OPTIMUM PARAMETERS OF A GEOMETRIC MULTIGRID FOR A TWO-DIMENSIONAL PROBLEM OF TWO-EQUATIONS

Cosmo D. Santiago, cosmo@unibrasil.com.br
Faculdades do Brasil (Unibrasil), Curitiba, PR - Brazil

Carlos Henrique Marchi, marchi@ufpr.br
Department of Mechanical Engineering, Federal University of Paraná, Curitiba, PR, Brazil

Abstract. *In this work it is studied the effect of some Multigrid Method parameters on the necessary CPU time to obtain the numerical solution for a two-dimensional linear thermoelastic problem and the Laplace one. The considered parameters are: grid size, inner iteration and number of grids. The mathematical model is discretized by finite difference method using uniform grids and numerical approximations of second order. The systems of algebraic equations are solved by MSI solver associated to the Geometric Multigrid Method with V-Cicle and Correction Scheme (CS). The transfer of information among grids is made by injection process in restriction and by full weighting operator in prolongation. The obtained results are compared to the ones of the Laplace problem, and also to single-grid ones for both problems, and to other results from literature. It was verified that the coupling of two equations does not degenerate the performance of the multigrid method when compared with the case of one equation.*

Keywords*: solver, finite difference method, CFD, numerical methods, thermoelasticity, 2D-Laplace´s equation*

1. Introduction

For the development of new technologies, it is common the resolution of complex problems in both fluid mechanics and heat transfer. The modeling of such problems, in general, results in partial differential equations, whose analytical solution is unknown or difficult to be obtained, even if some simplifications are employed. In order to overpass these difficulties, numerical approaches can be used, transforming a continuous problem in a discrete one. Diverse methodologies can be employed for this purpose and one of the most used is the Finite Difference Method (FDM) (Tannehill et al., 1997). The discretization of equations applying the FDM leads to a system of algebraic equations of the type

$$\mathbf{A}\vec{x} = \vec{b} \tag{1}$$

where **A** is the coefficients matrix, $\vec{b}$ is the independent vector and $\vec{x}$ is the variable vector. When very fine grids are employed the matrix **A** becomes very large, and consequently, its resolution is impracticable by direct methods, due to the high computational cost to invert the matrix **A** (Golub and Van Loan, 1989). The alternative, therefore, is the use of iterative methods. Researches have been made in order to minimize the CPU time necessary to solve the system given by Eq. (1), without affecting the solution quality (Ferziger and Peric, 1999; Pinto *et al*. 2005; Pinto and Marchi, 2006). In this context, a method that is used frequently to improve the convergence rate is the multigrid one (Briggs *et al.*, 2000).

The multigrid method, studied originally by Fedorenko (1964), improves significantly the convergence rate of conventional iterative ones. Researches about the multigrid method made by Brandt (1977), Stüben (1999) and Wesseling and Oosterlee (2001) presented good numerical results, in respect to the convergence rate, for fluid dynamic problems. The multigrid method has become popular and, currently, is one of most efficient iterative methods for solution of system of equations, like the Eq. (1) (Hirsch, 1988; Tannehill *et al*, 1997, Ghia *et al.*, 1982; Rubini, 1992).

The basic idea of the method is to use a set of grids and to execute iterations in each grid level and approximations of the solutions of this equation in coarse grids (Briggs et al., 2000). The transfer of information from one grid to another requires the definition of transfer operators, called prolongation and restriction. The restriction operators transfer the information from the fine grid to the coarse one and the prolongation ones transfer the information from the coarse grid to the fine one. The system of equations is solved in each grid level with an iterative method, which has the property of reducing the oscillatory errors.

One efficient technique used to reduce the strong oscillations of the residue in each grid, defined by

$$\vec{R} = \vec{b} - \mathbf{A}\vec{x}\ , \tag{2}$$

is to smooth the oscillations for a relaxation method.

Pinto *et al. (*2005) defined transfer operators for any coarsening ratio.

