbers. The spectrum relation obtained is shown for beams and Levy plates in the following sections.

9. BEAM EXAMPLE

The methodology is first shown for a straight homogeneous beam example. First the Timoshenko frame spectral element matrix is used (Ahmida and Arruda, 2001) and results are compared with the theoretical Euler-Bernoulli and Timoshenko spectral relations (Doyle, 1997). Results obtained with a MATLAB® implementation are shown in Fig. 3 for a steel beam with the following properties: thickness=5mm, width=5mm and a slice length of 0.5mm. The figure shows in solid lines the theoretical spectral relations and in lines with circles and plus signs the wavenumbers extracted from the eigenvalues of the transfer matrix.

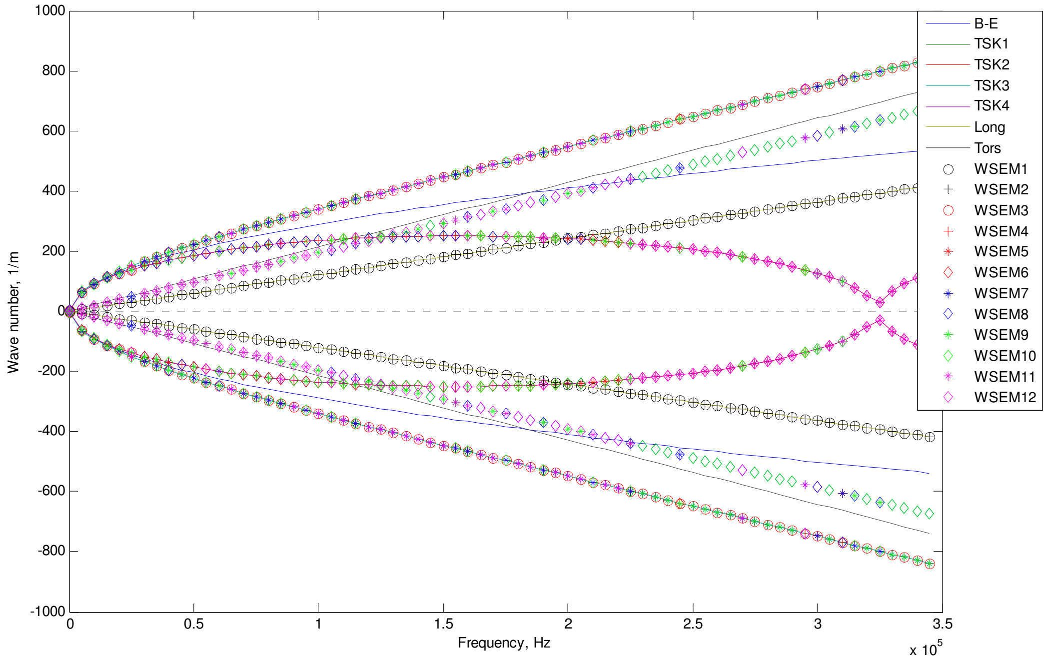

Figure 3. Spectrum relation for a Timoshenko beam; solid lines show the theoretical values and circles and plus signs the wavenumbers computed from the eigenvalues obtained from the spectral element dynamic matrix.

Then a slice of the beam was modeled with solid finite elements (brick) (Fig. 4) and the results obtained are shown in Fig. 5. Observe the validity of Bernoulli-Euler theory for beams for very low frequency ranges. The Timoshenko beam theory is valid for higher frequencies and the waveguide finite element approach is validated. Nevertheless, the WFEM presented frequency limits due to aliasing problems. The limit of the analyzed wavelength is determined by $\lambda \geq 2D_{slice}$, where D_{slice} is the maximum section dimension of the beam slice. In this example, the limit for flexural wavenumbers is at $k_{\lim it} = \pi / D_{slice} = 628m^{-1}$. This limit is indicated by a horizontal heavy black line in Fig. 5.

10. PLATE EXAMPLE

The same methodology is now applied to a homogeneous Levy plate example (simply supported along two parallel sides). The Kirchhoff spectral element matrix (see, for instance, Donadon et al., 2004) is used and the results obtained with a MATLAB® implementation are shown in Fig. 6 for a steel Levy plate with the following properties: thickness=1.8mm, width=180mm and a slice length of 2mm. The figure shows in lines with circles and plus signs the flexural wavenumbers extracted from the eigenvalues of the transfer matrix. A FEM model of a slice of the Levy plate has also been modeled but the pairing of corresponding DOFs has not been concluded and therefore, results are not shown. A general procedure for extracting paired DOFs for left and right surfaces of FEM slices of arbitrary sections in currently being implemented in commercial FEM software.

11. CONCLUSIONS

In this paper the basic concepts of the Waveguide Finite Element Method were reviewed and results for simple Timoshenko beams and Kirchhoff Levy plates were presented to illustrate the technique. Exact spectral elements of the Timoshenko beam and Kirchhoff plate were first used to validate the implementation and then some preliminary WFEM results for the beam were presented and discussed. In the SEM implementation, the high wavenumber limit determined by the length of the slice (see Mace et al., 2005) of π/L was clearly observed. In the case of the FEM slice, the other dimensions of the cross section of the beam also determine high wavenumber limits for the eigenvalues above which an alias phenomenon can be observed. Once the waveguide modes and spectrum relations (wavenumber versus frequency) have been computed, wave propagation solutions for the forced waveguide response can be predicted. The methods for forced response prediction will be treated in a future paper. These methods can be useful for the simulation of wave propagation damage detection techniques in structures such as pipes, frames, and reinforced plates and shells. This paper has treated the basic issues concerning the implementation of the WFEM method and aims at facilitating its comprehension and its use by a more large audience.

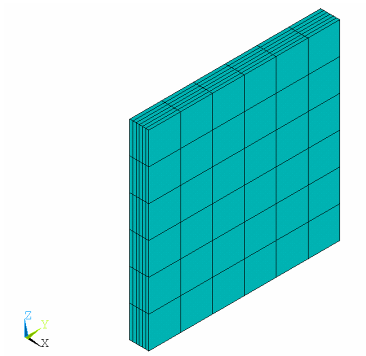

Figure 4: FE mesh of the beam slice.

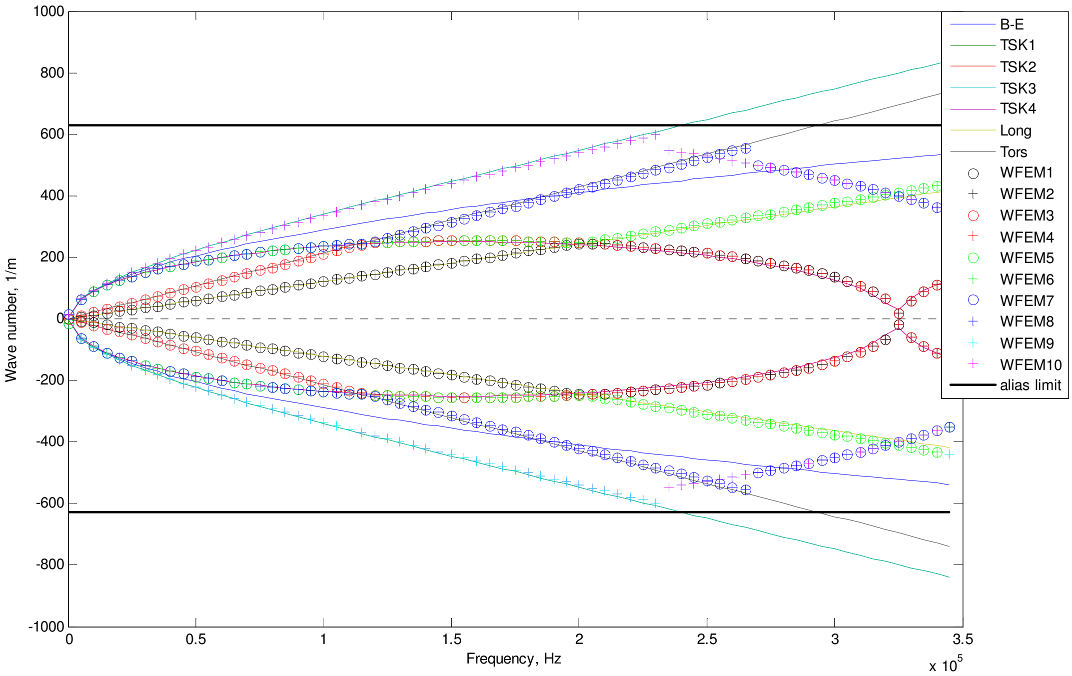

Figure 5: Spectrum relation for a Timoshenko beam. Solid lines show the theoretical values and circles and plus signs the wavenumbers computed from the eigenvalues obtained from the spectral element dynamic matrix.

12. ACKNOWLEDGEMENTS

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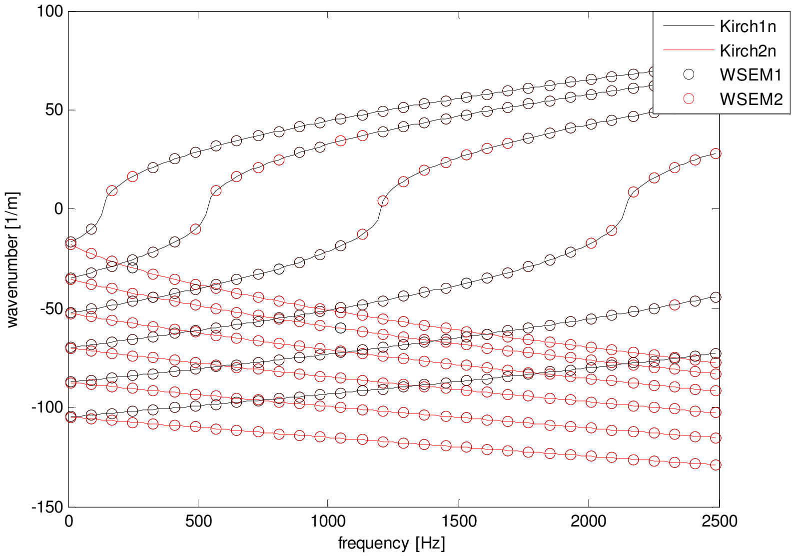

Figure 6. Spectrum relation for a Levy plate. Solid lines show the theoretical values and circles and plus signs show the wavenumbers computed from the eigenvalues of the transfer matrix obtained from the SEM dynamic matrix.

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