The ideal convergence rate for multigrid is independent of the grid size, i.e., it does not depend on the number of the grid nodes (Hirsch, 1988; Ferziger and Peric, 1999). Moreover, Ferziger and Peric affirm that the more important

property of multigrid method is the independence of the number of iterations from the number of grid nodes, in the finest grid, to obtain the convergence. The application of the multigrid method results in an approximately linear increase of CPU time with grid refinement, allowing the resolution of problems in much finer grids, and, therefore, more accurate solutions can be obtained (Hortmann and Peric, 1990).

Several multigrid algorithms can be found in literature, and they can be divided in two different schemes: CS (Correction Scheme) and FAS (Full Approximation Scheme). The two schemes can be implemented computationally with the V-Cycle, W-Cycle, F-Cycle, Full Multigrid (FMG) and other methodologies (Briggs et al, 2000; Trottenberg, 2001). CS-scheme is generally used in linear problems and FAS in nonlinear ones (Brandt, 1977).

All the algorithms are dependent on parameters that influence the CPU time. Manipulations in the parameter values of the multigrid method can improve the convergence rate by a factor next to 2, using the best combination of these parameters (Ferziger and Peric, 1999). Many of these parameters are studied and optimized by Pinto *et al*. (2005) for linear advection problems, advection-diffusion and Burger's equation. Pinto and Marchi (2006) made an analysis of CS and FAS schemes with some solvers and standard coarsening ratio for the Laplace's equation and suggested the use of the maximum possible number of grids. Tannehill *et al*. (1997) affirmed that the optimum performance of the multigrid method is obtained with diverse grids and suggested the use of 5 or 6 grids for the 2D-Laplace problem with a 129x129 nodes grid. Oliveira *et al*. (2006) also made a study to find the optimum values for some parameters of the method multigrid in linear and nonlinear one-dimensional problems.

In this work, it is proposed for the obtainment of the optimum values for the geometric multigrid method, for the CPU time optimization, for steady-state, two-dimensional linear model of thermoelasticity (TE), with two coupled equations and Dirichlet boundary conditions. It is intended to verify if the performance of multigrid has any change when compared to a one equation problem. The following parameters of the multigrid method are analyzed: the solver inner iterations number (*ITI*), the number of grids (*L*) and the number of variables (*N*). The results are compared to a two-dimensional diffusion problem, using Laplace problem (LP), which is solved by both multigrid and single-grid methods, and with other results from literature. The analyses are made considering the results of the multigrid method, methods for only one grid (single-grid) and the Gauss Elimination (direct) method. The effect of the functions/variables coupling u and v, that appears in the two equations, is also studied to verify if it interferes in the iterative procedure performance, when the multigrid method is used. The multigrid method properties are not preserved when applied to the Navier-Stokes equations, with high Reynolds numbers (Ferziger and Peric, 1999). Therefore, one also intends to apply the conclusions of this qualitative study to Navier-Stokes equations in alternative formulations without pressure-velocity coupling.

This text is organized as follows: in Section 2 the mathematical and numerical models are presented. In Section 3, the computational code is commented. In Section 4, the results are presented and in Section 5 the general conclusions are shown.

2. Mathematical and numerical models

The results of thermoelasticity problem are compared with the ones of a linear problem of two-dimensional heat conduction (Laplace problem), both in steady state and with Dirichlet boundary conditions, in cartesian coordinates.

Problem 1: The constitutive equations of two-dimensional steady state linear thermoelasticity problem, for elastic bodies, whose materials are homogeneous and isotropic (from the Hooke's law), can be reduced to two differential partial equations, written in terms of the displacements

$$C_\lambda \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2C_\lambda \alpha \frac{\partial T}{\partial x}+S^u \tag{3}$$

$$C_\lambda \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=2C_\lambda \alpha \frac{\partial T}{\partial y}+S^v \tag{4}$$

where $C_\lambda = \dfrac{1+\lambda}{1-\lambda}$, and λ is the Poisson's ratio, α is the coefficient of thermal expansion and u and v are displacements in coordinate directions x and y, respectively. The temperature field is given by the analytical solution of the two-dimensional problem diffusion,

$$T(x,y)=\sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi)} \tag{5}$$

The analytical solution proposed for the system formed for Eqs. (3) and (4) is

$$u(x,y) = \beta \sin(\pi x)\frac{\left(e^{2x}-1\right)\left(e^{y}-1\right)}{\left(e^{2}-1\right)(e-1)} \tag{6}$$

$$v(x,y) = \gamma x y^{2} \tag{7}$$

in which $\gamma = 1$ and $\beta = 0.01$ are parameters.

The analytical solution is given by Eqs. (6) and (7), which was obtained by the Manufactured Method Solutions (Roache, 2002). In general, the use of the fabricated solutions is recommended to verify the existence of eventual errors of programming and numerical errors. The substitution of the derivatives of the functions u and v in the Eqs. (3) and (4) results in the parcels of the source term, S^u and S^v, respectively.

The problem is solved in the domain $\Omega = \{0 \le x \le 1;\ 0 \le y \le 1\}$, which is partitioned in a number of nodes given by

$$N = N_x N_y \tag{8}$$

where N_x and N_y are the number of nodes in the directions *x* and *y*, respectively. Each grid node is defined as

$$(x_i, y_i) = ((i-1)h_x, (j-1)h_y), \text{ with } h_x = \frac{1}{N_x - 1} \text{ and } h_y = \frac{1}{N_y - 1} \tag{9}$$

where $i = 1,..., N_x$, $j = 1,..., N_y$, h_x and h_y are the grid size of elements in the directions *x* and *y*, in this order.

The governing equations are discretizated using the Finite Difference Method (FDM), for orthogonal cartesian coordinates and uniform grid in the two coordinate directions. The second order derivatives and mixed derivatives that appear in the first term of each equation are approximated with the Central Difference Scheme (CDS), which possesses second order of accuracy. The discretization of the Eqs. (3) and (4) results in

$$a_{P,P}\phi_{i,j} + a_{P,n}\phi_{i,j+1} + a_{P,s}\phi_{i,j-1} + a_{P,w}\phi_{i-1,j} + a_{P,e}\phi_{i+1,j} = b_P^{\phi} \tag{10}$$

where the coefficients are evaluated by the following expressions: $a_{P,e} = a_{P,w} = -1/h_x^2$, $a_{P,n} = a_{P,s} = -(1 + C_\lambda)/h_y^2$ and $a_{P,P} = -a_{P,e} - a_{P,w} - a_{P,n} - a_{P,s}$, in which $P = (j-1)N_x + i$; these coefficients for both equations are very similar and does not depend on u and v. The numerical approximation for the displacements u and v is represented by ϕ. The numerical approximation for the source term of Eq. (2) is given by

$$b_p^u = C_\lambda \left[\frac{\left(v_{i+1,j+1} + v_{i-1,j-1} - v_{i-1,j+1} - v_{i+1,j-1}\right)}{4h_x h_y} - 2\alpha\pi \frac{\cos(\pi x)\sinh(\pi y)}{\sinh(\pi)} \right] - S^u \tag{11}$$

where $v_{i+1,j+1}$, $v_{i-1,j-1}$, $v_{i-1,j+1}$ and $v_{i+1,j-1}$ are numerical approximations of the mixed-derivative that appears in Eq. (3). The approximation for the term b_p^v of the Eq. (4) is made in a similar form. The Eq. (10) is valid for inner nodes of the calculation domain and Dirichlet boundary conditions are applied taking into account the analytical solution.

Equation (10) represents an algebraic system of equations that has the form of Eq. (1), which is a five-diagonal matrix N_x times N_y, symmetrical and positive-definite (Briggs *et al.*, 2000). In the approximation of the mixed derivative of Eq. (3) appears the function v, that is placed in the b_P^u source term, as can be observed in Eq. (11). The same procedure occurs with the approximation of the mixed derivative of Eq. (4); therefore, the numerical solutions of Eq. (10) is coupled because the solution of u is dependent on the solution of v. The parcels S^u and S^v, that appear in Eqs. (3) and (4), are added to the respective source terms.

Problem 2: The mathematical model of the Laplace problem is given by (Ferziger and Peric, 1999) as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \text{ in } 0 < x < 1 \text{ and } 0 < y < 1 \tag{12}$$

$T(x,0) = T(0,y) = 0$, $T(x,1) = x$ and $T(1,y) = y$

where T represents the temperature. The analytical solution of this problem is given by

$$T(x,y) = xy \tag{13}$$

The numerical model adopted to solve the Laplace problem is the same used for the thermoelasticity problem.

In this work, the geometric multigrid (Wesseling and Oosterlee, 2001) is adopted, with the V-Cycle CS scheme for both problems. In this scheme the transferred information in each grid level are the residue (in the restriction) and the correction of the solution (in the prolongation). In the CS scheme, the Eq. (1) is solved only in the finest grid; in coarse grids, only the residual equation is solved (Briggs et al., 2000). The correction is transferred to be added to the solution of the current grid and, therefore, the next refined grid has its initial estimate trought up to date with a correction value that will contribute for the elimination of the low frequency errors (Trottenberg, 2001). In this work, the injection restriction and the prolongation by bilinear interpolation are adopted.

The ideal solver to be used with the multigrid method is one which has good smooth properties, for example, the Gauss-Seidel method (Briggs et al., 2000). Modified Strongly Implicit Method – MSI (Schneider and Zedan, 1981), presented better performance than the Gauss-Seidel one, as previous results of Pinto and Marchi (2006). Here, the MSI method was chosen as standard solver, and the coarsening ratio is equal to 2 (standard value in literature); this means that the size of the element on a finer grid is the half size of the element in an immediately coarser grid. Other coarsening ratios were studied by Pinto *et al*. (2005) for one-dimensional problems of advection, advection-diffusion and Burgers' equation. For the test-problem of thermoelasticity, an algorithm of the multigrid method, CS scheme with V-Cycle, for two grids is described

Table 1. Scheme CS for two grids with V-Cycle (Adapted from Briggs *et al*., 2000)

LMG$(u_0, u, v_0, v, b_u, b_v, h)$

Begin

1. Smooth $A_u^h u^h = b_u^h$ *ITI* times with initial guest u_0^h;
2. Calculate the residue: $R_u^h = b_u^h - A_u^h u^h$;
3. Smooth $A_v^h v^h = b_v^h$ *ITI* times with initial guest v_0^h;
4. Calculate the residue: $R_v^h = b_v^h - A_v^h v^h$;
5. Restrict the residue: $b_u^{2h} = I_h^{2h} R_u^h$ and $b_v^{2h} = I_h^{2h} R_v^h$
6. Smooth: $A_u^{2h} e_u^{2h} = b_u^{2h}$ *ITI* times with initial guest $e_u^{2h} = 0$;
7. Smooth: $A_v^{2h} e_v^{2h} = b_v^{2h}$ *ITI* times with initial guest $e_v^{2h} = 0$;
8. Obtain: $e_u^h = I_{2h}^h e_u^{2h}$ and $e_v^h = I_{2h}^h e_v^{2h}$;
9. Correct the solution: $u^h \leftarrow u^h + e_u^h$ and $v^h \leftarrow v^h + e_v^h$;
10. Smooth: $A_u^h u^h = b_u^h$ *ITI* times with initial guest u^h;
11. Smooth: $A_v^h v^h = b_v^h$ *ITI* times with initial guest v^h;

and of **LMG**$(u_0, u, v_0, v, b_u, b_v, h)$

The described algorithm in Tab. 1 is applied for two grids but it can be extended for some grids. In order to facilitate the notation, the vectorial notation for $\vec{u}$, $\vec{v}$ and $\vec{b}$ was omitted only in the algorithm. The restriction and prolongation operations are represented by I_h^{2h} and I_{2h}^h, respectively. The systems of equations for $\vec{u}$ and $\vec{v}$ are smoothed in the finest grid Ω^h (steps 1 and 3), in order to obtain an approximation of the solution with *ITI* iterations. The residue is calculated in steps 2 and 4, as indicated in the Eq. (2), and then it is transferred to the residual source terms (step 5) of the coarser grid Ω^{2h} and the system of equations is solved (steps 6 and 7). In step 9, the correction is transferred to the finest grid and the initial guest is re-estimated. In steps 10 and 11, the systems are solved for the finest grid with a corrected initial guess. The described algorithm covers only one V-Cycle CS scheme. The following procedure develops diverse calls of **LMG** until the achievement of a stop criterion:

Choose u_0^h, v_0^h and $ite_{\max}$

1. $i = 1$

while (no converged) or ($i \le ite_{\max}$), do

2. **LMG**$(u_0, u, v_0, v, b_u, b_v, h)$
3. $u_0 = u$ and $v_0 = v$

$i = i + 1$

end while

In step 2, the algorithm is called as many times as necessary until achieving the established convergence criterion or achieving the maximum number of cycles chosen ($ite_{\max}$); while it does not occur, the solution is updated (step 3) with the approximated solution of the previous cycle (Adapted from Wesseling, 1992).

3. Verification of the computational code

The algorithm described in the previous Section was implemented in FORTRAN 95 language, using the Visual Compaq FORTRAN 6.6. The simulations were made in a PC with Processor Intel Duo Core 2.66 GHz and 2 GB RAM, using double precision arithmetic.

Tests of coherency have been made for the verification of the computational code, to establish both the tolerance and the convergence criterion, although the analytical solution of the problem has been obtained, as cited in Section 2. The adopted procedure is described as follows: in the finest possible grid and taking into account the limitations of computational physical memory, the program is run until the elimination the iteration error. In this point, the solution of the algebraic system of equations is saved in a file and is taken as the exact solution of the system. With the average norm l_1 (Ferziger and Peric, 1999) calculated in each iteration, the number of significant figures was verified, without varying with the iterations. Considering this verification, a tolerance whose magnitude is 10^{-12} was obtained, that is two orders of magnitude over the significant figures verified for round-off-error. The tolerance, therefore, is defined as $\varepsilon = 10^{-12}$. The same procedure is used to generate the exact solution in all coarse grids. Thus, the convergence criterion is defined as

$$\bar{l}_1[E_k(\phi)]_k = \frac{\sum_{i=1}^{N} \left| (\phi_{k\to\infty} - \phi_k)_i \right|}{N} \tag{14}$$

where $\phi_{k\to\infty}$ is the exact solution of the system of equations, ϕ_k is the solution in k-iteration, N is the total number of nodes in the grid, E_k is the k-iteration error, $\bar{l}_1$ denotes the average norm of the error during the k-iteration and i denotes the node. Thus, one obtains minor possible relative error compared to the true solution of the system of equations, generating, therefore, greater confidence and quality of the solutions. The adopted initial estimate in this work is $\vec{u} = 0$ and $\vec{v} = 0$. The CPU time is measured using the function TIMEF of FORTRAN 95 PORTLIB library. The uncertainty of this function is approximately ± 0.05 s (Pinto *et al*, 2005).

4. Results

The results obtained for the multigrid method parameters considered in this work are presented in this section. About 180 simulations have been made. The number of inner iterations influences on the CPU time, number of grids and size of the problem in the two cases are analyzed. The considered simulation results are for the grids 257x257, 513x513 and 1025x1025. For the study about the size of the problem, the simulations have been made for all grids, since the coarsest grid, 5x5, until the finest grid, 2049x2049. The results of single-grid and the direct method (EG) also are presented and compared. Commentaries on the effect of the coupling of the equations will be made in the last Section.

4.1. Inner iterations – (*ITI*)

For each grid and each problem, simulations with number of iterations have been made varying from $ITI = 1$ until $ITI = 10$, and also $ITI = 15$. Figure 1 shows that, for all the tested grids, the minor CPU time occurred for $ITI = 2$, for both the thermoelasticity problem (TE) and the Laplace problem (LP). The CPU time increases almost linearly with the number of inner iterations, as can be observed in Fig. 1. The optimum number of inner iterations $(ITI_{optimum})$ for the

two cases can be established as $ITI_{optimum} = 2$. The CPU time observed for the of thermoelasticity problem in the finest grid is approximately 4 times larger than the time observed for the Laplace problem in the same grid size. The results show that the coupling of the equations does not influence the optimum number of inner iterations. The results are similar to those presented by Pinto and Marchi (2006) for the two-dimensional linear problem of heat conduction (Laplace problem), even using another criterion of tolerance for the convergence.

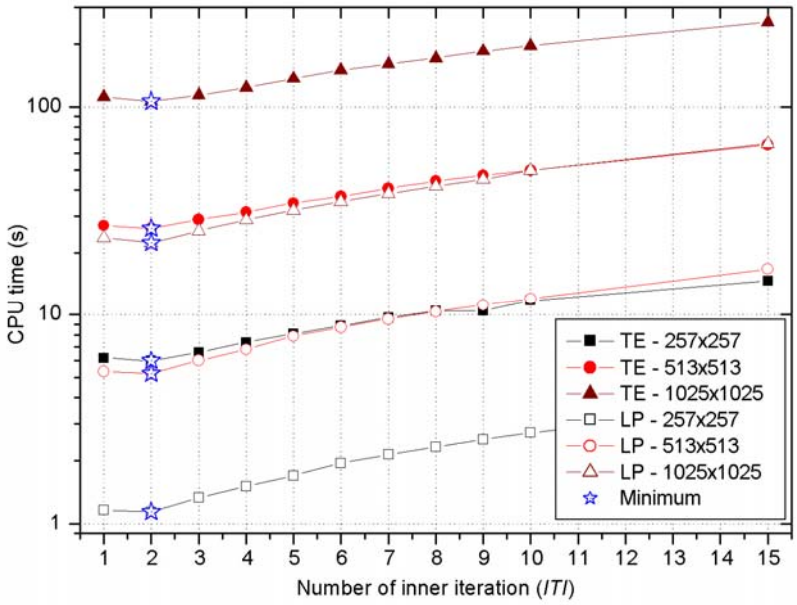

Figure 1. Influence of number of inner iterations on the CPU time.

4.2. Number of Grids – (*L*)

The study of the number of grids influence (*L*) takes into account the number of optimum inner iterations obtained previously. The intention is to optimize the CPU time. Figure 2 shows that the minor CPU time occurs with the number of grids around the maximum $(L_{maximum})$, or the maximum, that is, $t_{CPU}(L_{maximum}) \approx t_{CPU}(L_{optimum})$. For the finest grid considered in this analysis, i.e., 1025x1025 nodes, the difference between $L_{maximum}$ CPU time and $L_{optimum}$ CPU time is around 2%. For the Laplace's problem this difference is lower than 0,5%. One notices that when the number of grids is diminished in relation to the optimum, the CPU time increases very quickly. The simulations with L = 4, 3 and 2 grids, for the largest problem (largest N) for example, require a lot of CPU time. The conclusions of this analysis are the same ones for both problems studied TE and LP, except by the difference of the CPU time between $L_{optimum}$ and $L_{maximum}$. One notices, therefore, that the coupling of the equations of the thermoelastic problem does not influence the optimum number of grids. Similar results for linear and nonlinear one-dimensional problems have been obtained by Pinto *et al.* (2005) and Pinto and Marchi (2006). The results obtained in this work, for the 2D Laplace's equation with $N = 129\text{x}129$, are according to the conclusions of Tannehill *et al.* (1997), who affirmed that the use of 5 or 6 grids, for the same problem, practically results in the same performance of 7 grids. Hirsch (1988) cites, in his work, that generally 4 or 5 grids are used. Roache (1998) affirms that the use of only 2 levels of grids is not recommended.

4.3. Number of variables (*N*)

The optimum number of iterations $(ITI_{optimum})$ and the optimum number of grids $(L_{optimum})$, obtained previously, are considered in this study about the influence of the size of the problem on the CPU time. In this analysis, all sizes of grids are considered, i.e., from the minor 5x5 to the greatest supported by the PC memory, 2049x2049, using the multigrid method. The results obtained from single-grid method (unique grid) with MSI solver and with the Gauss Elimination for the thermoelasticity problem are also shown. In this case, the adopted grid were 5x5, 9x9,…, 257x257 and 5x5,…, 33x33, respectively. Very refined grids for the single-grid method require extremely high CPU time, taking hours or even days for achieving the convergence, using direct method. For the Laplace's problem, the results of the single-grid method and the Gauss Elimination are not presented, but they can be found in Pinto and Marchi (2006). For small grids, the CPU time is about zero. In this case, a methodology was adopted to obtain a time value which eliminated as possible the CPU time error due to uncertainty of measurement by the TIMEF function. The main idea is

to obtain an acceptable precision of the CPU time. Thus, for all grids whose time of simulation is less than 10 seconds, an external cycle in the main routine was made in order to the program make the number of necessary simulations until it gets 10 seconds or more. The time of a simulation, therefore, is the average consumed time for all simulations, in that grid. For example, in the thermoelastic problem, the size grid 5x5, simulated with the multigrid method, solver MSI, and optimum parameters *L* and *ITI*, needed 38175 simulations to reach 10,02 seconds, resulting in the average time of 0.000262 second by simulation.

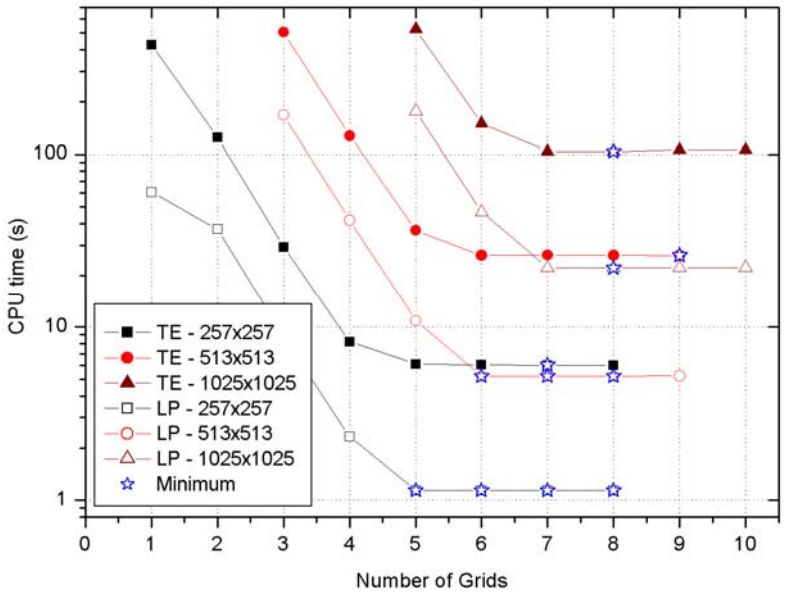

Figure 2. Influence of number of grids on the CPU time

Figure 3 shows the influence of the number of variables on the CPU time, for the two problems, in the multigrid case. It can be noticed that, in the multigrid method, when the size of the problem is increased, the CPU time grows almost linearly. For the thermoelastic problem, the multigrid method presents better results than the single-grid one starting from 33x33 nodes. For this grid, single-grid starts to present significantly larger CPU time than the multigrid method. The CPU time of the thermoelastic problem is on average 5.3 times larger than the CPU time for the Laplace's problem using the multigrid method. In comparison between the multigrid and single-grid methods, for the thermoelasticity problem and 257x257 grid, the multigrid method is approximately 70 times faster than the single-grid. For the 1025x1025 grid, by extrapolation, is obtained that the multigrid method is approximately 1000 times faster than the single-grid one and approximately 1.3×10^6 faster than the Gauss Elimination method. In comparison between the Gauss Elimination and the multigrid method, for the 33x33 grid, the last one is about 1400 times faster and comparing to the single-grid one, is about 1.4 time.

4.4. Analysis of methods

To determine the order (*p*) to the solver associated to the methods and the behavior of the curve in function of the time, a considered geometric least square fitting method is presented, given by

$$t_{CPU}(N) = cN^{p} \tag{15}$$

where p represents the order of *solver* associated to the used method, or inclination of the curve, and c is coefficient that depends of each method and each *solver*. The value of the exponent *p* in Eq. (15) gives important information to the analysis of *solve*r. The ideal multigrid method is that one whose *p* = 1, so the nearer the unit the value of *p*, the better is its performance. The obtained values of c and p to all *N* are in Tab. 2 and to $N > 33\times33$ in Tab. 3. They confirm that the CPU time of the multigrid method and solver MSI, for the two tested problems, increase almost linearly if all size of problems is considered, as observed in Fig. 3. The order of solver is not much affected when the two problems are solved with MG-MSI. In fact, the exponents *p* of the Eq. (15), with the multigrid method are 1.17 and 1.18, for the thermoelastic and Laplace problems, respectively, i.e., around the ideal. When $N > 33\times33$, *p* is obtained far better in two problems, Tab 3, with 1.05 and 1.06.

The values of *p* obtained for the single-grid method with MSI and the Gauss Elimination method demonstrate a weak performance when increase the size of the problem.

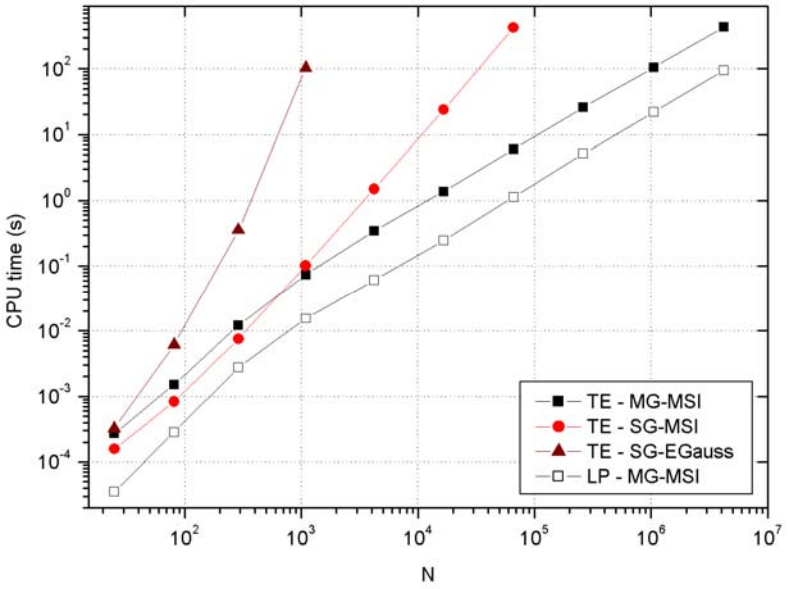

Figure 3. Influence of size of problem on the CPU time for the methods MG-MSI, SG-MSI and Gauss Elimination

Table 2. Values of *c* and *p* obtained from geometric least square fitting for solvers MSI and Gauss Elimination in two problems.

Problem	***Solver***	**MG**		**SG**	
		c	***p***	***c***	***p***
Laplace	**MSI**	$2.02\text{x}10^{-6}$	1.18	$1.60\text{x}10^{-8}$	1.97
Thermoelasticity	**MSI**	$1.26\text{x}10^{-5}$	1.17	$2.16\text{x}10^{-7}$	1.90
	EGauss	----	----	$3.90\text{x}10^{-9}$	3.35

Table 3. Values of *c* and *p* obtained from geometric least square fitting for solvers MSI in two problems to $N > 33 \times 33$.

Problem	***Solver***	**MG**		**SG**	
		c	***p***	***c***	***p***
Laplace	**MSI**	$8.71\text{x}10^{-6}$	1.05	$6.62\text{x}10^{-9}$	2.06
Thermoelasticity	**MSI**	$5.23\text{x}10^{-5}$	1.06	$8.03\text{x}10^{-8}$	2.01

5. Conclusion

In this work, it was analyzed the influence of diverse parameters of the geometric multigrid method, with CS scheme, on the necessary CPU time to solve problems with two coupled equations and Laplace's problem (only one equation). The analyzed parameters were: number of inner iterations (ITI), number of grids (L) and number of nodes (N). To discretizate the equations the Finite Difference Method with Central Difference Scheme and Dirichlet boundary conditions was adopted.

Based on the results of this work, it was verified that:

1) The optimum number of inner iterations is 2, in any grid, in the two problems. The *ITI* can affect the CPU time significantly.
2) The optimum number of grids is around the maximum, i.e., $L_{optimum} \approx L_{maximum}$. The number of grids can affect the CPU time significantly.
3) The coupling of two equations does not degenerate the performance of the multigrid method when compared to the case of one equation.

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